Efficient simulation techniques of the Lattice Boltzmann equation on unstructured meshes [1]

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Abstract.

In this paper, we apply special techniques from Numerics for PDE's to the Lattice Boltzmann equation. In [5] the concept of the *generalized mean intensity* has been proposed for radiative transfer equations. Here, we adapt this concept to the LBE, treating it as an analogous integro differential equation with constant characteristics. Thus, we combine an efficient finite difference-like discretization based on short-characteristic upwinding techniques on unstructured, locally adapted grids with Krylow-space methods (Bi-CGSTAB, GMRES, etc.). The implicit treatment of the LBE leads to nonlinearities which are efficiently solved with the Newton method even for a direct, stationary solution of the LBE. With our direct preconditioning by the transport part we obtain an efficient linear solver for transport dominated configurations, while collision dominated cases are treated with a block-Jacobian preconditioning. Due to our new *generalized equilibrium formulation* (GEF) we can combine the advantages of both preconditioners, i.e. independence of the number of unknowns for convection-dominated cases with robustness against stiff configurations. We further improve the GEF approach by using multigrid algorithms to obtain very good convergence rates for a wide range of problem parameters, and demonstrate the results via various benchmark problems.

1 Introduction

In this paper we consider the Lattice Boltzmann equation (LBE) in contrast to the Lattice Boltzmann method (LBM) which is thoroughly exploited in the literature. State of the art for the LBM is working on structured grids and mainly performing two explicit steps; first, particles stream along lattice vectors to the neighbouring nodes and, second, the particles are redistributed after collision. However, the LBE *is* a partial differential equation, and talking about modern Numerics for PDE's one has to be aware of advances w.r.t. implicit time discretizations and unstructured, locally adapted grids. In comparison to the LBM an implicit treatment of the LBE appears in very few places. In [12] Tölke has applied an upwind finite difference discretization of second order on structured grids and applied multigrid to solve the resulting linear equation systems. In [8] Mavriplis also took advantage of the multigrid method after trying various iterative solvers for the resulting equation systems and encountering problems due to the high stiffness of the system and bad conditioning for realistic configurations. In [9] Noble introduced an implicit discretization of the LB method and used a direct solver from LAPACK which proved to be efficient for the resulting sparse banded matrices. The resulting

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nonlinearities were therein successfully treated with Newton method. He demonstrated for his scheme to be superior to the explicit method, with runtimes of the Newton/LAPACK combination scaling like $N^{1.2}$ compared to N^2 for the LBM with N describing the number of unknowns. However, none of the authors looked beyond the structured framework. Here, we treat the LBE as a special integro-differential equation which consists of (linear) partial differential operators of transport-reaction type with constant characteristics. We apply a special, higher order Finite Difference discretization on unstructured grids and combine it with implicit time discretization.

The PDE which describes the continuous, stationary Lattice Boltzmann equation is given for the specific distributions f_i in the form

$$e_i \cdot \nabla f_i = \frac{1}{\tau}(f_i^{eq} - f_i) \tag{1}$$

while the nonstationary variant adds a time derivative:

$$\frac{\partial f_i}{\partial t} + e_i \cdot \nabla f_i = \frac{1}{\tau}(f_i^{eq} - f_i) \tag{2}$$

The LBE is discretized in phase space with the following lattice vectors for the D2Q9 model:

$$e_i \in \{\begin{pmatrix} c \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ c \end{pmatrix}, \begin{pmatrix} 0 \\ c \end{pmatrix}, \begin{pmatrix} -c \\ c \end{pmatrix}, \begin{pmatrix} -c \\ 0 \end{pmatrix}, \begin{pmatrix} -c \\ -c \end{pmatrix}, \begin{pmatrix} 0 \\ -c \end{pmatrix}, \begin{pmatrix} c \\ -c \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

This approach approximates the Navier-Stokes equation, whereas the viscosity ν is included via $\tau = \frac{3}{\nu\ c^2}$ and the distributions are summed up to get the moments of density and velocity:

$$\rho = \sum_i f_i \quad , \quad \rho_0 \cdot u = \sum_i f_i e_i$$

The parameter c determines the speed of sound, $c_s = c/\sqrt{3}$, of the system. The Mach-number $Ma = U/c_s$ is supposed to determine an additional error component of the system, namely the compressibility error of the order $O(Ma^2)$ (see [10]).

The equilibrium term f^{eq} of the incompressible model (i.e. $\rho_0 = 1$, see [3]) introduces a nonlinear coupling of the distributions, iff an implicit time-discretization is used:

$$f_i^{eq} = \omega_i(\rho + \rho_0(\frac{3}{c^2}(e_i \cdot u) + \frac{9}{2c^4}(e_i \cdot u)^2 - \frac{3}{2c^2}(u_1^2 + u_2^2))) \tag{3}$$

We want to emphasize that the parameter c appears as a linear scaling factor to the differential operator, and quadratic in the collision term through τ. This means that for small c and viscosity ν the equation is transport dominated, while increasing c and ν makes it collision dominated. However, we discovered a significant interplay between c, h, ν in the simulations, vastly influencing the approximation (see Sec. 2.4).

