Isogeometric Analysis of the Navier–Stokes–Cahn–Hilliard equations with application to incompressible two-phase flows

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Motivation

Objective: simulate **two-phase flow** problems with **large density and viscosity ratios** using the **Cahn–Hilliard phase field model**.

We would like to use a method that has the following advantages:

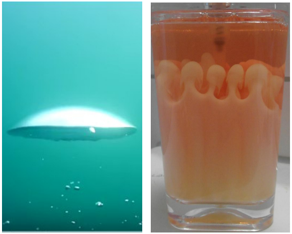


Figure : ©[7, 8]

- Complex geometries → Isogeometric Analysis
- Systematic physical approach to address interface dynamics (fluid free energy model)
- Natural handling of topological transitions
- Implicit fluid-fluid interface representation
- Modeling of interfacial forces as volume forces
- Reinitialization-free

Cahn–Hilliard (CH) phase field model

Equilibrium interface profiles **minimize** fluid free energy functional

$$\mathcal{E}(\varphi) := \int_\Omega f \, \mathrm{d}\Omega = \int_\Omega \frac{1}{2}\alpha|\nabla\varphi|^2 + \beta\psi(\varphi)\, \mathrm{d}\Omega$$

$$\frac{\partial\varphi}{\partial t} + \boldsymbol{v}\cdot\nabla\varphi = \nabla\cdot\left(m(\varphi)\nabla\underbrace{\left(-\alpha\nabla^2\varphi + \beta\psi'(\varphi)\right)}_{\eta := \frac{\delta\mathcal{E}}{\delta\varphi}}\right),$$

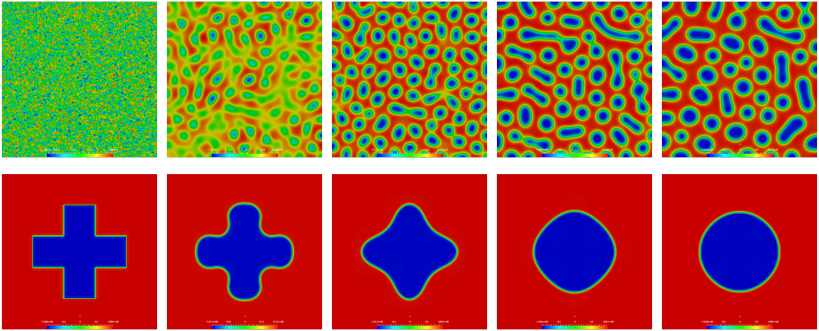
$$\nabla\eta\cdot\boldsymbol{n} = 0, \qquad \text{(no flux b.c.)}$$

$$\nabla\varphi\cdot\boldsymbol{n} = \frac{1}{\epsilon\sqrt{2}}\cos(\theta)(1-\varphi^2) \qquad \text{(contact angle b.c.)}$$

$$\frac{\partial\varphi}{\partial t} + \boldsymbol{v}\cdot\nabla\varphi - \nabla\cdot(m(\varphi)\nabla\eta) = 0,$$

$$\eta - \beta\,\frac{d\psi(\varphi)}{d\varphi} + \alpha\,\nabla^2\varphi = 0$$

Minimization of Ginzburg-Landau free energy



- Volume fraction of the first fluid in the mixture: $\vartheta = \frac{dV_1}{dV}$
- $\varphi = 2\vartheta - 1$
- Volume averaged density and viscosity:

$$\rho(\varphi(x, y)) = \rho_1(1 + \varphi)/2 + \rho_2(1 - \varphi)/2,$$
$$\mu(\varphi(x, y)) = \mu_1(1 + \varphi)/2 + \mu_2(1 - \varphi)/2$$

Governing equations

Navier–Stokes–Cahn–Hilliard system:

$$
\begin{aligned}
\rho(\varphi)\left(\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right)-\nabla\cdot\left(\mu(\varphi)\left(\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T\right)\right) &= -\nabla p+\rho(\varphi)\boldsymbol{g}+\eta\nabla\varphi && \text{in } \Omega_T,\\
\nabla\cdot\boldsymbol{v} &= 0 && \text{in } \Omega_T,\\
\frac{\partial\varphi}{\partial t}+\boldsymbol{v}\cdot\nabla\varphi-\nabla\cdot(m(\varphi)\nabla\eta) &= 0 && \text{in } \Omega_T,\\
\eta-\beta\,\frac{d\psi(\varphi)}{d\varphi}+\alpha\,\nabla^2\varphi &= 0 && \text{in } \Omega_T,\\
\varphi(\boldsymbol{x},0)=\varphi_0(\boldsymbol{x}),\quad \boldsymbol{v}(\boldsymbol{x},0) &= \boldsymbol{v}_0(\boldsymbol{x}) && \text{in } \Omega,\\
\frac{\partial\varphi}{\partial\boldsymbol{n}}=\frac{\partial\eta}{\partial\boldsymbol{n}}=0,\quad \boldsymbol{v} &= \boldsymbol{v}_D && \text{on } (\partial\Omega_T)_D,\\
\left(-p\boldsymbol{I}+\mu(\varphi)\left(\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T\right)\right)\cdot\boldsymbol{n} &= \boldsymbol{t} && \text{on } (\partial\Omega_T)_N
\end{aligned}
$$

Time discretization: one-step θ-scheme (implicit)

