Stabilized Extended One-Step Schemes for Stiff and Non-Stiff Delay Differential Equations

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Outline

- Extended One-Step Methods for DDEs
- Stability Analysis
- Numerical results

Part 1: Extended One-Step Methods for DDEs

Let us first provide EOSM applied to the initial value problem of the ODE form

$$\begin{aligned} y\prime(t) &= f(t, y(t)), \quad 0 < t \leq b, \\ y(0) &= y_0, \qquad\qquad t = 0. \end{aligned} \tag{1}$$

an A-stable *linear multistep method* (LMM) can not exceed 2. To overcome this "order barrier" imposed by A-stability, so called extended one-step A-stable methods of order up to five had constructed by coupling several LMMs

After discretization of the problem (1), one can get

$$y_{n+1} = y_n + h[\ \alpha_0 f_n + \alpha_1 f_{n+1} + \sum_{j=2}^{m-1} \alpha_j f_{n+j}] + \kappa_n(h)\ , \qquad (2)$$

In order to make the discretization in (2) one-step, they introduce, for $j = 2, ..., m-1$,

$$y_{n+j} = \beta_{j0} y_n + \beta_{j1} y_{n+1} + h[\ \gamma_{j0} f_n + \gamma_{j1} f_{n+1} + \sum_{i=2}^{j-1} \gamma_{ji} f_{n+i}] + E_{nj}(h)$$

The extended one-step scheme of such problem takes the form

$$y_{n+1} = y_n + h[\ \alpha_0 f_n + \alpha_1 f_{n+1} + \sum_{j=2}^{m-1} \alpha_j \hat{f}_{n+j}] + T_n(h),$$

$$\hat{y}_{n+j} = \beta_{j0} y_n + \beta_{j1} y_{n+1} + h[\ \gamma_{j0} f_n + \gamma_{j1} f_{n+1} + \sum_{i=2}^{j-1} \gamma_{ji} \hat{f}_{n+i}]$$

where $\alpha_j,\ \beta_{j0},\ \beta_{j1},\ \gamma_{j0},\ \gamma_{j1}$ and $\gamma_{ji},\ j = 2, 3, \ldots m-1$ are real coefficients

α_0	α_1	α_2	$\ldots$		α_{m-1}	
β_{20} β_{30} $\vdots$ $\beta_{m-1,0}$	β_{21} β_{31} $\beta_{m-1,1}$	γ_{20} γ_{30} $\gamma_{m-1,0}$	γ_{21} γ_{31} $\vdots$ $\gamma_{m-1,1}$	 γ_{32} $\vdots$ $\gamma_{m-1,2}$	 $\ddots$ $\cdots$	 $\gamma_{m-1,m-2}$

Table: Coefficients of the extended one-step methods.

- Jacques (1989) obtained an extended one-step third order L-stable method by requiring that $E_{n2}(h) = O(h^3)$ and this method depend on a free parameter.

- Chawla et al.(1994) obtained a two-parameter family of fourth order and A-stable methods by requiring that $E_{n2}(h)$ and $E_{n3}(h) = O(h^3)$;there exists a one-parameter sub-family of these methods which are, in addition, L-stable.

- Chawla et al.(1995) extended these ideas to obtain a two-parameter family of fifth order and gave sub-families of A-stable and L-stable methods.

Third order methods ($m = 3$)

$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$	
5	-4	2	4

$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$	
1	0	0	2

A-stable scheme of Usmani and Agarwal(left) and L-stable scheme of Jacques(right) .

Fourth order methods ($m = 4$)

$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$	
5	-4	2	4	
28	-27	12	18	0

$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$	
1	0	0	2	
2	-1	$-\frac{1}{2}$	4	$\frac{1}{2}$

A-stable scheme (left) and L-stable scheme (right) of order four for the ODEs (1) .

Fifth order methods ($m = 5$)

$\frac{251}{720}$	$\frac{323}{360}$	$-\frac{11}{30}$	$\frac{53}{360}$	$-\frac{19}{720}$	
5	-4	2	4		
28	-27	12	18	0	
$\frac{1611}{19}$	$-\frac{1592}{19}$	$\frac{712}{19}$	$\frac{966}{19}$	$-\frac{12}{19}$	$\frac{2}{19}$

L-stable scheme of order 5 for the ODEs (1) .