2 Discretization of the Lattice Boltzmann equation

Our approach to the discretization of the LBE is concerned with various aspects which will be covered by four main parts in this section. The first two parts are what we believe to be the essentials of modern Numerics: Implicit time discretization will be described in Section 2.1; the second aspect is high order space discretization on unstructured grids which will be described in Section 2.2. The next part shortly describes our boundary treatment. In Section 2.4 we finally present how our space discretization is connected to the choice of parameter c and how the combination of both affects the consistency and overall convergence.

2.1 Time discretized, linearized equation system

To overcome the limitations that the explicit LBM is committed to (numerical instability for $\tau \to \frac{1}{2}$, see [7], or the CFL restriction), we can treat the LBE in (1) fully implicitly to solve directly for stationary, laminar flow. For configurations with high Reynolds number and for nonstationary flow problems, we introduce the following time-stepping scheme:

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + e_i \cdot \nabla f_i^{n+1} + \frac{1}{\tau}(f_i^{n+1} - f_i^{eq,n+1}) = 0 \tag{4}$$

With $h_i^n := e_i \cdot \nabla f_i^n + \frac{1}{\tau}(f_i^n - f_i^{eq,n})$ we can rewrite equation (4) as:

$$f_i^{n+1} + \theta \Delta t \; h_i^{n+1} = (\theta - 1)\Delta t \; h_i^n + f_i^n$$

So we can choose between the schemes of explicit Euler, Crank-Nicholson and implicit Euler by choosing $\theta \in \{0, \frac{1}{2}, 1\}$. In the latter two cases and for the stationary approach, the collision term on the right hand side of equation (1) or (2) becomes part of the system matrix to be solved. This step causes in every point a coupling of the distributions in each direction, that means a 9x9 collision block each. Due to the nonlinearity in the equilibrium term, we have to apply a nonlinear solver in every timestep, resp., in the stationary solution procedure. For this purpose we have to linearize the f_i^{eq} terms,

$$\begin{aligned} f_i^{eq} &= \omega_i\Big(\sum_k f_k + 3\sum_k D_{ik} f_k + \frac{9}{2c^2}\sum_k D_{ik} f_k (e_i \cdot u^{old}) - \frac{3}{2c}\Big(\sum D_{1k} f_k u_1^{old} + \sum D_{3k} f_k u_2^{old}\Big)\Big) \\ &=: \sum_k \tilde{\omega}_{ik} f_k \end{aligned} \tag{5}$$

with $\tilde{\omega}_{ik} = \tilde{\omega}(i, k, \tau, c, u)$ and constant coefficients D_{ik} resulting from:

$$(e_i \cdot u) = c^2 \sum_k D_{ik} f_k \quad , \quad u_1 = c \sum_k D_{1k} f_k \quad , \quad u_2 = c \sum_k D_{3k} f_k$$

The nonlinearity is best to be solved using the Newton method, wherein the Jacobian can be determined analytically. The derivation results in a multiplier of 2 in each quadratic term:

$$df_i^{eq} = \omega_i\Big(\sum_k f_k + 3\sum_k D_{ik} f_k + \frac{9}{c^2}\sum_k D_{ik} f_k (e_i \cdot u^{old}) - \frac{3}{c}\Big(\sum D_{1k} f_k u_1^{old} + \sum D_{3k} f_k u_2^{old}\Big)\Big)$$

Here we present Testcase 1, which is due to a direct solution procedure of equation (1), treating the boundary implicitly as in the following equation (6). In this case we solve the "flow around cylinder" benchmark for two Reynolds numbers in view of comparing fixed point and Newton scheme for solving the nonlinearity. From Table 1 it becomes obvious that the fixed point iteration is only able to cope with small Reynolds numbers, while the choice of bigger c also positively influences the solver. In contrast, for higher values of Re (see Table 2) the Newton scheme still needs only few iterations to reduce significantly the nonlinear defect.

	c=1			c=10			c=100		
	N=572	N=2184	N=8528	N=572	N=2184	N=8528	N=572	N=2184	N=8528
fixed point	13	15	16	11	14	16	6	11	15
Newton	4	4	4	4	4	4	3	4	4

Table 1: Testcase 1: No. of iterations to reduce the nonlinear defect by 10^{-6}, $Re = 2$

	c=1			c=10			c=100		
	N=572	N=2184	N=8528	N=572	N=2184	N=8528	N=572	N=2184	N=8528
fixed point	180	> 300	>300	30	211	>300	8	23	>300
Newton	5	5	5	4	5	5	3	4	6

Table 2: Testcase 1: No. of iterations to reduce the nonlinear defect by 10^{-6}, $Re = 20$