Operator splitting solution algorithm

while $t \leq T$ **do**

1. **Solve nonlinear Advective Cahn–Hilliard system**:
   Find $(\varphi(\boldsymbol{x},t),\, \eta(\boldsymbol{x},t)) \in \mathcal{S} \times \mathcal{S} \times (0,T)$, s.t. $\forall\, q, v \in \mathcal{V}$ it holds:

   $$\mathcal{F}_{\mathsf{CH}}((\varphi,\eta);(q,v)) = 0 \qquad \text{Semilinear form}$$

   In each Newton iteration,

   $$\begin{aligned}
   &\text{Find } (\delta\varphi, \delta\eta) \in \mathcal{S} \times \mathcal{S} \times (0,T), \text{ s.t.} \\
   &\mathcal{F}'_{\mathsf{CH}}((\varphi^k,\eta^k);(\delta\varphi,\delta\eta),(q,v)) = -\mathcal{F}_{\mathsf{CH}}((\varphi^k,\eta^k);(q,v)) \qquad \forall\, q, v \in \mathcal{V} \\
   &(\varphi^{k+1},\eta^{k+1}) = (\varphi^k,\eta^k) + \omega\,(\delta\varphi,\delta\eta)
   \end{aligned}$$

2. **Solve nonlinear two-phase Navier–Stokes system**:
   Find $\boldsymbol{v}(\boldsymbol{x},t) \in \boldsymbol{\mathcal{S}} \times (0,T)$ and $p(\boldsymbol{x},t) \in \mathcal{Q} \times (0,T)$, s.t. $\forall\,(\boldsymbol{w},q) \in \boldsymbol{\mathcal{V}} \times \mathcal{Q}$ it holds

   $$\mathcal{F}_{\mathsf{NS}}(\boldsymbol{u};(\boldsymbol{w},q)) = 0 \qquad \text{Semilinear form with } \boldsymbol{u} = (\boldsymbol{v},p)$$

   In each Newton iteration,

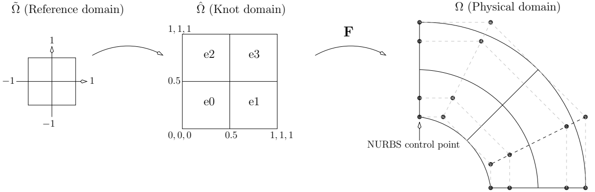
   $$\begin{aligned}
   &\text{Find } \delta\boldsymbol{v}(\boldsymbol{x},t) \in \boldsymbol{\mathcal{S}} \times (0,T) \text{ and } \delta p(\boldsymbol{x},t) \in \mathcal{Q} \times (0,T), \text{ s.t.} \\
   &\mathcal{F}'_{\mathsf{NS}}(\boldsymbol{u}^k;\delta\boldsymbol{u},(\boldsymbol{w},q)) = -\mathcal{F}_{\mathsf{NS}}(\boldsymbol{u}^k;(\boldsymbol{w},q)) \qquad \forall\,(\boldsymbol{w},q) \in \boldsymbol{\mathcal{V}} \times \mathcal{Q} \\
   &\boldsymbol{u}^{k+1} = \boldsymbol{u}^k + \omega\,\delta\boldsymbol{u}
   \end{aligned}$$

end

Discrete Isogeometric approximation spaces

Approximation of velocity and pressure functions with LBB-stable Taylor-Hood like non-uniform rational B-spline space pairs $\hat{\mathbf{V}}_h^{TH}/\hat{Q}_h^{TH}$

$$\hat{\mathbf{V}}_h^{TH} \equiv \hat{\mathbf{V}}_h^{TH}(\mathbf{p}, \boldsymbol{\alpha}) = \boldsymbol{\mathcal{N}}_{\alpha_1,\alpha_2}^{p_1+1,p_2+1} = \mathcal{N}_{\alpha_1,\alpha_2}^{p_1+1,p_2+1} \times \mathcal{N}_{\alpha_1,\alpha_2}^{p_1+1,p_2+1},$$
$$\hat{Q}_h^{TH} \equiv \hat{Q}_h^{TH}(\mathbf{p}, \boldsymbol{\alpha}) = \mathcal{N}_{\alpha_1,\alpha_2}^{p_1,p_2}.$$



Corresponding spaces $\mathbf{V}_h^{TH}$ and Q_h^{TH} in the physical domain Ω obtained via component-wise mapping using parametrization $\mathbf{F} : \hat{\Omega} \to \Omega$

$$\mathbf{V}_h^{TH} = \{\mathbf{v} : \mathbf{v} \circ \mathbf{F} \in \hat{\mathbf{V}}_h^{TH}\} \qquad Q_h^{TH} = \{q : q \circ \mathbf{F} \in \hat{Q}_h^{TH}\}$$

Discrete problems

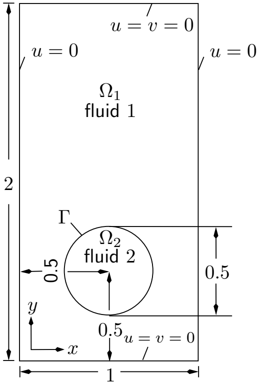
- Variational mixed formulation of CH problem

$$\begin{cases} \mathcal{S}^h = \mathcal{H}^1(\Omega) \cap V_h^{TH} \\ \mathcal{V}^h = \mathcal{H}_0^1(\Omega) \cap V_h^{TH} \\ \text{Find } (\varphi^h, \eta^h) \in \mathcal{S}^h \times \mathcal{S}^h \times (0,T), \text{ s.t.} \\ \mathcal{J}^h_{\mathsf{CH}}(\cdot; (\varphi^h, \eta^h), (q^h, v^h)) = \mathcal{F}_{\mathsf{CH}}(\cdot; (q^h, v^h)) \qquad \forall\, q^h, v^h \in \mathcal{V}^h \end{cases}$$

