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Isogeometric Analysis of Fluid–Structure Interaction based on a fully coupled Arbitrary Lagrangian–Eulerian variational formulation

Babak S. Hosseini[1], Stefan Turek[1], Matthias Möller[2]

[1]TU Dortmund University, Institute of Applied Mathematics, LS III
babak.hosseini@math.tu-dortmund.de

[2]Delft University of Technology, Delft Institute of Applied Mathematics

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Motivation

Objective: Develop an Isogeometric Analysis based fully coupled monolithic FSI simulation framework that in the sense of robustness/stability is particularly well suited for biomechanical applications.

We would like to use a method that has the following advantages:

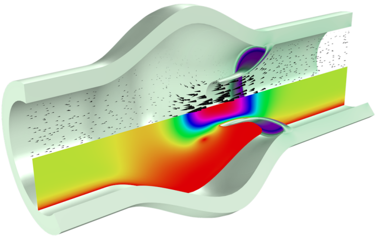


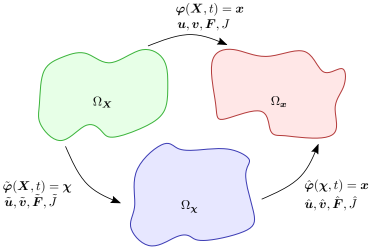
Figure : ©[12]

- Stable in biomechanical applications ($\rho^{\mathcal{F}}/\rho^{\mathcal{S}} \approx 1$)
- Fully coupled ALE variational formulation solves the difficulty of common variational description; Facilitates consistent Galerkin discretization of the FSI problem
- High continuity and regularity spaces; Complex geometries $\rightarrow$ Isogeometric Analysis

Compressible solid problem

Elastodynamics (Lagrangian perspective):

$$
\begin{aligned}
J\rho \frac{\partial \boldsymbol{v}}{\partial t} - \nabla_{\boldsymbol{X}} \cdot \boldsymbol{P} = J\rho \boldsymbol{b} \quad & \text{in } \Omega_{\boldsymbol{X}} \times I, \\
\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{v} = 0 \quad & \text{in } \Omega_{\boldsymbol{X}} \times I, \\
\boldsymbol{u}(\cdot, 0) = \mathring{\boldsymbol{u}}, \boldsymbol{v}(\cdot, 0) = \mathring{\boldsymbol{v}} \quad & \text{in } \Omega_{\boldsymbol{X}}, \\
\boldsymbol{u} = \boldsymbol{u}_D \quad & \text{on } \Gamma_{D,\boldsymbol{X}} \times I, \\
\boldsymbol{P}\,\boldsymbol{n}_0 = \boldsymbol{g}_0 \quad & \text{on } \Gamma_{N,\boldsymbol{X}} \times I.
\end{aligned}
$$



$$
\begin{aligned}
\boldsymbol{\varphi} \colon \overline{\Omega}_{\boldsymbol{X}} \times [0, T] & \longrightarrow \overline{\Omega}_{\boldsymbol{x}} \times [0, T] \\
(\boldsymbol{X}, t) & \longmapsto \boldsymbol{\varphi}(\boldsymbol{X}, t) = (\boldsymbol{x}, t)
\end{aligned}
$$

$$\boldsymbol{\varphi}(\boldsymbol{X}, t) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t)$$

$$\boldsymbol{v}(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{u}}{\partial t}\Big|_{\boldsymbol{X}}$$

$$\boldsymbol{F} := \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{u}) := \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{X}} = \nabla_{\boldsymbol{X}} \boldsymbol{\varphi}(\boldsymbol{X}) = \boldsymbol{I} + \nabla_{\boldsymbol{X}} \boldsymbol{u}$$

$$J := \det(\boldsymbol{F})$$

- St. Venant–Kirchhoff material

$$\boldsymbol{P} := \lambda \operatorname{tr}(\boldsymbol{E})\, \boldsymbol{F} + 2\mu \boldsymbol{F}\boldsymbol{E}$$

- Neo–Hookean material

$$\boldsymbol{P} := \mu(\boldsymbol{F} - \boldsymbol{F}^{-T}) + \lambda \log(\det \boldsymbol{F}) \boldsymbol{F}^{-T}$$

- Green–St. Venant strain tensor

$$\boldsymbol{E} := \frac{1}{2}\left(\nabla_{\boldsymbol{X}} \boldsymbol{u} + (\nabla_{\boldsymbol{X}} \boldsymbol{u})^T + (\nabla_{\boldsymbol{X}} \boldsymbol{u})^T \nabla_{\boldsymbol{X}} \boldsymbol{u}\right)$$

Incompressible fluid problem

Incompressible Newtonian flow (Eulerian perspective):

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t}\Big|_{\boldsymbol{x}} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) = \nabla\cdot\boldsymbol{\sigma} + \rho\boldsymbol{b} \quad \text{in } \Omega^{\mathcal{F}}_{\boldsymbol{x}}(t), t\in I,$$

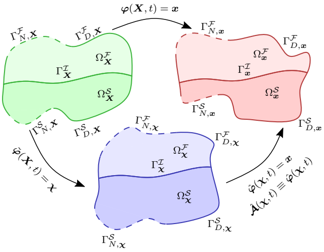
$$\nabla\cdot\boldsymbol{v} = 0 \quad \text{in } \Omega^{\mathcal{F}}_{\boldsymbol{x}}(t), t\in I,$$

$$p(\cdot,0) = \mathring{p}, \boldsymbol{v}(\cdot,0) = \mathring{\boldsymbol{v}} \quad \text{in } \Omega^{\mathcal{F}}_{\boldsymbol{x}}(0),$$

$$\boldsymbol{v} = \boldsymbol{v}_D \quad \text{on } \Gamma^{\mathcal{F}}_{D,\boldsymbol{x}}(t), t\in I,$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{g} \quad \text{on } \Gamma^{\mathcal{F}}_{N,\boldsymbol{x}}(t), t\in I.$$

$$\boldsymbol{\sigma} := -p\boldsymbol{I} + \mu(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T)$$



FSI coupling conditions:

- Geometric coupling: Fluid- and solid-domain never detach or overlap
- Continuity of velocity:
$$\boldsymbol{v}^{\mathcal{F}} = \boldsymbol{v}^{\mathcal{S}} \quad \text{on } \Gamma^{\mathcal{I}}_{\boldsymbol{x}}(t)$$
- Continuity of normal stresses:
$$\boldsymbol{\sigma}^{\mathcal{F}}\cdot\boldsymbol{n}^{\mathcal{F}}_{\boldsymbol{x}} = -\boldsymbol{\sigma}^{\mathcal{S}}\cdot\boldsymbol{n}^{\mathcal{S}}_{\boldsymbol{x}} \quad \text{on } \Gamma^{\mathcal{I}}_{\boldsymbol{x}}(t)$$

Strategy for combination into one conservation equation: **Rewrite fluid equations in a "structure-appropriate" framework (ALE)**

