Adaptive Space-Time Finite Element Approximations of Parabolic Optimal Control Problems

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- Existence and Uniqueness

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- Optimality Conditions

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- Numerical results

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References

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Part 1: Motivation Example

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Problem:

$$\inf_{y,u} J(y,u) = \frac{1}{2}||y - y^d||^2_{L^2(Q)} + \frac{\alpha}{2}||u - u^d||^2_{L^2(Q)} \qquad \text{(P)}$$

subject to state equation

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subject to state equation

$$y_t - \Delta y = u \quad \text{in } Q := \Omega \times (0,T)$$
$$y = 0 \,, \quad \text{on } \Sigma_{lat} := \Gamma \times (0,T)$$
$$y(.,0) = y^0 \,, \quad \text{on } \Sigma_{bot} := \Omega \times \{0\} \,.$$

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Notation and data

- J is the objective functional and $u^d \in L^2(Q)$ is control shift
- $\alpha > 0$ is regularization parameter

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Part 2: Existence and Uniqueness of optimal control problem

- Defining the control- to- state map $G : u \to y$ from $L^2(Q)$ to $W(0,T)$, a linear and bounded operator which assigns to a control $u \in L^2(Q)$ the unique solution $y = y(u)$ of the state equation,writing the state $y = Gu$
- introduce the reduced objective functional

$$\inf_{y,u} J_{red}(u) := \frac{1}{2}||Gu - y^d||^2_{L^2(Q)} + \frac{\alpha}{2}||u - u^d||^2_{L^2(Q)}$$

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Theorem

The reduced optimal control problem has a unique optimal solution $(y, u) \in W(0,T) \times L^2(Q)$.

Proof

Direct method of the calculus of variations(minimizing sequence) .

Part 3: Optimality Conditions

Theorem

Let $(y, u) \in W(0,T) \times L^2(Q)$ *be the optimal solution of the problem* $(\mathcal{P})$ *. Then,there exists an adjoint state* $p \in W(0,T)$ *as a weak solution of*

$$-p_t - \Delta p = y - y^d \quad \text{in} \;\; Q := \Omega \times (0,T)$$

$$p = 0 \qquad \text{on} \;\; \Sigma_{lat} := \Gamma \times (0,T)$$

$$p = 0 \qquad \text{on} \; \Sigma_{top} = \Omega \times T$$

and the gradient equation

$$p + \alpha(u - u^d) = 0 \;\; \text{in} \; Q$$

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We introduce the Lagrangian

$$L(y,u,p,q) := J(y,u) - \int_0^T \langle y_t - \Delta y - u, p\rangle_{H^{-1}\times H_0^1}\, dt + (y - y^0, q)_{0,\Omega}$$

Critical Points of the Lagrangian are characterized by

$$L_y(y,u,p,q) = 0,$$

$$L_u(y,u,p,q) = 0,$$

$$L_p(y,u,p,q) = 0,$$

$$L_q(y,u,p,q) = 0.$$

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$$\int_0^T \langle y_t, p \rangle_{H^{-1}\times H_0^1}\, dt = (y,p)|_0^T - \int_0^T \langle p_t, y \rangle_{H^{-1}\times H_0^1}\, dt$$

$$= (y|_{\Sigma_{top}}, p|_{\Sigma_{top}})_{0,\Omega} - (y|_{\Sigma_{bot}}, p|_{\Sigma_{bot}})_{0,\Omega} - \int_0^T \langle p_t, y \rangle_{H^{-1}\times H_0^1}\, dt$$

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Observing $y|_{\Sigma_{lat}} = 0$, applying Green's formula twice, we find

$$-\int_0^T \langle \Delta y, p\rangle_{H^{-1}\times H_0^1}\, dt = -\int_0^T \langle \Delta p, y\rangle_{H^{-1}\times H_0^1}\, dt + \int_0^T \langle n_\Gamma \cdot \nabla y, p\rangle\, dt$$

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and hence,

$$L(y,u,p,q) = J(y,u) + \int_0^T \langle -p_t - \Delta p, y\rangle \, dt + \int_0^T \langle n_\Gamma \cdot \nabla y, p\rangle \, dt$$

$$+(p|_{\Sigma_{top}}, y|_{\Sigma_{top}})_{0,\Omega} - (p|_{\Sigma_{bot}} - q, y|_{\Sigma_{bot}})_{0,\Omega} - (q, y^0)_{0,\Omega}$$

By taking $J_y = y - y^d$ and $J_u = \alpha(u - u^d)$ into account. this complet the proof.

