# PDE-constrained optimization

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# Overview

- **PDE-constrained optimization**
- **Distributed Control**
- **Boundary Control**
- **Fréchet  Derivative**
- **Lagrange Functional**
- **Connection Between L and J**

# Typical form of PDE-constrained optimization problems

$$\min_{y\in Y,\, u\in U} \quad J(y,u)$$

$$\text{subject to } c(y,u)=0,\ u\in U,\ y\in Y$$

where,

| | |
|---|---|
| $U$ | control space, |
| $Y$ | state space, |
| $\mathrm{J}:\mathrm{Y}\times\mathrm{U}\rightarrow R$ | objective function, |
| $\mathrm{c}:\mathrm{Y}\times\mathrm{U}\rightarrow Z,\ c(y,u)=0,$ | state equation. |

# Distributed Control

Optimal stationary heat source: Heating of a body $\Omega$ by a controlled heat source $u$ (say electromagnetic induction or microwaves) to reach the target $y_d$.

$$\min J(y,u) := \frac{1}{2}\left\| y - y_d \right\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\left\| u \right\|^2_{L^2(\Omega)}$$

subject to the state equation $(state\ y)$

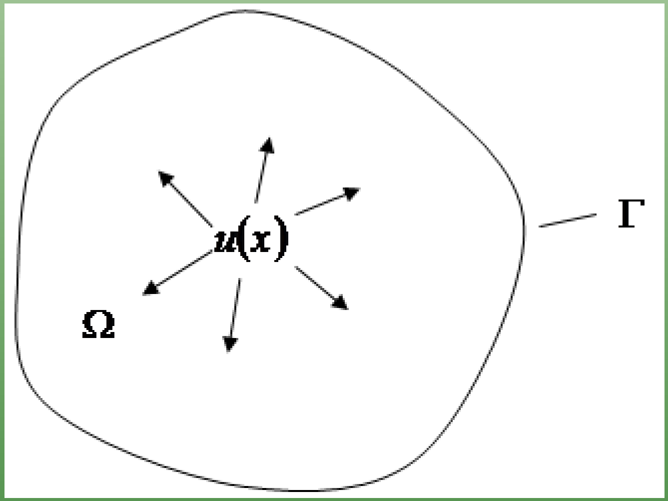
$$-\Delta y = u \quad \text{in } \Omega$$

$$y = 0 \quad \text{on } \Gamma$$

and control constraint $(control\ u)$

$$u_a(x) \le u(x) \le u_b(x) \quad \text{in } \Omega$$

# Distributed Control




*Controlled heat source*

# Boundary Control

Optimal stationary boundary temperature: Heating of a body $\Omega$ by a controlled boundary temperature $u$ to reach a target temperature $y_d$.

$$\min J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma)}$$

Subject to the state equation $(state\ y)$

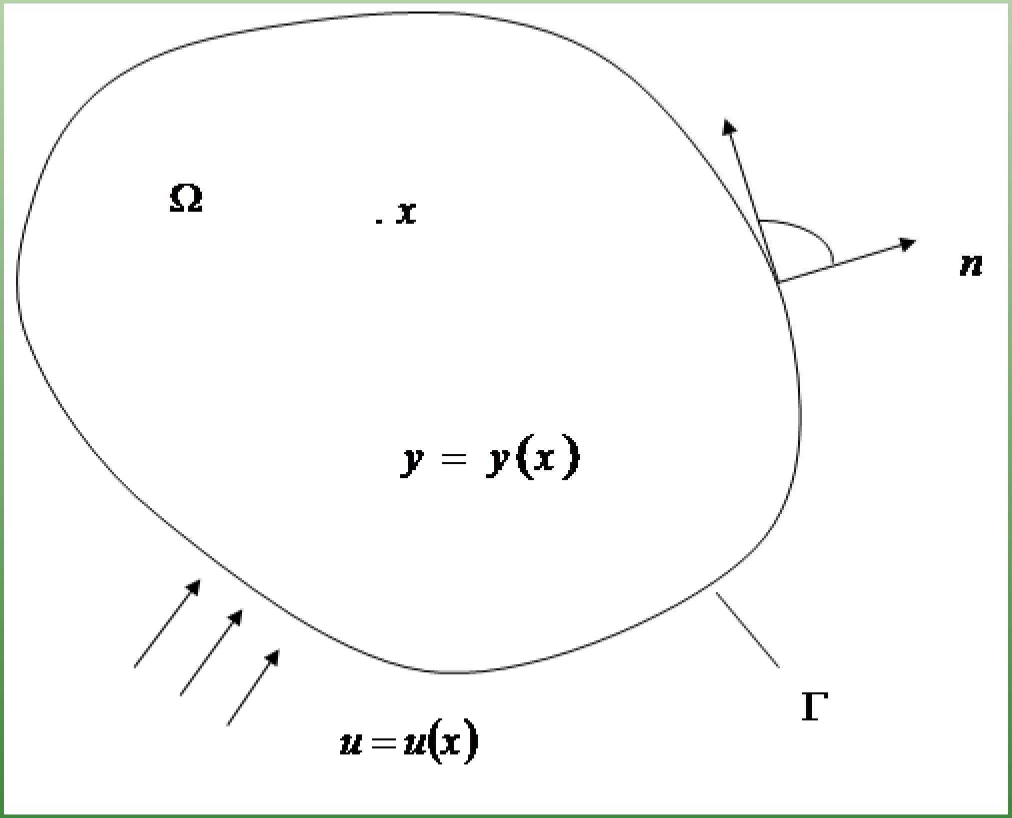
$$-\Delta y = 0 \qquad \text{in } \Omega$$