Governing equations

$$
\begin{aligned}
&\hat{J}\rho^{\mathcal{F}}\left(\frac{\partial \boldsymbol{v}^{\mathcal{F}}}{\partial t}\Big|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}}\boldsymbol{v}^{\mathcal{F}}\left(\hat{\boldsymbol{F}}^{-1}(\boldsymbol{v}^{\mathcal{F}} - \partial_t\hat{\boldsymbol{\mathcal{A}}})\right)\right) \\
-\nabla_{\boldsymbol{\chi}}\cdot\left(\hat{J}\left(-p^{\mathcal{F}}\boldsymbol{I} + \mu^{\mathcal{F}}\left(\nabla_{\boldsymbol{\chi}}\boldsymbol{v}^{\mathcal{F}}\hat{\boldsymbol{F}}^{-1} + \hat{\boldsymbol{F}}^{-T}(\nabla_{\boldsymbol{\chi}}\boldsymbol{v}^{\mathcal{F}})^T\right)\right)\hat{\boldsymbol{F}}^{-T}\right) &= \hat{J}\rho^{\mathcal{F}}\boldsymbol{b}^{\mathcal{F}} && \text{in } \Omega^{\mathcal{F}}_{\boldsymbol{\chi}}\times(0,T],\\
\nabla_{\boldsymbol{\chi}}\cdot\left(\hat{J}\hat{\boldsymbol{F}}^{-1}\boldsymbol{v}^{\mathcal{F}}\right) &= 0 && \text{in } \Omega^{\mathcal{F}}_{\boldsymbol{\chi}}\times[0,T],\\
p^{\mathcal{F}}(\cdot,0) = \mathring{p}^{\mathcal{F}}, \boldsymbol{u}^{\mathcal{F}}(\cdot,0) = \mathring{\boldsymbol{u}}^{\mathcal{F}}, \boldsymbol{v}^{\mathcal{F}}(\cdot,0) &= \mathring{\boldsymbol{v}}^{\mathcal{S}} && \text{in } \Omega^{\mathcal{F}}_{\boldsymbol{\chi}},\\
\boldsymbol{u}^{\mathcal{F}} = \boldsymbol{u}^{\mathcal{F}}_D, \boldsymbol{v}^{\mathcal{F}} &= \boldsymbol{v}^{\mathcal{F}}_D && \text{on } \Gamma^{\mathcal{F}}_{D,\boldsymbol{\chi}}\times(0,T],\\
\left(\hat{J}\boldsymbol{\sigma}^{\mathcal{F}}\hat{\boldsymbol{F}}^{-T}\right)\boldsymbol{n}^{\mathcal{F}}_0 &= \boldsymbol{g}^{\mathcal{F}}_0 && \text{on } \Gamma^{\mathcal{F}}_{N,\boldsymbol{\chi}}\times(0,T].\\
\hat{J}\rho^{\mathcal{S}}\frac{\partial \boldsymbol{v}^{\mathcal{S}}}{\partial t}\Big|_{\boldsymbol{\chi}} - \nabla_{\boldsymbol{\chi}}\cdot\hat{\boldsymbol{P}}^{\mathcal{S}} &= \hat{J}\rho^{\mathcal{S}}\boldsymbol{b}^{\mathcal{S}} && \text{in } \Omega^{\mathcal{S}}_{\boldsymbol{\chi}}\times(0,T],\\
\frac{\partial \boldsymbol{u}^{\mathcal{S}}}{\partial t} - \boldsymbol{v}^{\mathcal{S}} &= 0 && \text{in } \Omega^{\mathcal{S}}_{\boldsymbol{\chi}}\times(0,T],\\
\boldsymbol{u}^{\mathcal{S}}(\cdot,0) = \mathring{\boldsymbol{u}}^{\mathcal{S}}, \boldsymbol{v}^{\mathcal{S}}(\cdot,0) &= \mathring{\boldsymbol{v}}^{\mathcal{S}} && \text{in } \Omega^{\mathcal{S}}_{\boldsymbol{\chi}},\\
\boldsymbol{u}^{\mathcal{S}} &= \boldsymbol{u}^{\mathcal{S}}_D && \text{on } \Gamma^{\mathcal{S}}_{D,\boldsymbol{\chi}}\times(0,T],\\
\hat{\boldsymbol{P}}^{\mathcal{S}}\boldsymbol{n}^{\mathcal{S}}_0 &= \boldsymbol{g}^{\mathcal{S}}_0 && \text{on } \Gamma^{\mathcal{S}}_{N,\boldsymbol{\chi}}\times(0,T].\\
\nabla_{\boldsymbol{\chi}}\cdot\left(\alpha_u\hat{J}^{-1}\nabla_{\boldsymbol{\chi}}\boldsymbol{u}^{\mathcal{F}}\right) &= 0 && \text{in } \Omega^{\mathcal{F}}_{\boldsymbol{\chi}}\times(0,T],\\
\boldsymbol{u}^{\mathcal{F}} = \boldsymbol{u}^{\mathcal{S}}, \boldsymbol{v}^{\mathcal{F}} = \boldsymbol{v}^{\mathcal{S}}, \left(\hat{J}\boldsymbol{\sigma}^{\mathcal{F}}\hat{\boldsymbol{F}}^{-T}\right)\boldsymbol{n}_0 &= \hat{\boldsymbol{P}}^{\mathcal{S}}\boldsymbol{n}_0 && \text{on } \Gamma^{\mathcal{I}}_{\boldsymbol{\chi}}\times(0,T].
\end{aligned}
$$

Variational formulation

- Displacement trial and test spaces in the fluid domain:

$$\boldsymbol{\mathcal{T}}_{\boldsymbol{u}}^{\mathcal{F}} := \{\boldsymbol{u}^{\mathcal{F}} \in \boldsymbol{\mathcal{H}}^1(\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}) : \boldsymbol{u}^{\mathcal{F}} = \boldsymbol{u}^{\mathcal{S}} \text{ on } \Gamma_{\chi}^{\mathcal{I}}, \boldsymbol{u}^{\mathcal{F}} = \boldsymbol{u}_D^{\mathcal{F}} \text{ on } \Gamma_{D,\boldsymbol{\chi}}^{\mathcal{F}}\}$$
$$\boldsymbol{\mathcal{W}}_{\boldsymbol{u}}^{\mathcal{F}} := \{\boldsymbol{\phi}^{\boldsymbol{u},\mathcal{F}} \in \boldsymbol{\mathcal{H}}_0^1(\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}) : \boldsymbol{\phi}^{\boldsymbol{u},\mathcal{F}} = \boldsymbol{\phi}^{\boldsymbol{u},\mathcal{S}} \text{ on } \Gamma_{\chi}^{\mathcal{I}}\}$$

- Velocity trial and test spaces in the fluid domain:

$$\boldsymbol{\mathcal{T}}_{\boldsymbol{v}}^{\mathcal{F}} := \{\boldsymbol{v}^{\mathcal{F}} \in \boldsymbol{\mathcal{H}}^1(\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}) : \boldsymbol{v}^{\mathcal{F}} = \boldsymbol{v}^{\mathcal{S}} \text{ on } \Gamma_{\chi}^{\mathcal{I}}, \boldsymbol{v}^{\mathcal{F}} = \boldsymbol{v}_D^{\mathcal{F}} \text{ on } \Gamma_{D,\boldsymbol{\chi}}^{\mathcal{F}}\}$$
$$\boldsymbol{\mathcal{W}}_{\boldsymbol{v}}^{\mathcal{F}} := \{\boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}} \in \boldsymbol{\mathcal{H}}_0^1(\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}) : \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}} = \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{S}} \text{ on } \Gamma_{\chi}^{\mathcal{I}}\}$$

- Pressure trial and test space in the fluid domain:

$$\mathcal{L}^{\mathcal{F}} := \mathcal{L}^2(\Omega_{\boldsymbol{\chi}}^{\mathcal{F}})/\mathbb{R}$$

- Displacement trial and test space in the solid domain:

$$\boldsymbol{\mathcal{T}}_{\boldsymbol{u}}^{\mathcal{S}} := \{\boldsymbol{u}^{\mathcal{S}} \in \boldsymbol{\mathcal{H}}^1(\Omega_{\boldsymbol{\chi}}^{\mathcal{S}}) : \boldsymbol{u}^{\mathcal{S}} = \boldsymbol{u}_D^{\mathcal{S}} \text{ on } \Gamma_{D,\boldsymbol{\chi}}^{\mathcal{S}}\}, \qquad \boldsymbol{\mathcal{W}}_{\boldsymbol{u}}^{\mathcal{S}} := \boldsymbol{\mathcal{H}}_0^1(\Omega_{\boldsymbol{\chi}}^{\mathcal{S}})$$

- Velocity trial and test space in the solid domain:

$$\boldsymbol{\mathcal{T}}_{\boldsymbol{v}}^{\mathcal{S}} := \boldsymbol{\mathcal{H}}^1(\Omega_{\boldsymbol{\chi}}^{\mathcal{S}}), \qquad \boldsymbol{\mathcal{W}}_{\boldsymbol{v}}^{\mathcal{S}} := \boldsymbol{\mathcal{H}}_0^1(\Omega_{\boldsymbol{\chi}}^{\mathcal{S}})$$

- Pressure trial and test spaces in the solid domain:

$$\mathcal{L}^{\mathcal{S}} := \mathcal{L}^2(\Omega_{\boldsymbol{\chi}}^{\mathcal{S}})/\mathbb{R}$$

Variational formulation

Let $\boldsymbol{\mathcal{T}} := \{\boldsymbol{\mathcal{T}}_{\boldsymbol{v}}^{\mathcal{F}} \times \boldsymbol{\mathcal{T}}_{\boldsymbol{v}}^{\mathcal{S}} \times \boldsymbol{\mathcal{T}}_{\boldsymbol{u}}^{\mathcal{F}} \times \boldsymbol{\mathcal{T}}_{\boldsymbol{u}}^{\mathcal{S}} \times \mathcal{L}^{\mathcal{F}} \times \mathcal{L}^{\mathcal{S}}\}$, let $\boldsymbol{U} = \{\boldsymbol{v}^{\mathcal{F}}, \boldsymbol{v}^{\mathcal{S}}, \boldsymbol{u}^{\mathcal{F}}, \boldsymbol{u}^{\mathcal{S}}, p^{\mathcal{F}}, p^{\mathcal{S}}\}$, and let $\boldsymbol{\Phi} = \{\boldsymbol{\phi}^{v,\mathcal{F}}, \boldsymbol{\phi}^{v,\mathcal{S}}, \boldsymbol{\phi}^{u,\mathcal{F}}, \boldsymbol{\phi}^{u,\mathcal{S}}, \phi^{p,\mathcal{F}}, \phi^{p,\mathcal{S}}\}$.
Find $\boldsymbol{U} \in \boldsymbol{\mathcal{T}} \times I$ such that:

$$
\begin{aligned}
\mathcal{F}_1(\boldsymbol{U};\boldsymbol{\Phi}) &= 0 \quad \forall \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}} \in \boldsymbol{\mathcal{W}}_{\boldsymbol{v}}^{\mathcal{F}}\\
\mathcal{F}_2(\boldsymbol{U};\boldsymbol{\Phi}) &= 0 \quad \forall \phi^{p,\mathcal{F}} \in \mathcal{L}^{\mathcal{F}}\\
\mathcal{F}_3(\boldsymbol{U};\boldsymbol{\Phi}) &= 0 \quad \forall \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{S}} \in \boldsymbol{\mathcal{W}}_{\boldsymbol{v}}^{\mathcal{S}}\\
\mathcal{F}_4(\boldsymbol{U};\boldsymbol{\Phi}) &= 0 \quad \forall \boldsymbol{\phi}^{\boldsymbol{u},\mathcal{S}} \in \boldsymbol{\mathcal{W}}_{\boldsymbol{u}}^{\mathcal{S}}\\
\mathcal{F}_5(\boldsymbol{U};\boldsymbol{\Phi}) &= 0 \quad \forall \phi^{p,\mathcal{S}} \in \mathcal{L}^{\mathcal{S}}\\
\mathcal{F}_6(\boldsymbol{U};\boldsymbol{\Phi}) &= 0 \quad \forall \boldsymbol{\phi}^{\boldsymbol{u},\mathcal{F}} \in \boldsymbol{\mathcal{W}}_{\boldsymbol{u}}^{\mathcal{F}}
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{F}_1(\boldsymbol{U};\boldsymbol{\Phi}) :=\\
&\int_0^T \int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}} \hat{J}\rho^{\mathcal{F}} \left(\frac{\partial \boldsymbol{v}^{\mathcal{F}}}{\partial t}\bigg|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \boldsymbol{v}^{\mathcal{F}} \left(\hat{\boldsymbol{F}}^{-1} (\boldsymbol{v}^{\mathcal{F}} - \partial_t \hat{\boldsymbol{\mathcal{A}}}) \right) \right) \cdot \boldsymbol{\phi}^{\boldsymbol{v}} \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{F}} \, \mathrm{d}t\\
&+ \int_0^T \int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}} \hat{J} \left(-p^{\mathcal{F}} \boldsymbol{I} + \mu^{\mathcal{F}} \left(\nabla_{\boldsymbol{\chi}} \boldsymbol{v}^{\mathcal{F}} \hat{\boldsymbol{F}}^{-1} + \hat{\boldsymbol{F}}^{-T} (\nabla_{\boldsymbol{\chi}} \boldsymbol{v}^{\mathcal{F}})^T \right) \right) \hat{\boldsymbol{F}}^{-T} : \nabla_{\boldsymbol{\chi}} \boldsymbol{\phi}^{\boldsymbol{v}} \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{F}} \, \mathrm{d}t\\
&- \int_0^T \int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}} \hat{J}\rho^{\mathcal{F}} \boldsymbol{f}^{\mathcal{F}} \cdot \boldsymbol{\phi}^{\boldsymbol{v}} \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{F}} \, \mathrm{d}t - \int_0^T \int_{\Gamma_{N,\boldsymbol{\chi}}^{\mathcal{F}}} \boldsymbol{g}_0^{\mathcal{F}} \cdot \boldsymbol{\phi}^{\boldsymbol{v}} \, \mathrm{d}\Gamma_{N,\boldsymbol{\chi}}^{\mathcal{F}} \, \mathrm{d}t.
\end{aligned}
$$

Variational formulation

$$\mathcal{F}_2(\boldsymbol{U};\boldsymbol{\Phi}) := \int_0^T \int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}} \nabla_{\boldsymbol{\chi}} \cdot \left(\hat{J} \hat{\boldsymbol{F}}^{-1} \boldsymbol{v}^{\mathcal{F}} \right) \cdot \phi^p \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{F}} \, \mathrm{d}t.$$

$$\begin{aligned} \mathcal{F}_3(\boldsymbol{U};\boldsymbol{\Phi}) := & \int_0^T \int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{S}}} \hat{J} \rho^{\mathcal{S}} \left. \frac{\partial \boldsymbol{v}^{\mathcal{S}}}{\partial t} \right|_{\boldsymbol{\chi}} \cdot \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{S}} \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{S}} \, \mathrm{d}t + \int_0^T \int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{S}}} \hat{\boldsymbol{P}}^{\mathcal{S}} : \nabla_{\boldsymbol{\chi}} \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{S}} \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{S}} \, \mathrm{d}t \\ & - \int_0^T \int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{S}}} \hat{J} \rho^{\mathcal{S}} \boldsymbol{b}^{\mathcal{S}} \cdot \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{S}} \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{S}} \, \mathrm{d}t - \int_0^T \int_{\Gamma_{N,\boldsymbol{\chi}}^{\mathcal{S}}} \boldsymbol{g}_0^{\mathcal{S}} \cdot \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{S}} \, \mathrm{d}\Gamma_{N,\boldsymbol{\chi}}^{\mathcal{S}} \, \mathrm{d}t. \end{aligned}$$

$$\mathcal{F}_4(\boldsymbol{U};\boldsymbol{\Phi}) := \int_0^T \int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{S}}} \left(\left. \frac{\partial \boldsymbol{u}^{\mathcal{S}}}{\partial t} \right|_{\boldsymbol{\chi}} - \boldsymbol{v}^{\mathcal{S}} \right) \cdot \boldsymbol{\phi}^{\boldsymbol{u},\mathcal{S}} \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{S}} \, \mathrm{d}t.$$

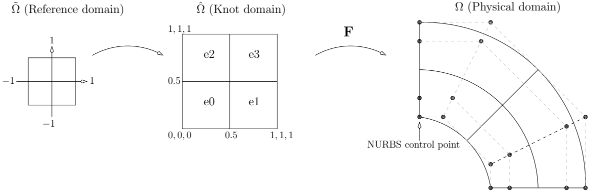
$$\mathcal{F}_5(\boldsymbol{U};\boldsymbol{\Phi}) := \int_0^T \int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{S}}} p^{\mathcal{S}} \cdot \phi^{p,\mathcal{S}} \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{S}} \, \mathrm{d}t.$$

$$\mathcal{F}_6(\boldsymbol{U};\boldsymbol{\Phi}) := \int_0^T \int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}} \alpha_u \hat{J}^{-1} \nabla_{\boldsymbol{\chi}} \boldsymbol{u}^{\mathcal{F}} : \nabla_{\boldsymbol{\chi}} \boldsymbol{\phi}^{\boldsymbol{u}} \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{F}} \, \mathrm{d}t.$$

Discrete Isogeometric approximation spaces

Approximation of velocity and pressure functions with LBB-stable Taylor-Hood like non-uniform rational B-spline space pairs $\hat{\mathbf{V}}_h^{TH}/\hat{Q}_h^{TH}$

$$\hat{\mathbf{V}}_h^{TH} \equiv \hat{\mathbf{V}}_h^{TH}(\mathbf{p}, \boldsymbol{\alpha}) = \boldsymbol{\mathcal{N}}_{\alpha,\alpha}^{p+1,p+1} = \mathcal{N}_{\alpha,\alpha}^{p+1,p+1} \times \mathcal{N}_{\alpha,\alpha}^{p+1,p+1}$$
$$\hat{Q}_h^{TH} \equiv \hat{Q}_h^{TH}(\mathbf{p}, \boldsymbol{\alpha}) = \mathcal{N}_{\alpha,\alpha}^{p,p}$$



Corresponding spaces $\mathbf{V}_h^{TH}$ and Q_h^{TH} in the physical domain Ω obtained via component-wise mapping using parametrization $\mathbf{F} : \hat{\Omega} \rightarrow \Omega$

$$\mathbf{V}_h^{TH} = \{\mathbf{v}_h = \hat{\mathbf{v}}_h \circ \mathbf{F}^{-1}, \hat{\mathbf{v}}_h \in \hat{\mathbf{V}}_h^{TH}\}, Q_h^{TH} = \{q_h = \hat{q}_h \circ \mathbf{F}^{-1}, \hat{q}_h \in \hat{Q}_h^{TH}\}$$

Solution algorithm and discrete problem

- Discrete spaces:

$$\begin{aligned}\boldsymbol{\mathcal{T}}^h := \{ &\left(\boldsymbol{\mathcal{T}}_{\boldsymbol{v}}^{\mathcal{F}} \cap \mathbf{V}_h^{TH}\right) \times \left(\boldsymbol{\mathcal{T}}_{\boldsymbol{v}}^{\mathcal{S}} \cap \mathbf{V}_h^{TH}\right) \times \left(\boldsymbol{\mathcal{T}}_{\boldsymbol{u}}^{\mathcal{F}} \cap \mathbf{V}_h^{TH}\right) \times \left(\boldsymbol{\mathcal{T}}_{\boldsymbol{u}}^{\mathcal{S}} \cap \mathbf{V}_h^{TH}\right) \\ &\times \left(\mathcal{L}^{\mathcal{F}} \cap Q_h^{TH}\right) \times \left(\mathcal{L}^{\mathcal{S}} \cap Q_h^{TH}\right)\}\end{aligned}$$

$$\begin{aligned}\boldsymbol{\mathcal{W}}^h := \{ &\left(\boldsymbol{\mathcal{W}}_{\boldsymbol{v}}^{\mathcal{F}} \cap \mathbf{V}_h^{TH}\right) \times \left(\boldsymbol{\mathcal{W}}_{\boldsymbol{v}}^{\mathcal{S}} \cap \mathbf{V}_h^{TH}\right) \times \left(\boldsymbol{\mathcal{W}}_{\boldsymbol{u}}^{\mathcal{F}} \cap \mathbf{V}_h^{TH}\right) \times \left(\boldsymbol{\mathcal{W}}_{\boldsymbol{u}}^{\mathcal{S}} \cap \mathbf{V}_h^{TH}\right) \\ &\times \left(\mathcal{L}^{\mathcal{F}} \cap Q_h^{TH}\right) \times \left(\mathcal{L}^{\mathcal{S}} \cap Q_h^{TH}\right)\}\end{aligned}$$