We extend the above schemes to the DDEs

$$\begin{aligned} y\prime(t) &= f(x, y(t), y(\alpha(t)), \quad & a \leq t \leq b, \\ y(t) &= g(t), & \nu \leq t \leq a. \end{aligned} \tag{3}$$

Here f , α and g denote given functions with $\alpha(t) \leq t$ for $t \geq a$, the function α is usually called the delay or lag function and y is unknown solution for $t > a$. If the delay is a constant, it is called the constant delay, if it is a function of only time, then it is called the time dependent delay, if it is a function of time and the solution $y(t)$, then it is called the state dependent delay.

The extended one-step scheme for DDE (3) is given by

$$y_{n+1} = y_n + h[\ \alpha_0 f_n + \alpha_1 f_{n+1} + \sum_{j=2}^{m-1} \alpha_j \hat{f}_{n+j}], n = 0, 1, \ldots, N-1, \tag{4}$$

where $\hat{f}_{n+j} = f(t_{n+j}, \hat{y}_{n+j}, y^h(\alpha(t_{n+j})))$ and $\alpha_j,\ j = 2, 3, \ldots m-1$ are real coefficients. The function y^h is computed from

$$\begin{cases} y^h(t) = g(t) \quad \text{for} \quad t \leq a \\ y^h(t) = \beta_{j0} y_k + \beta_{j1} y_{k+1} + h[\ \gamma_{j0} f_k \\ \qquad + \gamma_{j1} f_{k+1} + \sum_{i=2}^{j-1} \gamma_{ji} \hat{f}_{k+i}]\ , \\ \qquad t_k < t \leq t_{k+1} \quad k = 0, 1, \ldots \end{cases} \tag{5}$$

where $\beta_{j0},\ \beta_{j1},\ \gamma_{j0},\ \gamma_{j1}$ and γ_{ji} are real coefficients. The function $\hat{y}_{n+j}$ are computed from (5) when $t = t_{n+j}$. We will use $\tilde{}$ for the coefficients of $\hat{y}_{n+j}$ as in the following form

$$\hat{y}_{n+j} = \tilde{\beta}_{j0} y_n + \tilde{\beta}_{j1} y_{n+1} + h[\ \tilde{\gamma}_{j0} f_n + \tilde{\gamma}_{j1} f_{n+1} + \sum_{i=2}^{j-1} \tilde{\gamma}_{ji} \hat{f}_{n+i}] \tag{6}$$

Scheme of third order $(m = 3)$

In order to determine the coefficients α_0, α_1 and α_2, we rewrite (4) for $m = 3$ in the exact form

$$\begin{aligned} y(t_{n+1}) = y(t_n) + h\,[&\alpha_0 f(t_n, y(t_n), y(\alpha(t_n))) \\ &+ \alpha_1 f(t_{n+1}, y(t_{n+1}), y(\alpha(t_{n+1}))) \\ &+ \alpha_2 f(t_{n+2}, y(t_{n+2}), y(\alpha(t_{n+2})))] + \kappa(t_{n+1}). \end{aligned} \tag{7}$$

We expand the left and right sides of (7) in the Taylor series at the point t_{n+1}, equate the coefficients up to the third order terms $O(h^3)$ and solving the resulting system of equations, we obtain

$$\alpha_0 = \frac{5}{12}, \quad \alpha_1 = \frac{2}{3}, \quad \alpha_2 = -\frac{1}{12} \tag{8}$$

$$\kappa(t_{n+1}) = \frac{h^4}{24} y^{(4)}(\xi) \quad (9)$$

where $t_n < \xi < t_{n+2}$. Substituting from (8) into (4) for $m = 3$, we obtain

$$y_{n+1} = y_n + \frac{h}{12}\left[5f_n + 8f_{n+1} - \hat{f}_{n+2}\right] \quad (10)$$

$$y^h(t) = g(t) \quad \text{for} \quad t \leq a \quad (11)$$

and $y^h(t)$ with $t > a$ is defined by

$$y^h(t) = \beta_{20} y_k + \beta_{21} y_{k+1} + h\left[\gamma_{20} f_k + \gamma_{21} f_{k+1}\right], \text{ for } t_k < t \leq t_{k+1}; \ k = \quad (12)$$

In order to determine the coefficients β_{20}, β_{21}, γ_{20} and γ_{21}, we rewrite (12) in the exact form

$$\begin{aligned} y(t) = &\beta_{20} y(t_k) + \beta_{21} y(t_{k+1}) + h\left[\gamma_{20} f(t_k, y(t_k), y(\alpha(t_k)))\right. \\ &\left. + \gamma_{21} f(t_{k+1}, y(t_{k+1}), y(\alpha(t_{k+1})))\right] + E(t_{k+1}). \end{aligned} \quad (13)$$