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Part 4: Optimality system as a fourth order elliptic equation

Theorem

Assume that the state y and the djoint state p are sufficiently smooth. Then,the optimality sytem for problem $(\mathcal{P})$ is equivalent to the fourth order elliptic boundary value problem

$$
\begin{aligned}
-y_{tt} + \Delta^2 y + \alpha^{-1} y &= f \quad \text{in } Q \\
y &= 0\,, \quad \text{on } \Sigma_{lat} \\
y_t - \Delta y &= 0 \quad \text{on } \Sigma_{lat} \\
y &= y^0\,, \quad \text{on } \Sigma_{bot}\,. \\
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where the right hande side f is given by

$$f := \alpha^{-1} y^d - \Delta u^d - u_t^d.$$

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where the right hande side f is given by

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f := \alpha^{-1} y^d - \Delta u^d - u^d_t.
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By the gradient equation, we eliminat the control in the state equation, yields

$$y_t - \Delta y = -\alpha^{-1} p + u^d$$

Differentiating w.r.t. time t results in

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On the other hand, the adjoint equation gives

$$-\alpha^{-1} p_t = \alpha^{-1} \Delta p + \alpha^{-1}(y - y^d).$$

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$$\alpha^{-1}\Delta p = -\Delta y_t + \Delta^2 y + \Delta u^d,$$

Thus, by inserting

$$y_{tt} - \Delta y_t = \alpha^{-1}\Delta p + \alpha^{-1}(y - y^d) + u_t^d.$$

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which results in

$$\alpha^{-1}\Delta p = -\Delta y_t + \Delta^2 y + \Delta u^d,$$

$$-y_{tt} + \Delta^2 y + \alpha^{-1} y = f,$$

where the right-hand side f is given by

$$f := \alpha^{-1} y^d - \Delta u^d - u_t^d.$$

Observing supp $u^d \subset Q$. from the boundary condintion in the optimality system and the gradient equation,we deduce $u|_{\Sigma_{lat}} = 0$ and $u|_{\Sigma_{top}} = 0$.

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Setting $\omega := -\Delta y$, the fourth order boundary value problems reads as follows

$$
\begin{aligned}
-y_{tt} - \Delta w + \alpha^{-1} y = f, &\quad \text{in } Q,\\
\omega + \Delta y = 0 &\quad \text{in } Q,\\
y = y^0 \,, &\quad \text{on } \Sigma_{bot}\,,\\
y = 0 \,, &\quad \text{on } \Sigma_{lat}\\
y_t + \omega = 0 &\quad \text{on } \Sigma\\
\Delta y + \omega = 0 &\quad \text{on } \Sigma
\end{aligned}
$$

Setting

$$W := L^2((0,T);H^1(\Omega)) \cap C([0,T];L^2(\Omega)),$$

$$Y := \{y \in H^1(Q) \cap C([0,T];L^2(\Omega) \mid y|_{\Sigma_{bot}} = y^0 \,,\; y|_{\Sigma_{lat}} = 0\},$$

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The weak formulation of the system amounts to the compution of $(w,y) \in W \times Y$, such that for all $(v,z) \in W \times Y_0$ there holds

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The weak formulation of the system amounts to the compution of $(w,y) \in W \times Y$, such that for all $(v,z) \in W \times Y_0$ there holds

$$a_{11}(w,v_1) + a_{12}(y,v_1) = \ell_1(v_1),$$
$$-a_{21}(w,v_2) + a_{22}(y,v_2) = \ell_2(v_2).$$

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Here, the bilinear forms $a_{ij}(\cdot,\cdot), 1 \leq i,j \leq 2,$ are given by

$$a_{11}(w,v_1) := (\nabla w, \nabla v_1)_{L^2(Q)} + (w(\cdot,T), v_1(\cdot,T))_{L^2(\Omega)},$$

$$a_{12}(y,v_1) := (y_t, (v_1)_t)_{L^2(Q)} + \alpha^{-1}(y,v_1)_{L^2(Q)},$$

$$a_{21}(w,v_2) := (w,v_2)_{L^2(Q)},$$

$$a_{22}(y,v_2) := (\nabla y, \nabla v_2)_{L^2(Q)} - \int_0^T \langle \mathbf{n} \cdot \nabla y, v_2 \rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\, dt,$$