- Time discretization.: **Shifted Crank-Nicolson** $(\theta = \frac{1}{2} + O(\Delta t))$

while $t \leq T$ **do**

Solve the nonlinear monolithic FSI problem:

Find $\boldsymbol{U}^h \in \boldsymbol{\mathcal{T}}^h$, s.t. $\forall\, \boldsymbol{\Phi}^h \in \boldsymbol{\mathcal{W}}^h$ it holds

$$\mathcal{F}(\boldsymbol{U}^h; \boldsymbol{\Phi}^h) = \sum_i \mathcal{F}_i(\boldsymbol{U}^h; \boldsymbol{\Phi}^h) = 0 \qquad \text{Semilinear form}$$

In each Newton iteration,

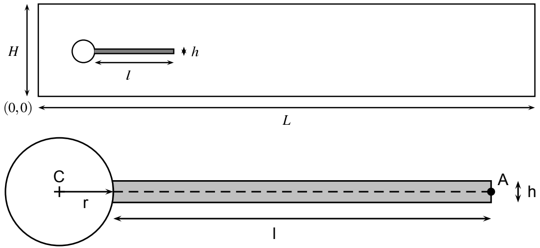
Find $\delta\boldsymbol{U}^h = \left\{\delta\boldsymbol{v}^{h,\mathcal{F}}, \delta\boldsymbol{v}^{h,\mathcal{S}}, \delta\boldsymbol{u}^{h,\mathcal{F}}, \delta\boldsymbol{u}^{h,\mathcal{S}}, \delta p^{h,\mathcal{F}}, \delta p^{h,\mathcal{S}}\right\} \in \boldsymbol{\mathcal{T}}^h$, s.t.

$$\mathcal{F}'(\boldsymbol{U}^{h,k}; \delta\boldsymbol{U}^h, \boldsymbol{\Phi}^h) = -\mathcal{F}(\boldsymbol{U}^{h,k}; \boldsymbol{\Phi}^h), \qquad \forall \boldsymbol{\Phi}^h \in \boldsymbol{\mathcal{W}}^h$$

$$\boldsymbol{U}^{h,k+1} = \boldsymbol{U}^{h,k} + \omega\, \delta\boldsymbol{U}^h,$$

end

FSI tests

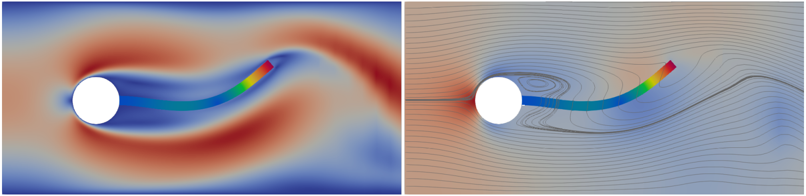


Parameter	Description	Unit	FSI 1	FSI 2	FSI 3
$\rho^{\mathcal{S}}$	Solid density	$[\frac{kg}{m^3}]$	1000	10000	1000
$\nu^{\mathcal{S}}$	Solid Poisson's ratio		0.4	0.4	0.4
$\mu^{\mathcal{S}}$	Solid Lamé constant	$[\frac{kg}{ms^2}]$	0.5×10^6	0.5×10^6	2×10^6
$\rho^{\mathcal{F}}$	Fluid density	$[\frac{kg}{m^3}]$	1000	1000	1000
$\nu^{\mathcal{F}}$	Fluid kinematic viscosity	$[\frac{m^2}{s}]$	0.001	0.001	0.001
$\bar{U}$	Average inflow velocity	$[\frac{m}{s}]$	0.2	1	2
$\beta = \frac{\rho^{\mathcal{S}}}{\rho^{\mathcal{F}}}$	Fluid-solid density ratio		1	10	1
$Re = \frac{\bar{U}d}{\nu^{\mathcal{F}}}$	Reynold's number		20	100	200

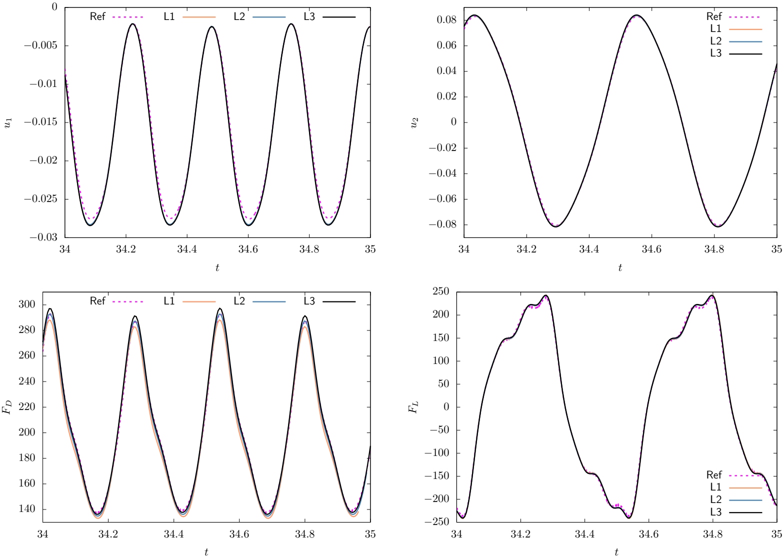
FSI 2

Taylor–Hood NURBS spaces $\hat{\mathbf{V}}_h^{TH} = \boldsymbol{\mathcal{N}}_{0,0}^{3,3}, \hat{Q}_h^{TH} = \mathcal{N}_{0,0}^{2,2}$ on multi-patch mesh

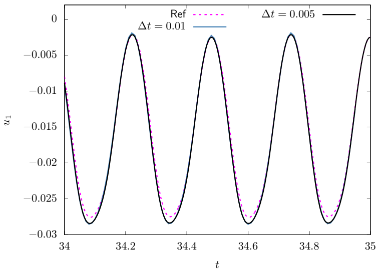
	Δt	level	ndof	$u_1(A)[\times 10^{-3}]$ [f]	$u_2(A)[\times 10^{-3}]$ [f]	F_D [f]	F_L [f]
	1×10^{-2}	1	25209	$-15.22 \pm 13.34[3.85]$	$1.23 \pm 82.1[1.92]$	$211.43 \pm 77.41[3.84]$	$1.1 \pm 237.6[1.92]$
		2	111573	$-15.14 \pm 13.28[3.85]$	$1.21 \pm 82.1[1.92]$	$214.53 \pm 78.80[3.84]$	$1.3 \pm 236.0[1.92]$
		3	468621	$-15.22 \pm 13.33[3.85]$	$1.27 \pm 82.4[1.92]$	$217.48 \pm 80.30[3.84]$	$1.2 \pm 236.9[1.93]$
	5×10^{-3}	1	25209	$-15.23 \pm 13.13[3.85]$	$1.23 \pm 82.4[1.92]$	$210.70 \pm 77.66[3.84]$	$0.9 \pm 243.0[1.93]$
		2	111573	$-15.21 \pm 13.10[3.86]$	$1.20 \pm 82.5[1.92]$	$213.91 \pm 79.13[3.85]$	$1.2 \pm 241.9[1.93]$
		3	468621	$-15.29 \pm 13.15[3.86]$	$1.26 \pm 82.8[1.92]$	$216.80 \pm 80.63[3.85]$	$0.9 \pm 242.8[1.93]$
Turek/Hron[1]	5×10^{-4}	$4+0$	304128	$-14.85 \pm 12.70[3.86]$	$1.30 \pm 81.6[1.93]$	$215.06 \pm 77.65[3.86]$	$0.6 \pm 237.8[1.93]$



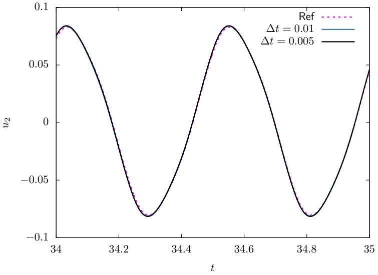
FSI 2



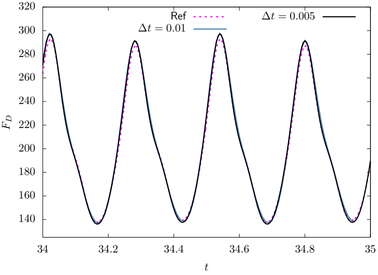
FSI 2



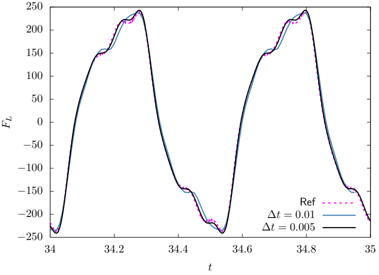
Ref
$\Delta t = 0.01$
$\Delta t = 0.005$
u_1
t