2.2 The 'short-characteristic' discretization procedure

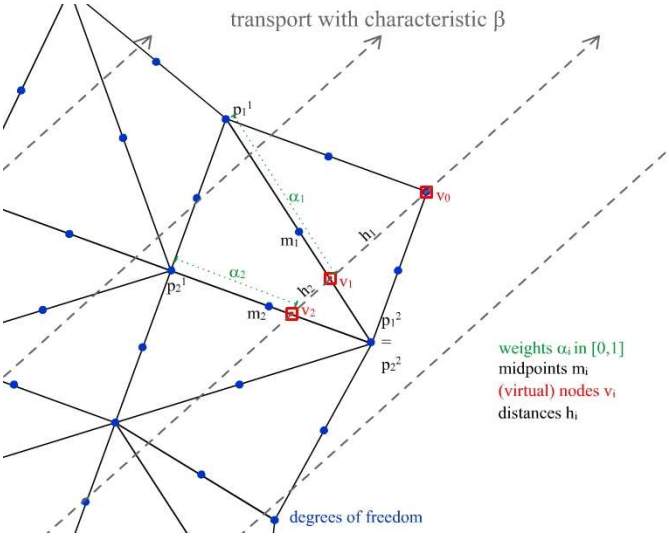


Figure 1: Setting for FD discretization of 1st and 2nd order

The transport operator in (1), which is described in the LBM as a trivial streaming process of particles, wil be discretized for each of the 8 constant characteristics using finite difference techniques. This task is performed very efficiently in [4], using a backward difference scheme of up to second order accuracy. The upwind discretization procedure works on unstructured triangular meshes as following: With the normal vector $\mathbf{n}_i$ in i-th direction, the constant characteristics upwinding of first order (to be denoted as upw 1) yields the difference quotient,

$$\mathbf{n}_i \cdot \nabla u(v_0) = u'(v_0) \sim \frac{u(v_0) - u(v_1)}{h_1} + O(h_1),$$

while the second order scheme (to be denoted as upw 2) yields (with $h_1 + h_2 = r \cdot h_1$):

$$\mathbf{n}_i \cdot \nabla u(v_0) = u'(v_0) \sim \frac{-(1-r^2)u(v_0) - r^2 u(v_1) + u(v_2)}{h_1(r^2 - r)} + O(h_1, h_2)^2$$

So the LB differential operator will be discretized (scaled by parameter c) as:

$$e_i \cdot \nabla f_i = c\mathbf{n}_i \cdot \nabla f_i \quad \text{for cartesian lattice vectors } e_i$$

$$e_i \cdot \nabla f_i = \sqrt{2}c\mathbf{n}_i \cdot \nabla f_i \quad \text{for diagonal lattice vectors } e_i$$

Exploiting the grid information we can implement this discretization in a matrix-free style, i.e. we can calculate all matrix entries including the FD and collision coefficients 'on-the-fly'.

2.3 Boundary treatment

The boundary treatment is directly connected to the time discretization of the LBE. In the stationary approach, we have to treat the boundary implicitly and include the dependence

of the bounce-back distributions into the system matrix (only then we can achieve quadratic convergence of the Newton scheme):

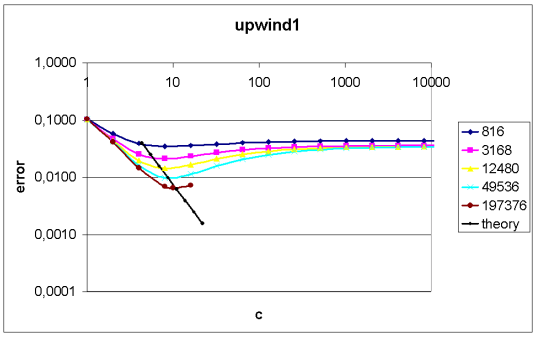
$$f_i = f_{-i} + 6\rho_0 \cdot \omega_i \frac{e_i \cdot u}{c^2} \tag{6}$$

Alternatively, for the time stepping we *can* make use of the distributions from the old timestep (thus faciliating the solution of the linear system):

$$f_i^{n+1} = f_{-i}^{n} + 6\rho_0 \cdot \omega_i \frac{e_i \cdot u}{c^2} \tag{7}$$

2.4 Convergence w.r.t. mesh size h and sound speed constant c

We cannot omit one important aspect while looking at the results so far. Starting the work on this topic it was not clear how to actually treat the parameter c, which appears throughout our discretization. The idea to choose c as big as possible to reduce the Macherror proved to be wrong in the end. Looking at the equations from the viewpoint of asymptotical analysis for the LBE gave some hints that we must choose c depending on h (see [6], [10]). Without the dependence of the two variables we would not achieve a consistent approximation of the Navier-Stokes equation in the asymptotic limit. Additionally the order of our FD discretization of the differential operator is supposed to determine our choice, i.e. $h = O((1/c)^3)$ for first order upwind, resp., $h = O((1/c)^{3/2})$ for second order upwind. We use various configurations for a numerical analysis of this theory. We compare to the given analytical solution for rotating Couette flow from [2]. We compute the L_2-error depending on c for various refinement levels and both upwind discretizations. Additionally we draw the theoretical slope of convergence into the graph, the points in the slope indicating the distance for one refinement step (see Figs. 2, 3). The results match sufficiently well with the theory, also showing that the steps in the slope are more expansive for the 2nd order scheme. Additional comparisons against CFD results are performed for the "flow around cylinder" benchmark (see [11]), resp., for the Driven Cavity problem at various Reynolds numbers (as in [8]). Both proved to be in accordance with the suggested theory. We can also derive from the results a rule of thumb for the obvious relevance of c_{opt}, thus obtaining a hint how to adapt our solvers, resp., choose the appropriate preconditioners. Obviously c_{opt} is getting smaller for increasing Re (see Fig. 4), that is when we can encounter problems due to bigger compressibility errors. As remedy it is common to use multiple-relaxation-time models (MRT, see [7]) which are subject of our current research. So far we can state that very big values for c are hardly necessary, especially for complex configurations at high Reynolds numbers. On the other hand, c_{opt} seems to be in an intermediate area, where we would need a combination of transport and collision preconditioning, as neither effect really dominates the other (see Table 3).