- Variational formulation of NS problem

$$\begin{cases} \boldsymbol{\mathcal{S}}^h = \boldsymbol{\mathcal{H}}^1(\Omega) \cap \mathbf{V}_h^{TH} \\ \boldsymbol{\mathcal{V}}^h = \boldsymbol{\mathcal{H}}_0^1(\Omega) \cap \mathbf{V}_h^{TH} \\ \mathcal{Q}^h = \mathcal{L}_2(\Omega)/\mathbb{R} \cap Q_h^{TH} \\ \boldsymbol{u}^h = (\boldsymbol{v}^h, p^h) \\ \text{Find } \ \boldsymbol{v}^h \in \boldsymbol{\mathcal{S}}^h \times (0,T) \ \text{ and } \ p^h \in \mathcal{Q}^h \times (0,T), \text{ s.t.} \\ \forall\, (\boldsymbol{w}^h, q^h) \in \boldsymbol{\mathcal{V}}^h \times \mathcal{Q}^h \\ \mathcal{J}_{\mathsf{NS}}(\cdot; \boldsymbol{u}^h, (\boldsymbol{w}^h, q^h)) = \mathcal{F}_{\mathsf{NS}}(\cdot; (\boldsymbol{w}^h, q^h)) \end{cases}$$

Rising bubble



$$A_b = \int_{\Omega_2} 1 \, \mathrm{d}\boldsymbol{x},$$

$$V_b = \int_{\Omega_2} \boldsymbol{v}.y \, \mathrm{d}\boldsymbol{x} / A_b,$$

$$Y_b = \int_{\Omega_2} \boldsymbol{x}.y \, \mathrm{d}\boldsymbol{x} / A_b,$$

$$\not{c} = \frac{P_a}{P_b} = \frac{2\pi\sqrt{A_b/\pi}}{P_b},$$

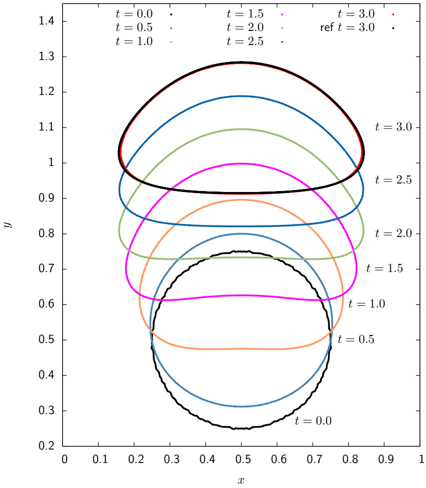
$$\mathsf{EOC}_{(\cdot)} = \frac{\log(\|e_{i-1}\|_{(\cdot)} / \|e_i\|_{(\cdot)})}{\log(h_{i-1}/h_i)}$$

$$\|e\|_1 = \frac{\sum_{t=1}^{N} |q_{t,\mathrm{ref}} - q_t|}{\sum_{t=1}^{N} |q_{t,\mathrm{ref}}|}, \ \|e\|_2 = \left(\frac{\sum_{t=1}^{N} |q_{t,\mathrm{ref}} - q_t|^2}{\sum_{t=1}^{N} |q_{t,\mathrm{ref}}|^2} \right)^{1/2}, \ \|e\|_\infty = \frac{\max_t |q_{t,\mathrm{ref}} - q_t|}{\max_t |q_{t,\mathrm{ref}}|}$$

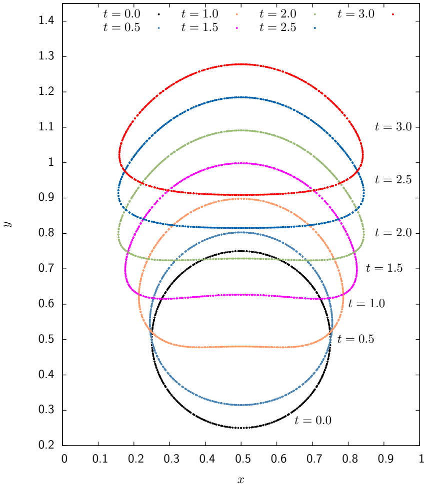
Test case	ρ_1	ρ_2	μ_1	μ_2	g	σ	Re	Eo	ρ_1/ρ_2	μ_1/μ_2
1	1000	100	10	1	0.98	24.5	49.5	9	10	10
2	1000	1	10	0.1	0.98	1.96	49.5	124.88	1000	100

Rising bubble, case 1, deformation over time

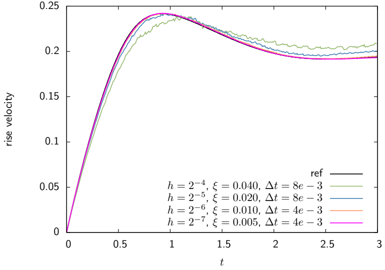
Phase field, IGA ($\mathcal{N}^{2,2}_{0,0} \times \mathcal{N}^{1,1}_{0,0}$)