Ref
$\Delta t = 0.01$
$\Delta t = 0.005$
u_2
t

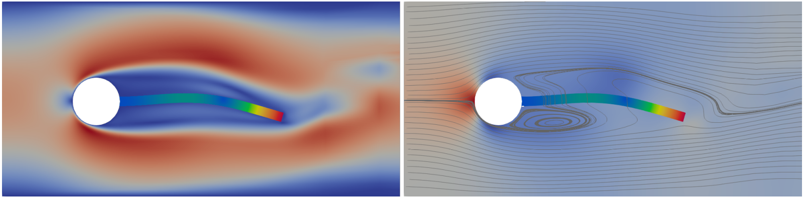


Ref
$\Delta t = 0.01$
$\Delta t = 0.005$
F_D
t



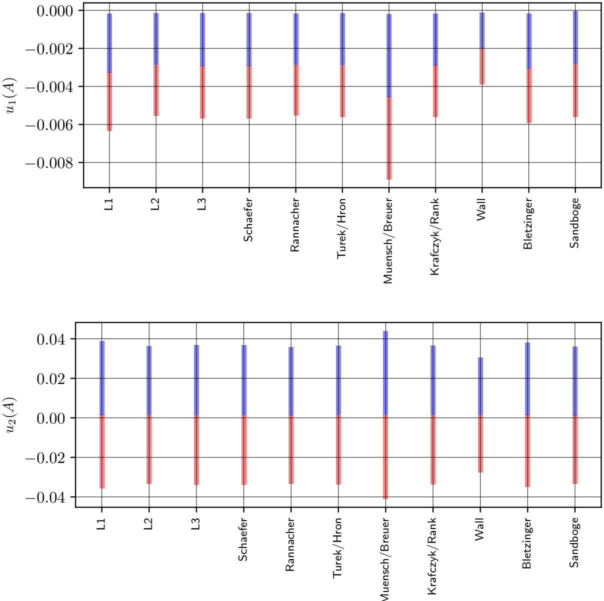
Ref
$\Delta t = 0.01$
$\Delta t = 0.005$
F_L
t

FSI 3



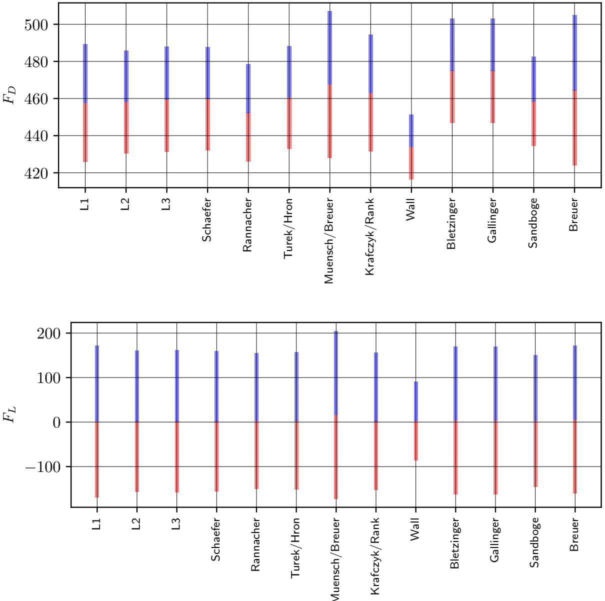
	Δt	ndof	$u_1(A)[\times 10^{-3}]$ [f]	$u_2(A)[\times 10^{-3}]$ [f]	F_D [f]	F_L [f]
Present	1.0e−3	25209	$-3.26 \pm 3.08[10.90]$	$1.48 \pm 37.21[5.46]$	$457.6 \pm 31.59[10.88]$	$1.29 \pm 169.74[5.43]$
		111573	$-2.85 \pm 2.69[10.92]$	$1.38 \pm 34.78[5.47]$	$458.1 \pm 27.65[10.90]$	$2.06 \pm 158.95[5.44]$
		468621	$-2.92 \pm 2.76[10.93]$	$1.45 \pm 35.25[5.47]$	$459.5 \pm 28.32[10.97]$	$2.15 \pm 159.57[5.51]$
Present	5.0e−4	111573	$-2.89 \pm 2.72[10.88]$	$1.49 \pm 34.99[5.44]$	$458.6 \pm 27.19[10.86]$	$2.43 \pm 159.59[5.42]$
1) Schäfer	1.0e−3	941158	$-2.91 \pm 2.77[11.63]$	$1.47 \pm 35.26[4.98]$	459.9 ± 27.92	1.84 ± 157.70
2b) Rannacher	5.0e−4	72696	$-2.84 \pm 2.67[10.84]$	$1.28 \pm 34.61[5.42]$	452.4 ± 26.19	2.36 ± 152.70
3) Turek/Hron[4]	2.5e−4	304128	$-2.88 \pm 2.72[10.93]$	$1.47 \pm 34.99[5.46]$	460.5 ± 27.74	2.50 ± 153.91
4) Münsch/Breuer[7]	2.0e−5	324480	$-4.54 \pm 4.34[10.12]$	$1.50 \pm 42.50[5.05]$	467.5 ± 39.50	16.2 ± 188.70
5) Krafczyk/Rank	5.1e−5	2480814	$-2.88 \pm 2.71[11.00]$	$1.48 \pm 35.10[5.50]$	463.0 ± 31.30	1.81 ± 154.00
6) Wall	5.0e−4	27147	$-2.00 \pm 1.89[10.60]$	$1.45 \pm 29.00[5.30]$	434.0 ± 17.50	2.53 ± 88.60
7) Bletzinger	5.0e−4	271740	$-3.04 \pm 2.87[10.99]$	$1.55 \pm 36.63[5.51]$	474.9 ± 28.12	3.86 ± 165.90
Gallinger[8]					474.9 ± 28.10	3.90 ± 165.90
Sandboge[9]			$-2.83 \pm 2.78[10.8]$	$1.35 \pm 34.75[5.4]$	458.5 ± 24.00	2.50 ± 147.50
Breuer[10]					464.5 ± 40.50	6.00 ± 166.00

FSI 3



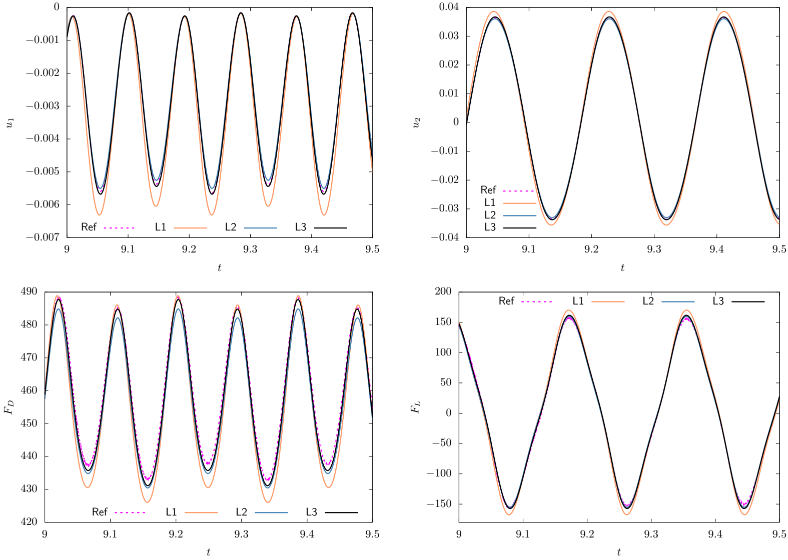
0.000
−0.002
−0.004
−0.006
−0.008
$u_1(A)$
L1
L2
L3
Schaefer
Rannacher
Turek/Hron
Muensch/Breuer
Krafczyk/Rank
Wall
Bletzinger
Sandboge
0.04
0.02
0.00
−0.02
−0.04
$u_2(A)$
L1
L2
L3
Schaefer
Rannacher
Turek/Hron
Muensch/Breuer
Krafczyk/Rank
Wall
Bletzinger
Sandboge

FSI 3



500
480
460
440
420
F_D
L1
L2
L3
Schaefer
Rannacher
Turek/Hron
Muensch/Breuer
Krafczyk/Rank
Wall
Bletzinger
Gallinger
Sandboge
Breuer
200
100
0
−100
F_L
L1
L2
L3
Schaefer
Rannacher
Turek/Hron
Muensch/Breuer
Krafczyk/Rank
Wall
Bletzinger
Gallinger
Sandboge
Breuer

FSI 3



Conclusions and outlook

Isogeometric Analysis $\oplus$ fully coupled monolithic ALE-FSI model

- robust numerical method
- successful (benchmarks)

Extension to

- 3D
- Complex geometries from biomechanical contexts
- Local refinement (Hierarchical B-splines, T-splines, etc.)

Work already done:

- ALE "Binary-fluid"–Structure Interaction based on the Cahn–Hilliard phase field model

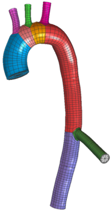
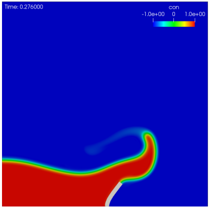


Figure : ©[5]



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http://www.featflow.de/en/benchmarks/cfdbenchmarking/fsi_benchmark.html

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A Monolithic FEM/Multigrid Solver for an ALE Formulation of Fluid–Structure Interaction with Applications in Biomechanics.