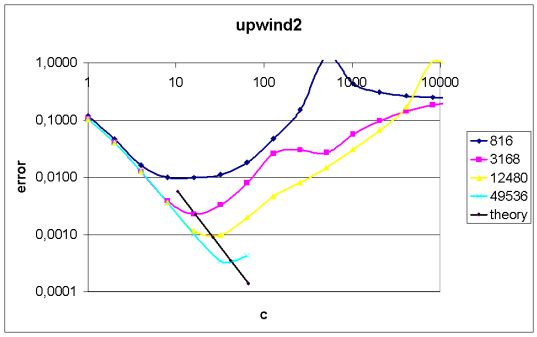
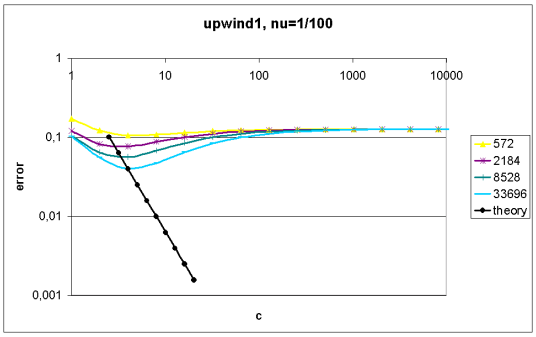


Figure 2: Rotating Couette flow, L_2-error against c for upw 1 (left), resp., upw 2 (right)



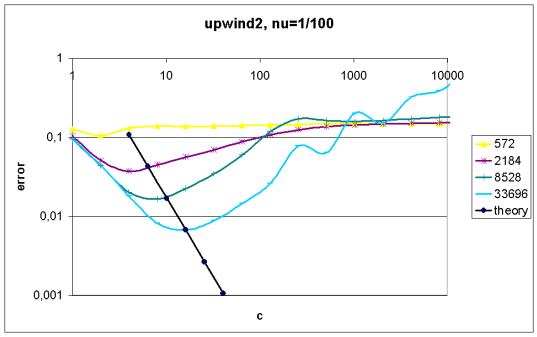


Figure 3: "Flow around cylinder" benchmark for $Re = 2$, L_2-error against c for upw 1 (left), resp., upw 2 (right)