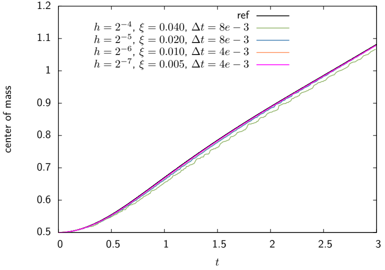


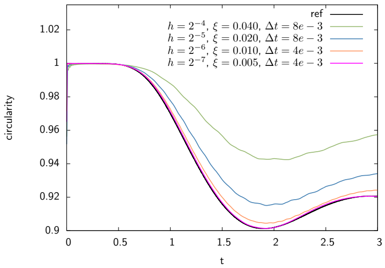
TP2D reference, Level Set, Q_2P_1 FE



Rising bubble, case 1, quantities over time





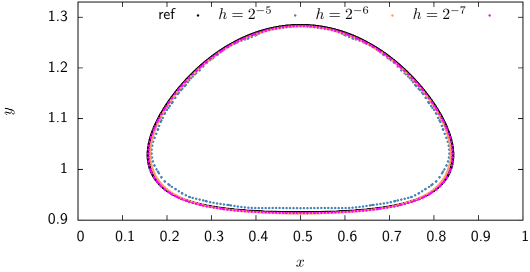


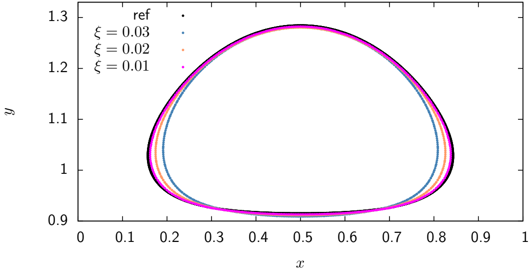
Rising bubble, case 1

h	ξ	Δt	$\not{c}_{\min}$	$t\|_{\not{c}=\not{c}_{\min}}$	$V_{b,\max}$	$t\|_{V_b=V_{b,\max}}$	$Y_b(t=3)$
2^{-4}	0.040	0.008	0.9425	2.1281	0.2384	1.2000	1.0665
2^{-5}	0.020	0.008	0.9151	1.9280	0.2423	0.9520	1.0778
2^{-6}	0.010	0.004	0.9044	1.9240	0.2422	0.9120	1.0792
2^{-7}	0.005	0.004	0.9013	1.9200	0.2420	0.9200	1.0794
ref			0.9013	1.9041	0.2417	0.9213	1.0813

Table : Minimum circularity and maximum rise velocity with corresponding incidence times and final center of mass position for test case 1.

Rising bubble, case 1, influence of h and ξ



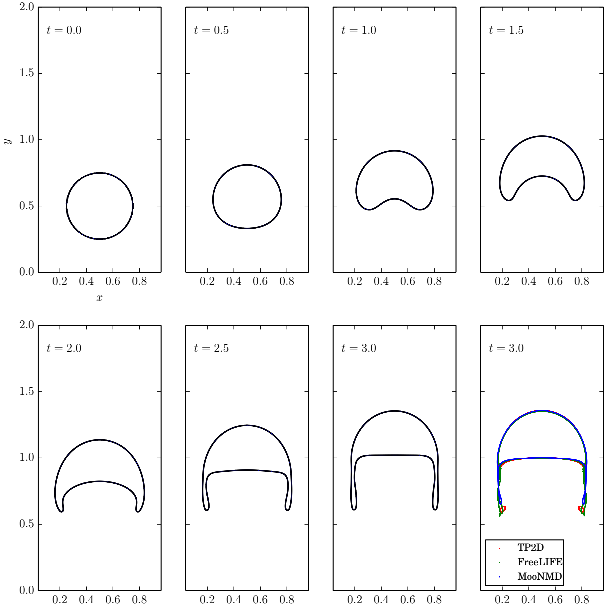


Rising bubble, case 1, convergence orders

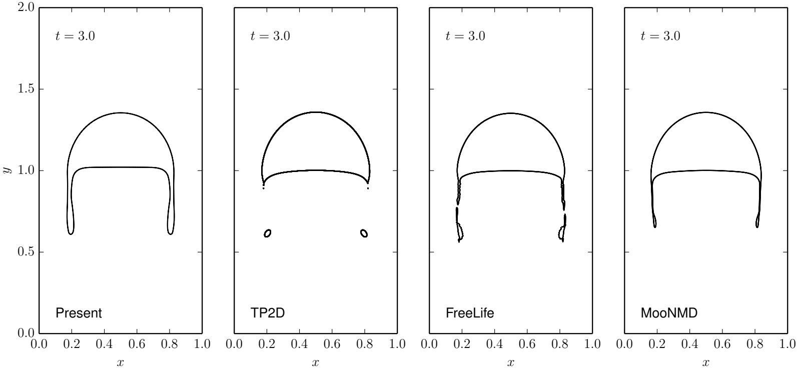
q	h	EOC_1	$\text{EOC}_1^{\text{self,L7}}$	EOC_2	$\text{EOC}_2^{\text{self,L7}}$	EOC_∞	$\text{EOC}_\infty^{\text{self,L7}}$
Y_b	2^{-5}	1.7049	2.0024	1.6818	1.9263	1.4755	1.6026
	2^{-6}	1.4633	2.5718	1.5127	2.5136	1.5947	2.2132
	2^{-7}	0.5312		0.5706		0.8730	
V_b	2^{-5}	1.3263	1.3883	1.3518	1.3969	1.2714	1.2100
	2^{-6}	2.0064	2.3780	1.8934	2.3116	1.4112	2.0974
	2^{-7}	1.1755		1.0575		0.9790	
$\not{c}$	2^{-5}	1.4927	1.5363	1.5095	1.5463	1.5051	1.1128
	2^{-6}	2.0446	2.3055	2.0443	2.2597	1.9111	1.6609
	2^{-7}	2.1778		2.0334		1.6871	