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Turek, S. and Hron, J. and Razzaq, M. and Wobker, H. and Schäfer, M.
Numerical Benchmarking of Fluid-Structure Interaction: A comparison of different discretization and solution approaches

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Lecture Notes on Continuum Mechanics on Arbitrary Moving Domains

Münsch, M. and Breuer, M.
Numerical Simulation of Fluid–Structure Interaction Using Eddy–Resolving Schemes

Gallinger, T.G.
Effiziente Algorithmen zur partitionierten Lösung stark gekoppelter Probleme der Fluid-Struktur-Wechselwirkung

Sandboge, R.
Fluid-structure interaction with openfsitm and md nastrantm structural solver

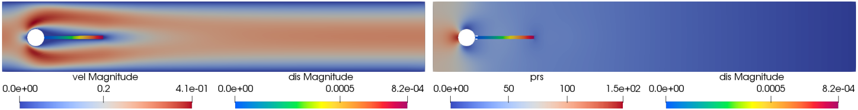
M. Breuer and G. De Nayer and M. Münsch and T. Gallinger and R. Wüchner
Fluid–structure interaction using a partitioned semi-implicit predictor–corrector coupling scheme for the application of large-eddy simulation

Aortic valve
https://en.wikipedia.org/wiki/Aortic_valve

Simulation of flow in the heart valve
https://cdn.comsol.com/wordpress/2018/04/heart-valve-comsol-simulation-results.png

Appendix, FSI 1

	ndof	$u_1(A)[\times 10^{-5}]$ (%-Err)	$u_2(A)[\times 10^{-4}]$ (%-Err)	F_D (%-Err)	F_L (%-Err)
Present	5067	2.240935(1.30)	8.455798(3.01)	14.28377(0.073)	0.774193(1.368)
	25209	2.261569(0.39)	8.201354(0.09)	14.28930(0.035)	0.765377(0.214)
	111573	2.266417(0.18)	8.196860(0.15)	14.29256(0.012)	0.764979(0.161)
	468621	2.268144(0.10)	8.194405(0.18)	14.29334(0.006)	0.764847(0.144)
	1919997	2.268989(0.07)	8.191383(0.21)	14.29367(0.004)	0.764798(0.138)
1) Schäfer	322338			14.2890	0.76900
2b) Rannacher	351720	2.2695	8.1556	14.2603	0.76388
3) Turek/Hron[4]	19320832	2.270493	8.208773	14.29426	0.76374
5) Krafczyk/Rank	14155776	2.2160	8.2010	14.3815	0.75170
6) Wall	164262	2.2680	8.2310	14.2940	0.76487
7) Bletzinger	217500	2.2640	8.2800	14.3510	0.76351



Appendix, Mesh motion models

- **Harmonic mesh motion model**

$$
\begin{aligned}
-\nabla_{\boldsymbol{\chi}} \cdot (\boldsymbol{\sigma}_{\text{mesh}}) &= 0 && \text{in } \Omega_{\boldsymbol{\chi}}^{\mathcal{F}}, \\
\boldsymbol{u}^{\mathcal{F}} &= \boldsymbol{u}^{\mathcal{S}} && \text{on } \Gamma_{\boldsymbol{\chi}}^{\mathcal{I}}, \\
\boldsymbol{u}^{\mathcal{F}} &= 0 && \text{on } \partial\Omega_{\boldsymbol{\chi}}^{\mathcal{F}} \setminus \Gamma_{\boldsymbol{\chi}}^{\mathcal{I}},
\end{aligned}
$$

$$
\boldsymbol{\sigma}_{\text{mesh}} = D\nabla_{\boldsymbol{\chi}}\boldsymbol{u}.
$$

- **Linear elastic mesh motion model**

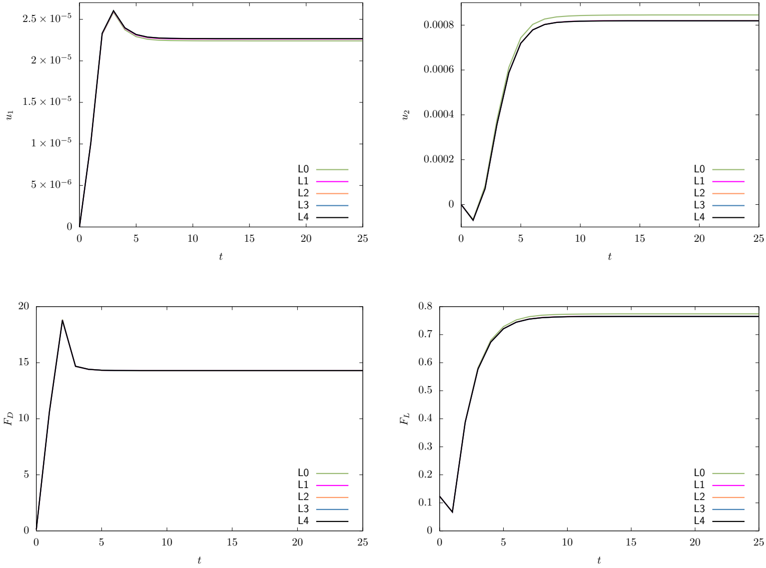
$$
\begin{aligned}
-\nabla_{\boldsymbol{\chi}} \cdot (\boldsymbol{\sigma}_{\text{mesh}}) &= 0 && \text{in } \Omega_{\boldsymbol{\chi}}^{\mathcal{F}}, \\
\boldsymbol{u}^{\mathcal{F}} &= \boldsymbol{u}^{\mathcal{S}} && \text{on } \Gamma_{\boldsymbol{\chi}}^{\mathcal{I}}, \\
\boldsymbol{u}^{\mathcal{F}} &= 0 && \text{on } \partial\Omega_{\boldsymbol{\chi}}^{\mathcal{F}} \setminus \Gamma_{\boldsymbol{\chi}}^{\mathcal{I}},
\end{aligned}
$$

$$
\boldsymbol{\sigma}_{\text{mesh}} = 2\alpha_{\mu}\boldsymbol{\varepsilon} + \alpha_{\lambda}\,\mathrm{tr}\,(\boldsymbol{\varepsilon})\,\boldsymbol{I}.
$$

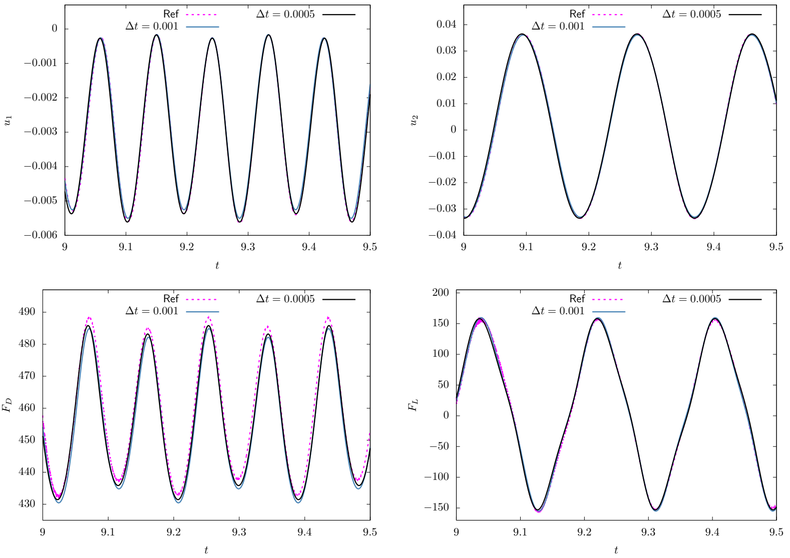
- **Biharmonic mesh motion model**

$$
\begin{aligned}
\nabla_{\boldsymbol{\chi}}^4\boldsymbol{u} = \nabla_{\boldsymbol{\chi}}^2\nabla_{\boldsymbol{\chi}}^2\boldsymbol{u} = \Delta_{\boldsymbol{\chi}}^2\boldsymbol{u} &= 0 && \text{in } \Omega_{\boldsymbol{\chi}}^{\mathcal{F}}, \\
\boldsymbol{u}^{\mathcal{F}} = \boldsymbol{u}^{\mathcal{S}}, \partial_{\boldsymbol{n}}\boldsymbol{u}^{\mathcal{F}} &= \partial_{\boldsymbol{n}}\boldsymbol{u}^{\mathcal{S}} && \text{on } \Gamma_{\boldsymbol{\chi}}^{\mathcal{I}}, \\
\boldsymbol{u}^{\mathcal{F}} = 0, \partial_{\boldsymbol{n}}\boldsymbol{u}^{\mathcal{F}} &= 0 && \text{on } \partial\Omega_{\boldsymbol{\chi}}^{\mathcal{F}} \setminus \Gamma_{\boldsymbol{\chi}}^{\mathcal{I}}.
\end{aligned}
$$

Appendix, FSI 1



Appendix, FSI 3



Appendix, Newton linearization (PDE level)