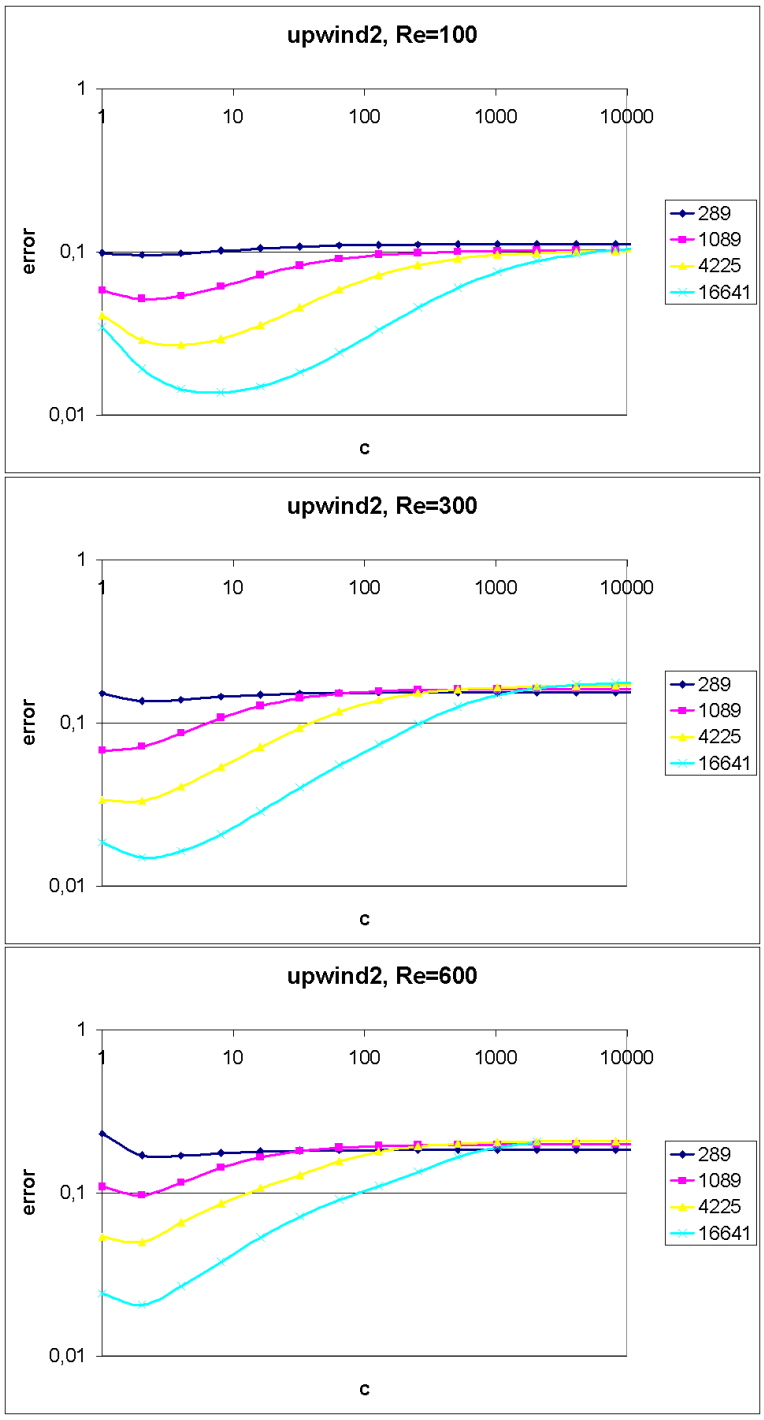


Figure 4: Driven Cavity, L_2-error against c for various Re, upw 2

3 Special preconditioners

As described in [5], for each constant characteristic, resp., lattice vector e_i, it is possible to find a numbering of the unknowns so that the resulting transport matrix is lower triagonal. This procedure has to be conducted once for each direction in preprocessing. Afterwards, whenever one has to solve a transport step, it is possible without actually inverting a matrix but by simple backward insertion of the solution starting at the inflow boundary. Instead of describing the actual algorithm, which is based on topological sorting from the field of graph theory [5], we will visualize the result of the renumbering. After applying the according permutation matrices to the 8 transport parts situated on the block-diagonal of the system matrix, we see a change of the original allocation. The entries that belong to the finite difference discretization of the differential operator are permutated into a lower triangular allocation in the matrix (Fig. 5). Beside each of these diagonal blocks there appear 8 offdiagonal blocks containing the collision terms of the LBE. For collision dominated configurations it makes sense to use the collision operators as preconditioning. Thus we can bring the offdiagonal entries onto the diagonal, which then consist of a 9x9 block for each node (see Fig. 6). A block-Jacobian preconditioner can be constructed by simply calculating the inverse of each 9x9 system. Unfortunately, this approach does not sustain the lower triangular form of the transport matrices.

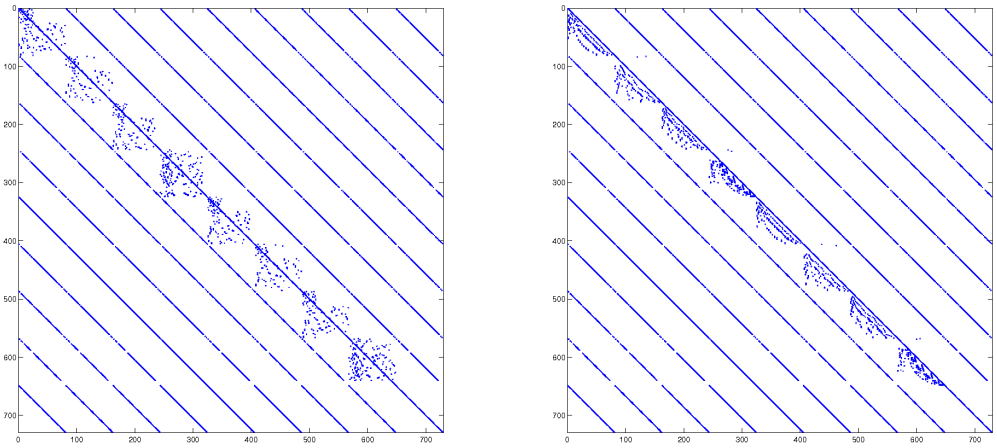


Figure 5: Matrix allocation: ordering by space index for each direction (left) vs. effect of renumbering algorithm in a symbolic representation (right)

We verify our preconditioners with the following numerical results. For this purpose we introduce Testcase 2, which is the solution of equation (2) choosing the implicit Euler scheme as time discretization (with $\Delta t = 1$) and treating the boundary explicitly as in equation (7). As flow problem we choose the Driven Cavity configuration at various Reynolds numbers. We provide the number of iterations to gain 6 digits using a GMRES solver, as it delivers safe results due to its monotone convergence behaviour. For $c = 1$ the transport preconditioner (tr-pre) gives good results on all levels, showing the independence of h for transport dominated problems due to