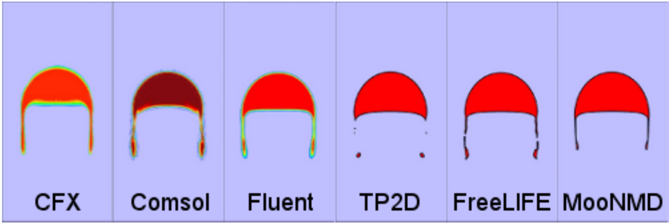
- Order of convergence between 1 and ~ 2 in all norms
- Order of convergence even exceeds 2 when compared to own result

Rising bubble, case 2, deformation over time

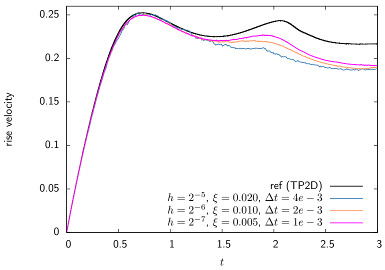


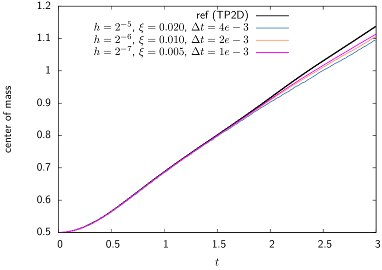
Rising bubble, case 2, final time shapes

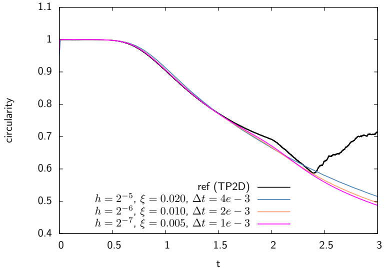




Rising bubble, case 2, quantities over time





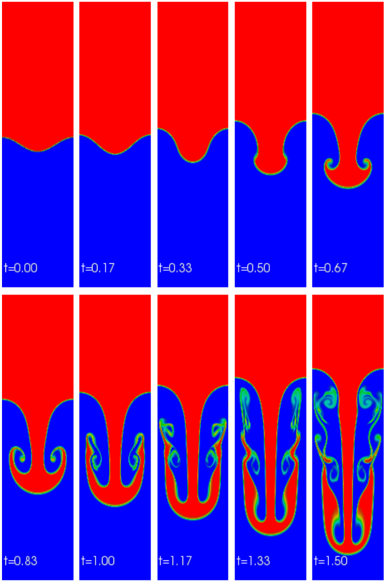


Rising bubble, case 2, convergence orders

q	h	EOC_1	$\text{EOC}_1^{\text{self,L7}}$	EOC_2	$\text{EOC}_2^{\text{self,L7}}$	EOC_∞	$\text{EOC}_\infty^{\text{self,L7}}$
Y_b	2^{-5}		2.0024		1.9263		1.6026
	2^{-6}	1.4620	2.5718	1.5072	2.5136	1.4755	2.2132
	2^{-7}	1.5358		1.5600		1.5947	
V_b	2^{-5}		1.3883		1.3969		1.2100
	2^{-6}	1.3060	2.3780	1.3710	2.3116	1.2714	2.0974
	2^{-7}	1.5620		1.5643		1.4112	
$\not c$	2^{-5}		1.5363		1.5463		1.1128
	2^{-6}	1.4147	2.3055	1.4478	2.2597	1.5051	1.6609
	2^{-7}	1.8389		1.8880		1.9111	

- Time interval confined to $[0, 2]$
- Order of convergence between 1 and ~ 2 in all norms
- Order of convergence even exceeds 2 when compared to own result

Rayleigh–Taylor instability



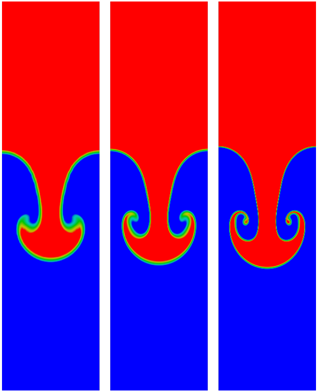


Figure : Evolution of a single wavelength initial condition for different ref. levels at $t \approx 0.8$

Rayleigh–Taylor instability

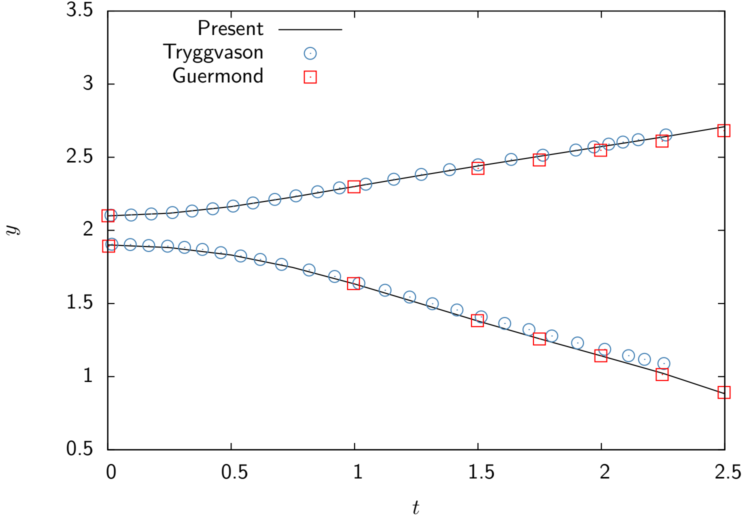


Figure : The y-coordinate of the tip of the rising and falling fluid versus time.