Linear form (for fixed $\boldsymbol{U}^k$):

$$\mathcal{F}(\boldsymbol{U}^k;\boldsymbol{\Phi}) = \sum_i \mathcal{F}_i(\boldsymbol{U}^k;\boldsymbol{\Phi})$$

$$\begin{aligned}
&\mathcal{F}_1(\boldsymbol{U}^k;\boldsymbol{\Phi}) := \\
&\left(\rho^{\mathcal{F}}\hat{J}^{n,\theta}\left(\boldsymbol{v}^{k,\mathcal{F}}-\boldsymbol{v}^{0,\mathcal{F}}\right),\boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}}\right)_{\Omega^{\mathcal{F}}_{\boldsymbol{\chi}}} + \left(\Delta t\theta\rho^{\mathcal{F}}\hat{J}\nabla_{\boldsymbol{\chi}}\boldsymbol{v}^{k,\mathcal{F}}\hat{\boldsymbol{F}}^{-1}\boldsymbol{v}^{k,\mathcal{F}},\boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}}\right)_{\Omega^{\mathcal{F}}_{\boldsymbol{\chi}}} \\
&+\left(\Delta t(1-\theta)\,\rho^{\mathcal{F}}\hat{J}^0\nabla_{\boldsymbol{\chi}}\boldsymbol{v}^{0,\mathcal{F}}\left(\hat{\boldsymbol{F}}^0\right)^{-1}\boldsymbol{v}^{0,\mathcal{F}},\boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}}\right)_{\Omega^{\mathcal{F}}_{\boldsymbol{\chi}}} \\
&-\left(\rho^{\mathcal{F}}\hat{J}\nabla_{\boldsymbol{\chi}}\boldsymbol{v}^{k,\mathcal{F}}\hat{\boldsymbol{F}}^{-1}\cdot\left(\boldsymbol{u}^{k,\mathcal{F}}-\boldsymbol{u}^{0,\mathcal{F}}\right),\boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}}\right)_{\Omega^{\mathcal{F}}_{\boldsymbol{\chi}}} \\
&+\left(\Delta t\theta\hat{J}\mu^{\mathcal{F}}\left(\nabla_{\boldsymbol{\chi}}\boldsymbol{v}^{k,\mathcal{F}}\hat{\boldsymbol{F}}^{-1}+\hat{\boldsymbol{F}}^{-T}\cdot\left(\nabla_{\boldsymbol{\chi}}\boldsymbol{v}^{k,\mathcal{F}}\right)^T\right)\hat{\boldsymbol{F}}^{-T},\nabla_{\boldsymbol{\chi}}\boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}}\right)_{\Omega^{\mathcal{F}}_{\boldsymbol{\chi}}} \\
&+\left(\Delta t(1-\theta)\,\hat{J}^0\,\mu^{\mathcal{F}}\left(\nabla_{\boldsymbol{\chi}}\boldsymbol{v}^{0,\mathcal{F}}\left(\hat{\boldsymbol{F}}^0\right)^{-1}+(\hat{\boldsymbol{F}}^0)^{-T}\cdot(\nabla_{\boldsymbol{\chi}}\boldsymbol{v}^{0,\mathcal{F}})^T\right)\left(\hat{\boldsymbol{F}}^0\right)^{-T},\nabla_{\boldsymbol{\chi}}\boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}}\right)_{\Omega^{\mathcal{F}}_{\boldsymbol{\chi}}} \\
&+\left(\Delta t\hat{J}\left(-p^{k,\mathcal{F}}\boldsymbol{I}\right)\hat{\boldsymbol{F}}^{-T},\nabla_{\boldsymbol{\chi}}\boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}}\right)_{\Omega^{\mathcal{F}}_{\boldsymbol{\chi}}} - \left(\rho^{\mathcal{F}}\,\Delta t\hat{J}^{n,\theta}\boldsymbol{b}^{\mathcal{F}},\boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}}\right)_{\Omega^{\mathcal{F}}_{\boldsymbol{\chi}}} - \left(\Delta t\boldsymbol{g}_0^{\mathcal{F}},\boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}}\right)_{\Gamma^{\mathcal{F}}_{N,\boldsymbol{\chi}}}
\end{aligned}$$

Appendix, Newton linearization (PDE level)

$$\mathcal{F}_2(\boldsymbol{U}^k;\boldsymbol{\Phi}) := \left(\hat{J}\hat{\boldsymbol{F}}^{-1} : \left(\nabla_{\boldsymbol{\chi}}\boldsymbol{v}^{k,\mathcal{F}}\right)^T, \phi^{p,\mathcal{F}}\right)_{\Omega^{\mathcal{F}}_{\boldsymbol{\chi}}}$$

$$\begin{aligned}\mathcal{F}_3(\boldsymbol{U}^k;\boldsymbol{\Phi}) := &\left(\hat{J}\rho^{\mathcal{S}}\left(\boldsymbol{v}^{k,\mathcal{S}} - \boldsymbol{v}^{0,\mathcal{S}}\right), \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{S}}\right)_{\Omega^{\mathcal{S}}_{\boldsymbol{\chi}}}\\ &+ \left(\Delta t\theta\,\hat{\boldsymbol{P}}\left(\boldsymbol{u}^{k,\mathcal{S}}\right), \nabla_{\boldsymbol{\chi}}\boldsymbol{\phi}^{\boldsymbol{v},\mathcal{S}}\right)_{\Omega^{\mathcal{S}}_{\boldsymbol{\chi}}} + \left(\Delta t(1-\theta)\,\hat{\boldsymbol{P}}\left(\boldsymbol{u}^{0,\mathcal{S}}\right), \nabla_{\boldsymbol{\chi}}\boldsymbol{\phi}^{\boldsymbol{v},\mathcal{S}}\right)_{\Omega^{\mathcal{S}}_{\boldsymbol{\chi}}}\\ &- \left(\Delta t\theta\,\hat{J}^{n,\theta}\rho^{\mathcal{S}}\boldsymbol{b}^{\mathcal{S}}, \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{S}}\right)_{\Omega^{\mathcal{S}}_{\boldsymbol{\chi}}} - \left(\Delta t\boldsymbol{g}^{\mathcal{S}}_0, \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{S}}\right)_{\Gamma^{\mathcal{S}}_{N,\boldsymbol{\chi}}}\end{aligned}$$

$$\mathcal{F}_4(\boldsymbol{U}^k;\boldsymbol{\Phi}) := \left(\boldsymbol{u}^{k,\mathcal{S}} - \boldsymbol{u}^{0,\mathcal{S}}, \boldsymbol{\phi}^{\boldsymbol{u},\mathcal{S}}\right)_{\Omega^{\mathcal{S}}_{\boldsymbol{\chi}}} - \left(\Delta t\theta\,\boldsymbol{v}^{k,\mathcal{S}}, \boldsymbol{\phi}^{\boldsymbol{u},\mathcal{S}}\right)_{\Omega^{\mathcal{S}}_{\boldsymbol{\chi}}} - \left(\Delta t(1-\theta)\,\boldsymbol{v}^{0,\mathcal{S}}, \boldsymbol{\phi}^{\boldsymbol{u},\mathcal{S}}\right)_{\Omega^{\mathcal{S}}_{\boldsymbol{\chi}}}$$

$$\mathcal{F}_5(\boldsymbol{U}^k;\boldsymbol{\Phi}) := \left(p^{k,\mathcal{S}}, \phi^{p,\mathcal{S}}\right)_{\Omega^{\mathcal{S}}_{\boldsymbol{\chi}}}$$

$$\mathcal{F}_6(\boldsymbol{U}^k;\boldsymbol{\Phi}) := \left(\alpha_u\hat{J}^{-1}\nabla_{\boldsymbol{\chi}}\boldsymbol{u}^{k,\mathcal{F}}, \nabla_{\boldsymbol{\chi}}\boldsymbol{\phi}^{\boldsymbol{u}}\right)_{\Omega^{\mathcal{F}}_{\boldsymbol{\chi}}}$$

$$\hat{J}^{n,\theta} := \theta\hat{J}^n + (1-\theta)\hat{J}^{n-1} = \theta\hat{J}(\boldsymbol{u}^k) + (1-\theta)\hat{J}(\boldsymbol{u}^0)$$

Appendix, Newton linearization (PDE level)

Bilinear form $\mathcal{J} = \mathcal{F}'(\boldsymbol{U}^k; \cdot, \cdot)$ from linearization of $\mathcal{F}$ around $\boldsymbol{U} = \boldsymbol{U}^k$:

$$\mathcal{F}'(\boldsymbol{U}^k; \delta\boldsymbol{U}, \boldsymbol{\Phi}) = \sum_i \mathcal{F}_i'(\boldsymbol{U}^k; \delta\boldsymbol{U}, \boldsymbol{\Phi})$$