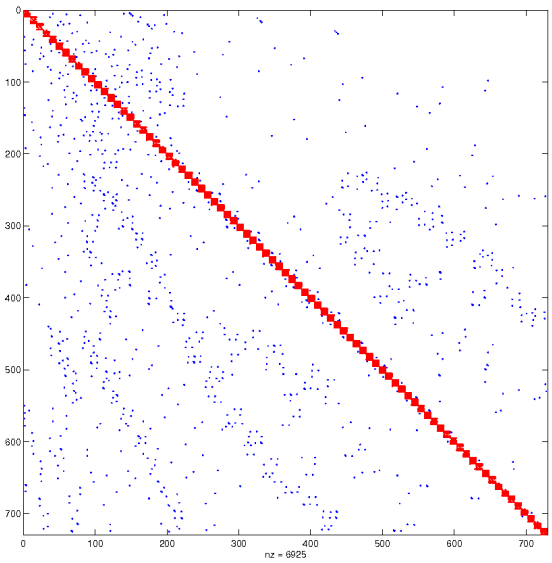


Figure 6: Matrix allocation: ordering by directional index for each space node results in 9x9 collision blocks on the diagonal

		plain	bl-jac	tr-pre
c=1	N=81	62	25	**45**
	N=289	101	51	**52**
	N=1089	190	110	**62**
	N=4225	363	196	**58**
c=10	N=81	91	26	91
	N=289	181	65	157
	N=1089	487	181	304
	N=4225	875	412	440
c=100	N=81	147	**35**	174
	N=289	299	**75**	339
	N=1089	675	**195**	720
	N=4225	1859	**511**	1834
c=1000	N=81	152	**35**	203
	N=289	324	**81**	423
	N=1089	734	**203**	946
	N=4225	2110	**507**	2555

Table 3: Testcase 2: No. of iterations to gain 6 digits, using GMRES linear solver

the almost exact preconditioner. With increasing c the iteration numbers of the block-Jacobian preconditioner (bl-jac) become stable, but strongly depend on the number of unknowns. It is obvious that a combination of both preconditioners is desired, but only possible by a special technique described in the following chapter.

4 Generalized equilibrium formulation (GEF) of the LBE

In [5],[13] the concept of the *generalized mean intensity* has been successfully combined with a direct transport solver to treat radiative transfer problems. The idea arose to convey this technique from one integro-differential equation to another, i.e. to the LBE. In this case we cannot improve the storage cost like in the GMI due to the varying coefficients. What we can do, though, is to combine the advantages of an efficient transport solver on the one hand with special preconditioning to deal with stiff problems on the other hand.

We illustrate the procedure by reformulating the implicit Euler approach for the LBE; nonetheless the other time discretizations and even the stationary approach can be modified, too. We start from the time discretized LBE, with $\mathbf{T}_k f_k \sim f_k + \Delta t\ e_k \cdot \nabla f_k + \frac{\Delta t}{\tau} f_k$. Then the equation

$$\mathbf{T}_k f_k^{n+1} = \frac{\Delta t}{\tau} f_k^{eq,n+1} + f_k^n \quad k = 0, \ldots, 8 \tag{8}$$

can be rewritten as

$$f_k^{n+1} = \mathbf{T}_k^{-1}(\frac{\Delta t}{\tau} f_k^{eq,n+1} + f_k^n) \quad k = 0, \ldots, 8 \tag{9}$$

and for each $i \in \{0, 8\}$ we multiply with the appropriate weights $\tilde{\omega}_{ik}$ from equation (5), which yields:

$$\tilde{\omega}_{ik} f_k^{n+1} = \tilde{\omega}_{ik} \mathbf{T}_k^{-1}(\frac{\Delta t}{\tau} f_k^{eq,n+1} + f_k^n) \quad k = 0, \ldots, 8$$

Summing up over k finally gives us for each equilibrium term $f_i^{eq,n+1}$:

$$f_i^{eq,n+1} = \sum_k \tilde{\omega}_{ik} f_k^{n+1} = \sum_k \tilde{\omega}_{ik} \mathbf{T}_k^{-1}(\frac{\Delta t}{\tau} f_k^{eq,n+1} + f_k^n) \quad i = 0, \ldots, 8$$

This means that in order to solve equation (8), we first solve for the terms f_i^{eq} and afterwards we obtain the f_i from a simple post-processing step according to equation (9). Thus, we have obtained the following linear system in matrix form:

$$\begin{bmatrix} Id - \tilde{\omega}_{11}\frac{\Delta t}{\tau}\mathbf{T}_1^{-1} & Id - \tilde{\omega}_{12}\frac{\Delta t}{\tau}\mathbf{T}_2^{-1} & \cdots & Id - \tilde{\omega}_{10}\frac{\Delta t}{\tau}\mathbf{T}_0^{-1} \\ Id - \tilde{\omega}_{21}\frac{\Delta t}{\tau}\mathbf{T}_1^{-1} & Id - \tilde{\omega}_{22}\frac{\Delta t}{\tau}\mathbf{T}_2^{-1} & & \vdots \\ \vdots & & \ddots & \vdots \\ Id - \tilde{\omega}_{01}\frac{\Delta t}{\tau}\mathbf{T}_1^{-1} & \cdots & \cdots & Id - \tilde{\omega}_{00}\frac{\Delta t}{\tau}\mathbf{T}_0^{-1} \end{bmatrix} \begin{bmatrix} f_1^{eq} \\ f_2^{eq} \\ \vdots \\ f_0^{eq} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_k \tilde{\omega}_{1k} \mathbf{T}_k^{-1} f_k^n \\ \sum_k \tilde{\omega}_{2k} \mathbf{T}_k^{-1} f_k^n \\ \vdots \\ \sum_k \tilde{\omega}_{0k} \mathbf{T}_k^{-1} f_k^n \end{bmatrix}$$