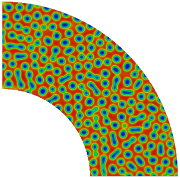
Conclusions and outlook

Isogeometric Analysis $\oplus$ Phase field based two-phase flow model

- robust numerical method
- successful (benchmarks)

Extension to

- Complex geometries



- Alternative Navier–Stokes–Cahn–Hilliard models (Abels, Boyer, ..)
- Multiphysics (FSI, Non-Newtonian fluids, ..)
- Local refinement (Hierarchical B-splines, T-splines, etc.)
- Multigrid

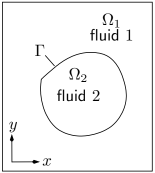
Appendix, Two-phase flow model based on fluid free energy

- Force balance boundary condition

$$\left[-p\boldsymbol{I} + \mu\left(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T\right)\right]\Big|_\Gamma \cdot \boldsymbol{n} = \sigma\kappa\boldsymbol{n}$$

- Internal force b.c. as volumetric surface tension force

$$\boldsymbol{f}_{st} = \sigma\kappa\boldsymbol{n}\delta(\Gamma, \boldsymbol{x})$$



Phase field methods are based on models of fluid free energy

Simplest free energy density model for isothermal fluids yielding two phases is

$$f(\varphi) = \frac{1}{2}\alpha|\nabla\varphi|^2 + \beta\psi(\varphi) \qquad \alpha > 0, \beta > 0$$

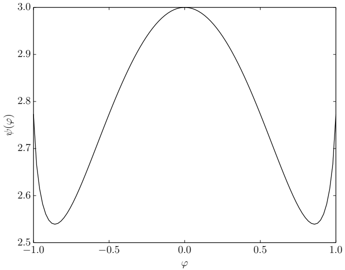
- Order parameter $\varphi \in [-1, 1]$ measure of phase
- $\frac{1}{2}\alpha|\nabla\varphi|^2$: interfacial (surface) free energy density
  - Penalizes large gradients
- $\psi(\varphi)$: homogeneous free energy density

Appendix, Principle: Energy minimization

$$\psi(\varphi) = T((1+\varphi)\log(1+\varphi) + (1-\varphi)\log(1-\varphi)) + T_c(1-\varphi^2)$$

Equilibrium interface profiles **minimize** fluid free energy (Ginzburg-Landau free energy) functional

$$\mathcal{E}(\varphi) := \int_\Omega f \,\mathrm{d}\Omega = \int_\Omega \frac{1}{2}\alpha|\nabla\varphi|^2 + \beta\psi(\varphi)\,\mathrm{d}\Omega$$



Variational derivative of $\mathcal{E}(\varphi)$ w.r.t. φ yields **chemical potential**

$$\eta = \frac{\delta\mathcal{E}}{\delta\varphi} = -\alpha\nabla^2\varphi + \beta\psi'(\varphi)$$

Equilibrium interface profiles satisfy

$$\beta\psi'(\varphi) - \alpha\nabla^2\varphi \equiv \eta = \text{const.}$$

Appendix, Cahn–Hilliard system

Idea: Generalize problem to a mass diffusion equation in a binary system applying the principle of conservation of mass with a local diffusion mass flux

$$\boldsymbol{J} = -m(\varphi)\nabla\eta,$$
$$m(\varphi) = D(\varphi^2 - 1)^2$$

Mass conservation for φ requires

$$\frac{d\varphi}{dt} + \nabla \cdot \boldsymbol{J} = \frac{\partial \varphi}{\partial t} + \nabla \cdot (\boldsymbol{v}\varphi) + \nabla \cdot \boldsymbol{J} = 0$$

Advective Cahn–Hilliard (CH) equation

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{v} \cdot \nabla\varphi = \nabla \cdot (m(\varphi)\nabla\eta) = \nabla \cdot \left(m(\varphi)\nabla\left(-\alpha\nabla^2\varphi + \beta\psi'(\varphi)\right)\right),$$

$$\nabla\eta \cdot \boldsymbol{n} = 0, \qquad \text{(no flux b.c.)}$$

$$\nabla\varphi \cdot \boldsymbol{n} = \frac{1}{\epsilon\sqrt{2}}\cos(\theta)(1 - \varphi^2) \qquad \text{(contact angle b.c.)}$$

Appendix, State of equilibrium, surface tension

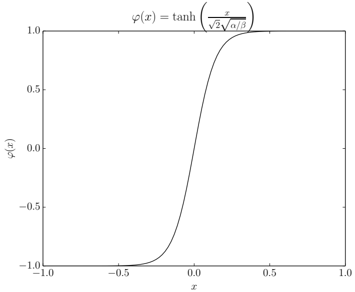
Equilibrium interface profile given by solution of equation

$$\eta(\varphi) = -\alpha\nabla^2\varphi + \beta\psi'(\varphi)^1 = -\alpha\Delta\varphi + \beta\varphi(\varphi^2 - 1) = 0$$