the functionals $\ell_i(\cdot), 1 \leq i \leq 2,$ are given by

$$\ell_1(v) := \ (f, v)_{0,Q},$$

$$\ell_2(z) := \ 0.$$

The operator theoretic formulation reads

$$\mathcal{L}(w, y) = (\ell_1, \ell_2)^T,$$

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The operator theoretic formulation reads

$$\mathcal{L}(w, y) = (\ell_1, \ell_2)^T,$$

where the operator $\mathcal{L} : W \times Y \to W^* \times Y^*$ is given by

$$\langle \mathcal{L}(w, y), (v, z) \rangle := a_{11}(w, v) + a_{21}(y, v) - a_{21}(w, z) + a_{22}(y, z)$$

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Theorem

The operator $\mathcal{L}$ is a continuous, bijective linear operator. Hence, for any $(\ell_1, \ell_2) \in W^* \times Y^*$, the system admits a unique solution $(y, w) \in W \times Y$. The solution depends on the data according to

$$\|(w, y)\|_{W \times Y} \lesssim \|(\ell_1, \ell_2)\|_{W^* \times Y^*}$$

Proof

Inf-Sup condition .

Corollary

Let $(y_h, w_h) \in Y_h \times W_h, Y_h \subset Y, W_h \subset W,$ *be an approximate solution of* $(y, w) \in Y \times W$. *Then, there holds*

$$\|(y - y_h, w - w_h)\|_{Y \times W} \lesssim \|(Res_1, Res_2)\|_{Y^* \times W^*},$$

Corollary

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$$\|(y - y_h, w - w_h)\|_{Y \times W} \lesssim \|(Res_1, Res_2)\|_{Y^* \times W^*},$$

where the residuals $Res_1 \in V^*, Res_2 \in W^*$ *are given by*

$$Res_1(v) := \ell_1(v) - ((y_h)_t, v_t)_{0,Q} - a(w_h, v)$$

$$-\alpha^{-1}\,(y_h, v)_{0,Q} - (w_h(\cdot, T), v)_{0,\Omega},\ v \in Y,$$

$$Res_2(z) := \ell_2(z) - a(y_h, z) + (w_h, z)_{0,Q}.\ z \in W.$$

Proof

The assertion is an immediate consequence of the previous

Corollary

Let $(y_h, w_h) \in Y_h \times W_h, Y_h \subset Y, W_h \subset W,$ *be an approximate solution of* $(y, w) \in Y \times W$. *Then, there holds*

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$$Res_2(z) := \ell_2(z) - a(y_h, z) + (w_h, z)_{0,Q}.\; z \in W.$$

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The assertion is an immediate consequence of the previous

Part 4: Space-time finite element discretization

Basic Definitions and Notations

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- $\mathcal{T}_h(Q)$: Shape regular simplical triangulation of the space-time domain Q.
- $\mathcal{N}_h(D)$: Set of vertices $a_D^{(i)}, 1 \leq i \leq \text{card}\, \mathcal{N}_h(D)$

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- $h_K :=$ diam$(K), K \in \mathcal{T}_h(Q)$, $h_F :=$ diam$(F), F \in \mathcal{F}_h(\bar{Q})$

Basic Definitions and Notations

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- $\mathcal{T}_h(Q)$: Shape regular simplical triangulation of the space-time domain Q.
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N_{Q_p} &:= \text{card } \mathcal{N}_h(Q_p) \\
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N_{\Sigma_P} &:= N_{\Sigma_{bot}} + N_{\Sigma_{lat}}.
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We use P1 conforming finite elements with respect to the triangulation $\mathcal{T}_h(Q)$. Denoting by $\varphi_{Q_P}^{(i)}, 1 \leq i \leq N_{Q_P}$, and by $\varphi_{\Sigma_{bot}}^{(i)}, 1 \leq i \leq N_{\Sigma_{bot}}$, as well as $\varphi_{\Sigma_P}^{(i)}, 1 \leq i \leq N_{\Sigma_P}$, the nodal basis functions associated with the nodal points in $\mathcal{N}_h(Q_P)$ and $\mathcal{N}_h(\Sigma_{bot}), \mathcal{N}_h(\Sigma_P)$, respectively, we introduce the finite element spaces