So, the resulting system matrix contains the transport steps in an inverse manner, which means that we obtain an *implicit* matrix only as we do not calculate the actual inverses, however we can apply the matrix to a vector. Due to our special discretization and the lower triangular form of the transport matrices, the whole procedure is very cost efficient. For the same reason we can apply additional preconditioning to the system, by collecting the entries in the 9x9 diagonals, which are known to us due to $diag(T_i^{-1}) = diag(T_i)^{-1}$ with T_i lower triagonal. These diagonal entries are weighted with the $\tilde{\omega}(i, k, \tau, c, u)$, so we expect a stabilizing effect in the case of large c, while the plain GEF should perform well for transport dominated configurations due to its construction. We complement the previous results and show how our GEF performs for Testcase 2 (see Table 4). For $c = 1$ the GEF still yields us very good convergence on all levels. With our special preconditioning we gain additionally robustness against high values of c. This means that our preconditioned GEF combines two positive effects, which is only possible due to our special discretization technique. Nevertheless it remains the dependence on h for moderate and large values of c. That is why we decided to use the multigrid method to overcome this last drawback.

		plain	bl-jac	tr-pre	GEF	GEF(diag)
c=1	N=81	62	25	45	32	19
	N=289	101	51	52	42	29
	N=1089	190	110	62	55	42
	N=4225	363	196	58	70	56
c=10	N=81	91	26	91	65	25
	N=289	181	65	157	120	60
	N=1089	487	181	304	246	156
	N=4225	875	412	440	387	297
c=100	N=81	147	35	174	110	35
	N=289	299	75	339	233	84
	N=1089	675	195	720	497	203
	N=4225	1859	511	1834	1219	522
c=1000	N=81	152	35	203	111	34
	N=289	324	81	423	243	83
	N=1089	734	203	946	536	215
	N=4225	2110	507	2555	1629	515

Table 4: Same as Tab. 3: Comparison of direct solution for f_i (left) against GEF (right)

5 Multigrid solver for LBE

We showed that the GEF solver performs well for configurations with dominating differential operator, independent of the number of unknowns. Additionally it became robust against large

values of c with the block-Jacobian preconditioning, but with level-dependent convergence rates for the relevant, intermediate c-range. That is why we implemented a prototypical version of the multigrid method, and show the results for the second testcase in comparison to Table 4. For this purpose we have to implement grid-transfer operators: We apply linear and quadratic prolongation P for the 1st and 2nd order discretization, respectively. However, so far we use only constant restriction $R = Id$, that is what still needs further optimization. As smoothing we use the simple Richardson scheme in Table 5 and GMRES in the following tables. The results show throughout independence of the refinement level (even some impovement for finer levels), nevertheless the transport preconditioning and plain GEF encounter significant problems for the case $c = 100$, as was expected. The block-Jacobian variants yield significant stabilization here. The pure block-Jacobian smoother without any transport preconditioning gives the worst rates, which underlines the importance of our special discretization technique. If we concentrate on the approximate area of c_{opt}, i.e. $c \sim 10$, we see that even without this stabilization the rates are quite good, and improve when more smoothing steps s are applied. However, increasing of s does not improve the rates as much as expected, which means some slight aberration from common multigrid theory which might be due to the non-optimal restriction. Also, we are not completely satisfied with the values for the 2nd order upwinding, but expect significant improvement after implementing a quadratic restriction operator.

	single grid		mg s=4		mg s=8		mg s=16	
upw 1	GEF	GEF(diag)	GEF	GEF(diag)	GEF	GEF(diag)	GEF	GEF(diag)
1089	0.9983	0.9842	0.84	0.39	0.73	0.35	0.56	0.30
4225	0.9988	0.9938	0.73	0.36	0.56	0.33	0.38	0.31
16641	0.9991	0.9971	0.57	0.30	0.36	0.30	0.31	0.28
66049	0.9992	0.9984	0.40	0.23	0.23	0.22	0.23	0.22
	single grid		mg s=4		mg s=8		mg s=16	
upw 2	GEF	GEF(diag)	GEF	GEF(diag)	GEF	GEF(diag)	GEF	GEF(diag)
1089	0.9992	-	-	-	-	-	-	-
4225	0.9993	0.9961	-	-	0.88	-	0.50	0.57
16641	0.9994	0.9978	-	0.45	0.40	0.52	0.36	0.36
66049	0.9994	0.9984	0.59	0.34	0.29	0.28	0.21	0.21