Solution characterized by

- two stable minima at $\varphi \approx \pm 1$ standing for the two phases
- $1D$ transition region given by

$$\varphi(x) = \tanh\left(\frac{x}{\sqrt{2}\sqrt{\alpha/\beta}}\right)$$



Surface tension of the interface of an isothermal fluid system in equilibrium is equal to integral of free energy density through interface:

$$\sigma = \alpha \int_{-\infty}^{\infty} \left(\frac{d\varphi}{dx}\right)^2 \mathrm{d}\boldsymbol{x} = \frac{2\sqrt{2}}{3}\sqrt{\alpha\beta}$$

[1] $\psi(\varphi) = \frac{1}{4}(\varphi - 1)^2(\varphi + 1)^2$

Appendix, Mixed formulation

For general ψ:

- $\sigma \propto \sqrt{\alpha\beta}$
- $\epsilon \propto \sqrt{\alpha/\beta}$ (equilibrium interface thickness)

Introduce auxiliary interface thickness $\xi = \sqrt{\alpha/\beta}$ and set

$$\alpha = \frac{3}{2\sqrt{2}}\sigma\xi, \qquad \beta = \frac{3}{2\sqrt{2}}\frac{\sigma}{\xi}.$$

- Direct formulation of CH: 4th order spatial derivatives $\rightarrow \mathcal{C}^1$-FE
- High continuity spaces $\rightarrow$ Isogeometric Analysis
- Alternative: Mixed formulation

Mixed formulation of Advective Cahn–Hilliard equation

$$\frac{\partial\varphi}{\partial t} + \boldsymbol{v} \cdot \nabla\varphi - \nabla \cdot (m(\varphi)\nabla\eta) = 0,$$

$$\eta - \beta\,\frac{d\psi(\varphi)}{d\varphi} + \alpha\,\nabla^2\varphi = 0$$

Appendix, Newton linearization (PDE level)

Linear form (for fixed φ^k, η^k):

$$
\begin{aligned}
\mathcal{F}_{\mathsf{CH}}((\varphi^k, \eta^k); \cdot) = & \int_\Omega \varphi^k \, q + \theta \Delta t \left((\boldsymbol{v} \cdot \nabla) \varphi^k \, q + m \, \nabla \eta^k \cdot \nabla q \right) \, \mathrm{d}\boldsymbol{x} \\
& - \int_\Omega \varphi^n \, q - (1-\theta) \Delta t \left((\boldsymbol{v} \cdot \nabla) \varphi^n \, q + m \nabla \eta^n \cdot \nabla q \right) \, \mathrm{d}\boldsymbol{x}, \\
& + \int_\Omega \eta^k \, v \, \mathrm{d}\boldsymbol{x} - \int_\Omega \beta \, \psi'(\varphi^k) \, v \, \mathrm{d}\boldsymbol{x} - \int_\Omega \alpha \nabla \varphi^k \cdot \nabla v \, \mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Bilinear form $\mathcal{J}_{\mathsf{CH}} = \mathcal{F}'_{\mathsf{CH}}((\varphi^k, \eta^k); \cdot, \cdot)$ from linearization of $\mathcal{F}_{\mathsf{CH}}$ around $(\varphi, \eta) = (\varphi^k, \eta^k)$:

$$
\begin{aligned}
\mathcal{J}_{\mathsf{CH}} = & \int_\Omega \delta\varphi \, q + \theta \Delta t \left((\boldsymbol{v} \cdot \nabla) \delta\varphi \, q + m \nabla \delta\eta \cdot \nabla q \right) \, \mathrm{d}\boldsymbol{x} \\
& + \int_\Omega \delta\eta \, v \, \mathrm{d}\boldsymbol{x} - \int_\Omega \beta \, \psi''(\varphi^k) \, \delta\varphi \, v \, \mathrm{d}\boldsymbol{x} - \int_\Omega \alpha \, \nabla \delta\varphi \cdot \nabla v \, \mathrm{d}\boldsymbol{x}
\end{aligned}
$$

Appendix, Newton linearization (PDE level)

Linear form (for fixed $\boldsymbol{u}^k$):

$$\mathcal{F}_{\mathsf{NS}}(\boldsymbol{u}^k;\cdot) = {}^2\mathcal{R}(\boldsymbol{u}^k)$$

Bilinear form $\mathcal{J}_{\mathsf{NS}} = \mathcal{F}'_{\mathsf{NS}}(\boldsymbol{u}^k;\delta\boldsymbol{u},(\boldsymbol{w},q))$ from linearization of $\mathcal{F}_{\mathsf{NS}}$ around $\boldsymbol{u} = \boldsymbol{u}^k$:

$$\begin{aligned}\mathcal{J}_{\mathsf{NS}} = &\int_\Omega \rho(\varphi)\boldsymbol{w}\cdot\delta\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \mu(\varphi)\nabla\boldsymbol{w} : \left(\nabla\delta\boldsymbol{v} + (\nabla\delta\boldsymbol{v})^T\right)\,\mathrm{d}\boldsymbol{x}\\ &+ \int_\Omega \rho(\varphi)\boldsymbol{w}\cdot\delta\boldsymbol{v}\cdot\nabla\boldsymbol{v}^k\,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho(\varphi)\boldsymbol{w}\cdot\boldsymbol{v}^k\cdot\nabla\delta\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\\ &- \int_\Omega \nabla\cdot\boldsymbol{w}\,\delta p\,\mathrm{d}\boldsymbol{x} + \int_\Omega q\,\nabla\cdot\delta\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\end{aligned}$$

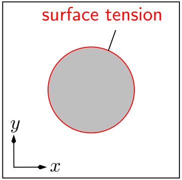
[2] Residual of the Navier–Stokes system