$$\begin{aligned}
&\mathcal{F}_1'(\boldsymbol{U}^k; \delta\boldsymbol{U}, \boldsymbol{\Phi}) := \\
&\int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}} \left(\rho^{\mathcal{F}} \theta \hat{J} \operatorname{tr}\left(\hat{\boldsymbol{F}}^{-1} \nabla_{\boldsymbol{\chi}} \delta \boldsymbol{u}^{\mathcal{F}}\right) \left(\boldsymbol{v}^{k,\mathcal{F}} - \boldsymbol{v}^{0,\mathcal{F}}\right) + \rho^{\mathcal{F}} \hat{J}^{n,\theta} \delta \boldsymbol{v}^{\mathcal{F}}\right) \cdot \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}} \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{F}} \\
+ &\int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}} \Bigg(\Delta t \theta \rho^{\mathcal{F}} \left(\hat{J} \operatorname{tr}\left(\hat{\boldsymbol{F}}^{-1} \nabla_{\boldsymbol{\chi}} \delta \boldsymbol{u}^{\mathcal{F}}\right) \nabla_{\boldsymbol{\chi}} \boldsymbol{v}^{k,\mathcal{F}} \hat{\boldsymbol{F}}^{-1} \boldsymbol{v}^{k,\mathcal{F}} + \hat{J} \nabla_{\boldsymbol{\chi}} \boldsymbol{v}^{k,\mathcal{F}} \left(-\hat{\boldsymbol{F}}^{-1} \nabla_{\boldsymbol{\chi}} \delta \boldsymbol{u}^{\mathcal{F}} \hat{\boldsymbol{F}}^{-1}\right) \boldsymbol{v}^{k,\mathcal{F}}\right) \\
&+ \Delta t \theta \rho^{\mathcal{F}} \hat{J} \left(\nabla_{\boldsymbol{\chi}} \delta \boldsymbol{v}^{\mathcal{F}} \hat{\boldsymbol{F}}^{-1} \boldsymbol{v}^{k,\mathcal{F}} + \nabla_{\boldsymbol{\chi}} \boldsymbol{v}^{k,\mathcal{F}} \hat{\boldsymbol{F}}^{-1} \delta \boldsymbol{v}^{\mathcal{F}}\right) \\
&- \rho^{\mathcal{F}} \hat{J} \operatorname{tr}\left(\hat{\boldsymbol{F}}^{-1} \nabla_{\boldsymbol{\chi}} \delta \boldsymbol{u}^{\mathcal{F}}\right) \nabla_{\boldsymbol{\chi}} \boldsymbol{v}^{k,\mathcal{F}} \hat{\boldsymbol{F}}^{-1} \cdot \left(\boldsymbol{u}^{k,\mathcal{F}} - \boldsymbol{u}^{0,\mathcal{F}}\right) \\
&- \rho^{\mathcal{F}} \hat{J} \left(\nabla_{\boldsymbol{\chi}} \boldsymbol{v}^{k,\mathcal{F}} \left(-\hat{\boldsymbol{F}}^{-1} \nabla_{\boldsymbol{\chi}} \delta \boldsymbol{u}^{\mathcal{F}} \hat{\boldsymbol{F}}^{-1}\right) \left(\boldsymbol{u}^{k,\mathcal{F}} - \boldsymbol{u}^{0,\mathcal{F}}\right) + \nabla_{\boldsymbol{\chi}} \boldsymbol{v}^{k,\mathcal{F}} \hat{\boldsymbol{F}}^{-1} \delta \boldsymbol{u}^{\mathcal{F}}\right) \\
&- \rho^{\mathcal{F}} \hat{J} \nabla_{\boldsymbol{\chi}} \delta \boldsymbol{v}^{\mathcal{F}} \hat{\boldsymbol{F}}^{-1} \cdot \left(\boldsymbol{u}^{k,\mathcal{F}} - \boldsymbol{u}^{0,\mathcal{F}}\right)\Bigg) \cdot \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}} \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}
\end{aligned}$$

Appendix, Newton linearization (PDE level)

$$
\boldsymbol{G}(\delta\boldsymbol{u}) := \begin{pmatrix} \partial\delta u_2/\partial y & -\partial\delta u_2/\partial x \\ -\partial\delta u_1/\partial y & \partial\delta u_1/\partial x \end{pmatrix}
$$

$$
\begin{aligned}
&\mathcal{F}_2'(\boldsymbol{U}^k; \delta\boldsymbol{U}, \boldsymbol{\Phi}) := \\
&\int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}} \Bigg(\boldsymbol{\sigma}_{uv}^{\mathcal{F}} \boldsymbol{G}(\delta\boldsymbol{u}) \\
&+ \mu^{\mathcal{F}} \left(\nabla_{\boldsymbol{\chi}} \boldsymbol{v}^{k,\mathcal{F}} \left(-\hat{\boldsymbol{F}}^{-1} \nabla_{\boldsymbol{\chi}} \delta\boldsymbol{u}^{\mathcal{F}} \hat{\boldsymbol{F}}^{-1} \right) + \left(-\hat{\boldsymbol{F}}^{-T} \cdot (\nabla_{\boldsymbol{\chi}} \delta\boldsymbol{u}^{\mathcal{F}})^T \hat{\boldsymbol{F}}^{-T} \right) (\nabla_{\boldsymbol{\chi}} \boldsymbol{v}^{k,\mathcal{F}})^T \right) \hat{J} \hat{\boldsymbol{F}}^{-T} \\
&+ \mu^{\mathcal{F}} \left(\nabla_{\boldsymbol{\chi}} \delta\boldsymbol{v}^{\mathcal{F}} \hat{\boldsymbol{F}}^{-1} + \hat{\boldsymbol{F}}^{-T} (\nabla_{\boldsymbol{\chi}} \delta\boldsymbol{v}^{\mathcal{F}})^T \right) \hat{J} \hat{\boldsymbol{F}}^{-T} \\
&- (p^{\mathcal{F}} \boldsymbol{I}) \boldsymbol{G}(\delta\boldsymbol{u}) - (\delta p^{\mathcal{F}} \boldsymbol{I}) \hat{J} \hat{\boldsymbol{F}}^{-T} - \left(\delta p^{\mathcal{F}} \boldsymbol{I} \right) \hat{J} \hat{\boldsymbol{F}}^{-T} \Bigg) : \nabla_{\boldsymbol{\chi}} \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{F}} \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{F}_3'(\boldsymbol{U}^k; \delta\boldsymbol{U}, \boldsymbol{\Phi}) := \\
&\int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}} \Bigg(\hat{J} \,\mathrm{tr} \left(\hat{\boldsymbol{F}}^{-1} \nabla_{\boldsymbol{\chi}} \delta\boldsymbol{u}^{\mathcal{F}} \right) \mathrm{tr} \left(\nabla_{\boldsymbol{\chi}} \boldsymbol{v}^{k,\mathcal{F}} \hat{\boldsymbol{F}}^{-1} \right) \\
&+ \hat{J} \,\mathrm{tr} \left(\nabla_{\boldsymbol{\chi}} \boldsymbol{v}^{k,\mathcal{F}} \left(-\hat{\boldsymbol{F}}^{-1} \nabla_{\boldsymbol{\chi}} \delta\boldsymbol{u}^{\mathcal{F}} \hat{\boldsymbol{F}}^{-1} \right) \right) + \hat{J} \,\mathrm{tr} \left(\nabla_{\boldsymbol{\chi}} \delta\boldsymbol{v}^{\mathcal{F}} \hat{\boldsymbol{F}}^{-1} \right) \Bigg) \cdot \phi^{p,\mathcal{F}} \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{F}_4'(\boldsymbol{U}^k; \delta\boldsymbol{U}, \boldsymbol{\Phi}) := \\
&\int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}} \left(-\alpha_u \hat{J}^{-1} \mathrm{tr} \left(\hat{\boldsymbol{F}}^{-1} \nabla_{\boldsymbol{\chi}} \delta\boldsymbol{u}^{\mathcal{F}} \right) \nabla_{\boldsymbol{\chi}} \boldsymbol{u}^{k,\mathcal{F}} + \alpha_u \hat{J}^{-1} \nabla_{\boldsymbol{\chi}} \delta\boldsymbol{u}^{\mathcal{F}} \right) \cdot \nabla_{\boldsymbol{\chi}} \boldsymbol{\phi}^{\boldsymbol{u},\mathcal{F}} \, \mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{F}}
\end{aligned}
$$

Appendix, Newton linearization (PDE level)

$$\mathcal{F}_5'(\boldsymbol{U}^k; \delta\boldsymbol{U}, \boldsymbol{\Phi}) := \int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{S}}} \rho^{\mathcal{S}}\, \delta\boldsymbol{v} \cdot \boldsymbol{\phi}^{\boldsymbol{v},\mathcal{S}} \,\mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{S}}$$

$$\mathcal{F}_6'(\boldsymbol{U}^k; \delta\boldsymbol{U}, \boldsymbol{\Phi}) := \int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{S}}} \Delta t\theta\lambda^{\mathcal{S}} \operatorname{tr}\left(\frac{1}{2}\left((\nabla_{\boldsymbol{\chi}}\delta\boldsymbol{u})^T \hat{\boldsymbol{F}} + \hat{\boldsymbol{F}}^T \nabla_{\boldsymbol{\chi}}\delta\boldsymbol{u}\right)\right) \hat{\boldsymbol{F}} : \nabla_{\boldsymbol{\chi}}\boldsymbol{\phi}^{\boldsymbol{v},\mathcal{S}} \,\mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{S}}$$

$$\mathcal{F}_7'(\boldsymbol{U}^k; \delta\boldsymbol{U}, \boldsymbol{\Phi}) := \int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{S}}} \delta\boldsymbol{u} \cdot \boldsymbol{\phi}^{\boldsymbol{u},\mathcal{S}} \,\mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{S}} - \int_{\Omega_{\boldsymbol{\chi}}^{\mathcal{S}}} \Delta t\theta\, \delta\boldsymbol{v} \cdot \boldsymbol{\phi}^{\boldsymbol{u},\mathcal{S}} \,\mathrm{d}\Omega_{\boldsymbol{\chi}}^{\mathcal{S}}$$