$$\begin{aligned}
Y_{h,\Sigma_{bot}} &:= \text{span}(\varphi_{\Sigma_{bot}}^{(1)}, \cdots, \varphi_{\Sigma_{bot}}^{(N_{\Sigma_{bot}})}), \\
Y_{h,0} &:= \text{span}(\varphi_{Q_P}^{(1)}, \cdots, \varphi_{Q_P}^{(N_{Q_P})}), \\
W_h &:= Y_{h,0} \oplus W_h^{\Sigma_P}, \quad W_h^{\Sigma_P} := \text{span}(\varphi_{\Sigma_P}^{(1)}, \cdots, \varphi_{\Sigma_P}^{(N_{\Sigma_P})}), \\
Y_h &:= \{y_h \in W_h \mid y_h|_{\Sigma_{bot}} = y_{h,0},\ y_h|_{\Sigma_{lat}} = 0\},
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The space-time finite element approximation of the solution $(w, y) \in W \times Y$ amounts to the computation of $(w_h, y_h) \in W_h \times Y_h$ such that $v_{h,1} \in Y_{h,0}$ and $v_{h,2} \in W_h$ there holds

$$a_{11}(w_h, v_{h,1}) + a_{12}(y_h, v_{h,1}) = \ell_1(v_{h,1}),$$
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Part 5: Residual-type a posteriori error estimation

Theorem

Let $(w, y) \in W \times Y$ *and* $(w_h, y_h) \in W_h \times Y_h$ *be the solution of the continuous and the space-time finite element approximation ,respectively. Then there holds*

$$||(w - w_h, y - y_h)||_{W \times Y} \lesssim \eta_h$$

where the estimator η_h *is given by*

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$$\eta_h := \Big(\sum_{K \in \mathcal{T}_h(Q)} (\eta_{K,1}^2 + \eta_{K,2}^2) + \sum_{F \in \mathcal{F}_h(Q)} (\eta_{F,1}^2 + \eta_{F,2}^2) + \sum_{F \in \mathcal{F}_h(\Sigma_{top})} \eta_{F,3}^2 \Big)^{1/2}$$

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In particular, the element residuals $\eta_{K,i}, 1 \leq i \leq 2$, are given by

$$\eta_{K,1} := h_K \|f - \alpha^{-1} y_h\|_{L^2(K)}, \quad K \in \mathcal{T}_h(Q),$$
$$\eta_{K,2} := h_K \|w_h\|_{L^2(K)}, \quad K \in \mathcal{T}_h(Q).$$

The face residuals $\eta_{F,i}, 1 \leq i \leq 3$, read as follows

$$\eta_{F,1} := h_F^{1/2} \; \|\mathbf{n}_F \cdot [\nabla w_h]_F\|_{L^2(F)}, \quad F \in \mathcal{F}_h(Q),$$
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Step MARK of Adaptive Cycle: Bulk Criterion

.

Given a universal constant $0 < \theta < 1$, we determine a set of elements $\mathcal{M}_K$ and a set of faces $\mathcal{M}_F$ such that

$$\theta\ \eta_h^2 \leq \sum_{K \in \mathcal{M}_K} (\eta_{K,1}^2 + \eta_{K,2}^2) + \sum_{F \in \mathcal{M}_F} (\eta_{F,1}^2 + \eta_{F,2}^2 + \eta_{F,3}^2)$$

Step REFINE of Adaptive Cycle: Refinement Rules

.

- Any $K \in \mathcal{M}_K$, $F \in \mathcal{M}_F$ is refined by bisection.

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- Any $K \in \mathcal{M}_K$, $F \in \mathcal{M}_F$ is refined by bisection.
- Further bisection is used to creat a geometrically conforming triangulation $\mathcal{T}_h$.

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- Further bisection is used to creat a geometrically conforming triangulation $\mathcal{T}_h$.