$$\frac{\partial y}{\partial n} = \sigma(u - y) \qquad \text{on } \Gamma$$

and control constraint $(control\ u)$

$$u_a(x) \le u(x) \le u_b(x) \quad \text{on } \Gamma$$

# Boundary Control




*Boundary control*

# Fréchet Derivative

An operator $F:U \rightarrow V$ is said to be differentiable at $u \in U,$ if there exit a linear and continuous operator $A:U \rightarrow V$ such that for all $h \in U,$

$$F(u+h) = F(u) + Ah + r(h)$$

and the remainder term $r$ satisfies

$$\frac{\|r(h)\|V}{\|h\|U} \rightarrow 0 \quad as \quad \|h\|U \rightarrow 0$$

$A$ is the Fréchet derivative of $F$ at $u,$ $A = F'(u).$

# Example

Let $U$ and $V$ be real Hilbert Spaces, $F : U \to V$ be a mapping defined by

$$F(u) := Au + b,$$ where $A$ is linear operator and $b \in V$

is Fréchet differentiable at $u$ and

$$F'(u)h := Ah$$

# Lagrange Functional

We have the distributed control problem

$$\min J(y,u) := \frac{1}{2}\left\| y - y_d \right\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\left\| u \right\|^2_{L^2(\Omega)}$$

subject to the state equation $(state\ y)$

$$\begin{aligned} -\Delta y &= u && \text{in } \Omega \\ y &= 0 && \text{on } \Gamma \end{aligned}$$

where $y \in Y$ (state space), $u \in U$ (control space).

# Continue

For the distributed control problem without control constraint,

$$\min_{y \in Y,\, u \in U} \quad J(y,u)$$

such that $\quad -\Delta y = u \quad \Leftrightarrow \quad u + \Delta y = 0$

and $\quad y = 0, \quad \text{on } \Gamma$

We define the Lagrange functional

$$L(y,u,p) := J(y,u) + \langle p, u + \Delta y \rangle$$

# Continue

Under some conditions:

$$DL = 0 \quad \Rightarrow \quad DJ = 0 \quad \text{and} \quad -\Delta y = u$$

so we define the optimality conditions (KKT-system) as:

$$DL = 0 \quad \Rightarrow \begin{cases} DL_y = 0, & (\text{adjoint equation}) \\ DL_u = 0, & (\text{gradient equation}) \\ DL_p = 0, & (\text{primal equation}) \end{cases}$$

# Lagrange Functional for Distributed Control problem

Langrange Functional for distributed control problems

$$L(y,u,p) := \frac{1}{2}\|y - z\|^2 + \frac{\alpha}{2}\|u\|^2 + \langle p, u + \Delta y \rangle$$

For $h = (h_1, h_2, h_3) \in \left( L^2(\Omega) \right)^3,$ we have the Fréchet derivative

$$DL(y,u,p)h := \left( \left\langle y - z + \Delta p, h_1 \right\rangle, \left\langle \alpha u + p, h_2 \right\rangle, \left\langle \Delta y + u, h_3 \right\rangle \right)$$

For finding critical points, we set

$$DL(y,u,p)h = 0, \qquad \forall h \in \left( L^2(\Omega) \right)^3$$

# Continue

$$\langle y - z + \Delta p, h_1 \rangle = 0 \qquad\qquad y - z + \Delta p = 0$$

$$\langle \alpha u + p, h_2 \rangle = 0 \qquad \Rightarrow \qquad \alpha u + p = 0$$

$$\langle \Delta y + u, h_3 \rangle = 0 \qquad\qquad \Delta y + u = 0$$

$$\Rightarrow \quad -\Delta p = y - z \qquad \rightarrow (\text{adjoint equation})$$

$$u = -\frac{1}{\alpha} p \qquad \rightarrow (\text{gradient equation})$$

$$-\Delta y = u \qquad \rightarrow (\text{primal equation})$$

# Continue

Elimination of $u$ leads to:

$$-\Delta y + \frac{1}{\alpha} p = 0$$

$$y - \Delta p = z$$

$$\Rightarrow \begin{pmatrix} -\Delta & \frac{1}{\alpha} \\ 1 & -\Delta \end{pmatrix} \begin{pmatrix} y \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ z \end{pmatrix}$$

$$y = 0, \quad p = 0 \quad \text{on } \Gamma$$

$\Rightarrow (y,\ p)$ is the critical point of $L$ and thus a candidate for min of $J$.

Thank You