Table 5: Testcase 2: Comparison of convergence rates using Richardson smoother, c=10

6 Conclusion and outlook

We have introduced a new approach for the LBE by applying highly sophisticated techniques from Numerics for PDE's: The implicit time discretization on general, unstructured meshes is supposed to overcome limitations of standard LB methods, which are already thoroughly

	single grid		mg s=4		mg s=8		mg s=16	
upw 1	GEF	GEF(diag)	GEF	GEF(diag)	GEF	GEF(diag)	GEF	GEF(diag)
1089	0.9814	0.9701	0.43	0.36	0.33	0.32	0.27	0.24
4225	0.9910	0.9887	0.36	0.33	0.28	0.29	0.27	0.27
16641	0.9949	0.9947	0.30	0.30	0.24	0.24	0.24	0.24
66049	0.9965	0.9969	0.23	0.24	0.21	0.22	0.19	0.20
	single grid		mg s=4		mg s=8		mg s=16	
upw 2	GEF	GEF(diag)	GEF	GEF(diag)	GEF	GEF(diag)	GEF	GEF(diag)
1089	0.9933	0.9881	-	0.63	0.53	0.61	0.49	0.54
4225	0.9960	0.9946	0.73	0.58	0.44	0.55	0.43	0.51
16641	0.9971	0.9969	0.38	0.47	0.35	0.45	0.30	0.40
66049	0.9976	0.9979	0.22	0.32	0.20	0.28	0.18	0.20

Table 6: Testcase 2: Comparison of convergence rates using GMRES smoother, c=10

		c=10				c=100			
		tr-pre	bl-jac	GEF	GEF(diag)	tr-pre	bl-jac	GEF	GEF(diag)
s=4	289	0.53	0.40	0.43	0.36	0.73	0.46	0.75	0.44
	1089	0.42	0.42	0.36	0.33	0.68	0.51	0.65	0.44
	4225	0.32	0.42	0.30	0.30	0.65	0.51	0.52	0.44
	66049	0.27	0.44	0.23	0.24	0.57	0.48	0.47	0.42
s=8	289	0.35	0.33	0.33	0.32	0.57	0.40	0.52	0.40
	1089	0.31	0.37	0.28	0.29	0.54	0.45	0.46	0.42
	4225	0.29	0.36	0.24	0.24	0.47	0.47	0.44	0.42
	66049	0.24	0.31	0.21	0.22	0.44	0.44	0.41	0.41
s=16	289	0.27	0.28	0.27	0.24	0.39	0.36	0.35	0.35
	1089	0.29	0.33	0.27	0.27	0.44	0.43	0.38	0.39
	4225	0.26	0.32	0.24	0.24	0.42	0.46	0.38	0.36
	66049	0.21	0.25	0.19	0.20	0.42	0.45	0.40	0.39

Table 7: Testcase 2: Comparison of convergence rates with different smoothers, upw 1

exploited. We have shown that it is possible to treat the LBE in a more general way as an integro-differential equation, and thus to carry over solution techniques that have been successful in the field of radiative transfer problems. A new GEF reformulation of the LBE was derived which can combine the efficiency of direct transport solvers on the one hand with special preconditioning for collision dominated cases on the other hand. The range of problem parameters was numerically analyzed to get a rule of thumb for an appropriate approximation to c_{opt} for various flow problems at different Reynolds numbers so that optimal approximation results can be achieved. Finally the performance was greatly improved using a multigrid algorithm and the achieved convergence

		c=10				c=100			
		tr-pre	bl-jac	GEF	GEF(diag)	tr-pre	bl-jac	GEF	GEF(diag)
s=4	289	-	0.71	-	0.63	-	0.83	-	0.81
	1089	0.88	0.89	0.73	0.58	-	0.84	-	0.80
	4225	0.57	-	0.38	0.47	-	0.84	-	0.77
	66049	0.27	-	0.22	0.32	-		-	0.71
s=8	289	0.58	0.62	0.53	0.61	-	0.79	-	0.79
	1089	0.48	0.59	0.44	0.55	-	0.81	-	0.79
	4225	0.31	0.51	0.35	0.45	-	0.80	0.87	0.76
	66049	0.19	0.37	0.20	0.28	-	0.80	0.62	0.74
s=16	289	0.49	0.56	0.49	0.54	0.76	0.76	0.73	0.75
	1089	0.42	0.53	0.43	0.51	0.76	0.78	0.71	0.75
	4225	0.28	0.43	0.30	0.40	0.75	0.77	0.67	0.75
	66049	0.15	0.37	0.18	0.20	0.63	0.76	0.60	0.70

Table 8: Testcase 2: Comparison of convergence rates with different smoothers, upw 2

rates are independent of the problem size. The ongoing research is concerned with an analysis of the MRT model and adjusting the solvers and preconditioners for the modified collision operator. We expect to get further understanding of the quality of our approximated solution, depending on the (variable) relaxation rates and choice of parameter c, and its interplay with corresponding preconditioning stategies which will be part of a forthcoming paper.

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