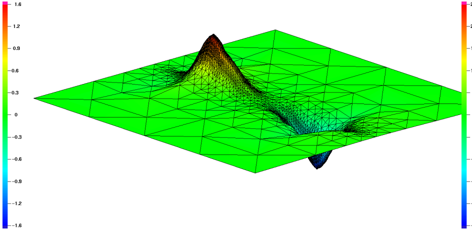
Part 6:Numerical results

Example

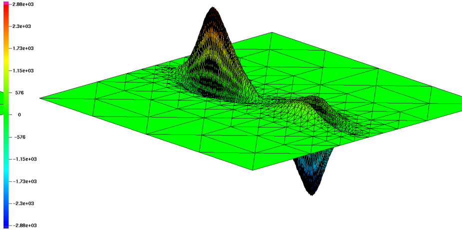
We choose $\Omega = (0,1), T = 1$, and $y^d = g - 0.1(g_{tt} - g_{xxxx})$,
$u^d = 0.9(g_t - g_{xx})$, $y^0 = g(x,0)$, $x \in \Omega$, $\alpha = 0.1$
where$g(x,t) = r(x)s(t)$, $(x,t) \in Q := \Omega \times (0,1)$

$$r(x) := \frac{10000x^4(1-x)^4}{1+1000(x-0.5)^2}$$

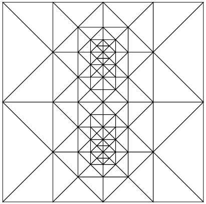
$$s(t) := \frac{1000t^2(1-t)^2}{1+100(t-0.25)^2} - \frac{1000t^2(1-t)^2}{1+100(t-0.75)^2}$$



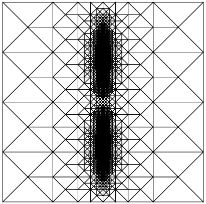
Optimal state



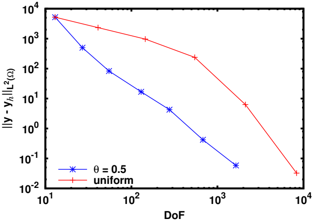
Optimal control



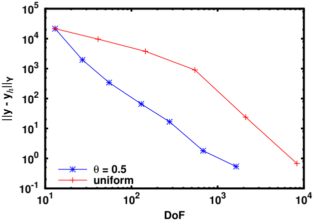
Adaptively refined triangulations after 4 cycles of the adaptive algorithm



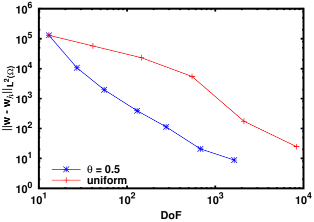
Adaptively refined triangulations after 8 cycles of the adaptive algorithm



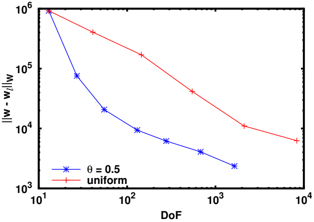
Adaptive versus uniform refinement: Error in y (L^2-norm)



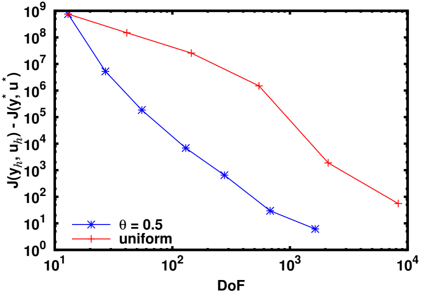
Adaptive versus uniform refinement: Error in y (Y-norm)



Adaptive versus uniform refinement: Error in w (L^2-norm)



Adaptive versus uniform refinement: Error in w (W-norm)



Adaptive versus uniform refinement: Error in the objective functional

Table: Convergence history of the AFEM. Discretization errors in y and w

l	DOF	$\|\|\|y-y_h\|\|\|_{L^2(Q)}$	$\|y-y_h\|_Y$	$\|w-w_h\|_{L^2(Q)}$	$\|w-w_h\|_W$
1	13	5.21e+03	2.16e+04	1.30e+05	9.24e+05
2	27	4.99e+02	1.96e+03	1.07e+04	7.55e+04
3	55	8.32e+01	3.42e+02	1.97e+03	2.07e+04
4	130	1.68e+01	6.61e+01	3.93e+02	9.40e+03
5	277	4.25e+00	1.67e+01	1.14e+02	6.18e+03
6	678	4.20e-01	1.81e+00	2.09e+01	4.08e+03
7	1639	5.84e-02	5.38e-01	8.87e+00	2.36e+03
8	4317	5.64e-02	4.18e-01	4.31e+00	1.42e+03
9	11391	4.16e-02	2.74e-01	1.94e+00	8.57e+02


HAPPY NEW YEAR
2014