Static bubble

- $\Omega = [-1, 1]^2$ with bubble centered at origin
- Finite element L^2–projection of bubble profile:

$$\int_\Omega (f - P_h f)\, w \,\mathrm{d}\boldsymbol{x} = 0, \quad \forall w \in \mathcal{W}_h$$

$$f = \varphi_{\text{bubble}}(\boldsymbol{x}) = \begin{cases} +1, & \text{for } \boldsymbol{x} \in \Omega_1 \\ -1, & \text{for } \boldsymbol{x} \in \Omega_2 \end{cases}$$



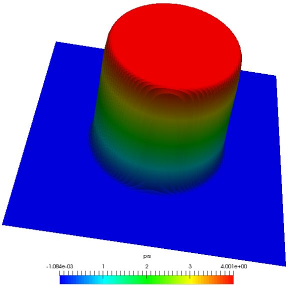
Pressure field expected to satisfy the **Laplace-Young law**

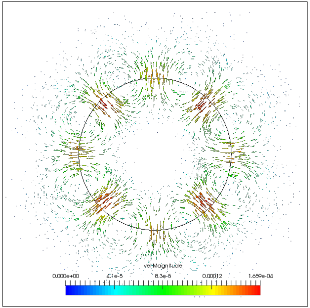
$$p_i = p_o + \sigma / r$$

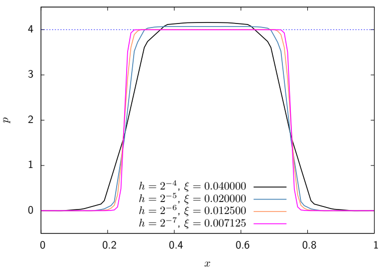
$\sigma = 1, r = 1/4$

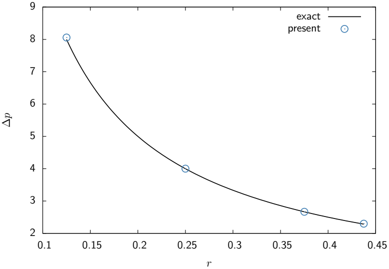
N_{el}	h	ξ	p_i	p_o	$\lvert\Delta p - (\frac{\sigma}{r})\rvert / (\frac{\sigma}{r})$	$\lvert p_i - p_o\rvert / (\frac{\sigma}{r})$	$\lVert p - p_h \rVert_{L^2}$	$\lVert \boldsymbol{v} - \boldsymbol{v_h} \rVert_{L^2}$
256	2^{-4}	0.0400	4.05234	-0.00431	0.01416	1.01416	0.539996	2.27e-04
1024	2^{-5}	0.0200	3.99563	-0.00233	0.00051	0.99949	0.379967	6.67e-05
4096	2^{-6}	0.0125	4.00642	-0.00139	0.00195	1.00195	0.296332	6.77e-05
16384	2^{-7}	0.007125	4.00067	-0.00074	0.00035	1.00035	0.222107	1.28e-04

Static bubble, pressure and spurious velocities









Rising bubble, case 2, influence of h

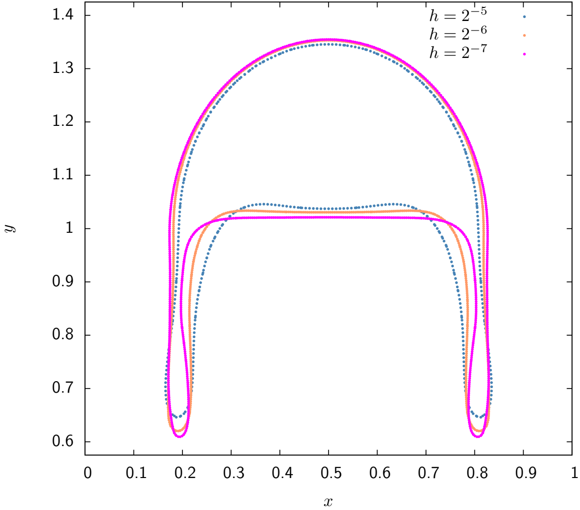


Figure : Shapes of the rising bubble at final time $t = 3$ for different h.

Appendix, Dimensionless numbers

Dimensionless numbers in the rising bubble setup:

- Reynolds number $Re = \rho_1 \tilde{v} L / \mu_1$
- Eötvös (or Bond) number $Eo = Bo = \Delta\rho g L^2 / \sigma$
- Characteristic velocity $\tilde{v} = \sqrt{g}$
- Characteristic length $L = 2r$

Appendix, Ostwald ripening

Ostwald ripening is an observed phenomenon in solid solutions or liquid sols that describes the change of an inhomogeneous structure over time, i.e., small crystals or sol particles dissolve, and redeposit onto larger crystals or sol particles [6].

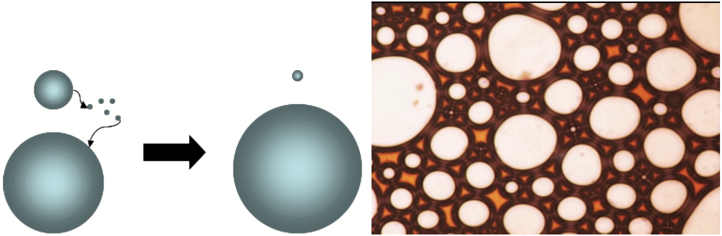


Figure : Images taken from [6]

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https://youtu.be/NjB7LXSQoQc