Similarly, we expand the left and right sides of (13) with Taylor series at point t_{k+1} and equate the coefficients up to the terms of second order $O(h^2)$. We obtain the resulting system of equations

$$\begin{cases} \beta_{20} + \beta_{21} = 1 \\ \beta_{20} - \gamma_{20} - \gamma_{21} = -\delta(t) \\ \beta_{20} - 2\gamma_{20} = \delta^2(t) \end{cases} \tag{14}$$

where

$$\delta(t) = \frac{1}{h}(t - t_{k+1}). \tag{15}$$

The solution of the above system (14) is

$$\begin{cases} \beta_{20} = 1 - \beta_{21} \\ \gamma_{20} = \frac{1}{2}(1 - \beta_{21} - \delta^2(t)) \\ \gamma_{21} = \frac{1}{2}(\delta^2(t) + 2\delta(t) - \beta_{21} + 1) \end{cases} \tag{16}$$

$$E(t_{k+1}) = \frac{h^3}{12}(2\delta^3(t) + 3\delta^2(t) + \beta_1 - 1)y^{(3)}(\eta) \qquad (17)$$

where β_{21} is a free parameter and $t_k < \eta < t_{k+1}$. Substituting from (16) into (12), we obtain

$$y^h(t) =(1-\beta_{21})y_k + \beta_{21}y_{k+1} + \frac{h}{2}\left[(1-\beta_{21}-\delta^2(t))f_k + (\delta^2(t) + 2\delta(t) - \beta_{21}+1)f_{k+1}\right], \quad \text{for} \quad t_k < t \leq t_{k+1};\ k = 0, 1 \qquad (18)$$

Finally, from (18), the approximation $\hat{y}_{n+2}$ is determined in the form

$$\hat{y}_{n+2} = (1-\beta_{21})y_n + \beta_{21}y_{n+1} - \frac{h}{2}\left[\beta_{21}f_n + (\beta_{21}-4)f_{n+1}\right]. \qquad (19)$$

Equations (10), (18) and (19) are the basis of the third order methods.

Part 3: Stability Analysis

Consider the test equation:

$$\begin{aligned} \acute{y}(t) &= \lambda y(t) + \mu y(t-\tau), \quad t \geq 0 \\ y(t) &= g(t), \quad -1 \leq t \leq 0. \end{aligned} \tag{20}$$

Definition

(P-stability)
A numerical method, applied to (20) is said to be P-stable if under the condition $Re(\lambda) < -\mu$, the numerical solution satisfies $y(t_n) \longrightarrow 0$ as $t \longrightarrow \infty$ for all stepsizes $h = \frac{\tau}{s}$, where s is positive integer.

Definition

(P-stability Region)
If λ and μ are real in (20), the region S_P in the $(\lambda\ ,\mu\ )$ plane is called the $P-$ stability region if for any $(\lambda\ ,\mu) \in S_P$ the numerical solution of (20) satisfies $y(t_n) \longrightarrow 0$ as $t \longrightarrow \infty$ for the stepsize h

Definition

(Q-stability Region)
If $\lambda = 0$ and μ is complex in (20), the region $Q(\mu)$ in the $\mu-$plane is called the $Q-$ stability region if for any $\mu \in Q(\mu)$ the numerical solution $y(t_n) \longrightarrow 0$ as $t_n \longrightarrow \infty$.

Applying the third order EOSM to the test problem (20), yields

$$[24 - 2\lambda h(8 - \beta_{21}) + (\lambda h)^2(4 - \beta_{21})]y_{n+1} = [24 + 2\lambda h(4 + \beta_{21}) + (\lambda h)^2\beta_{21}]y_n + \mu h[(10 + \lambda h\beta_{21})y(t_n - \tau) + (16 - \lambda h(4 - \beta_{21}))y(t_{n+1} - \tau) - 2\mu h y(t_{n+2} - \tau$$

The characteristic polynomial associated with (21) takes the form

$$\begin{aligned} W_s(z) &= [24 - 2X(8-\beta_{21}) + X^2(4-\beta_{21})]z^{s+1} \\ &\quad - [24 + 2X(4+\beta_{21}) + X^2\beta_{21}]z^s \\ &\quad - Y[10 + X\beta_{21} + (16 - X(4-\beta_{21}))z - 2z^2] \\ &= 0, \quad s = 1, 2, \ldots \end{aligned} \tag{22}$$

where $X = \lambda h$ and $Y = \mu h$.

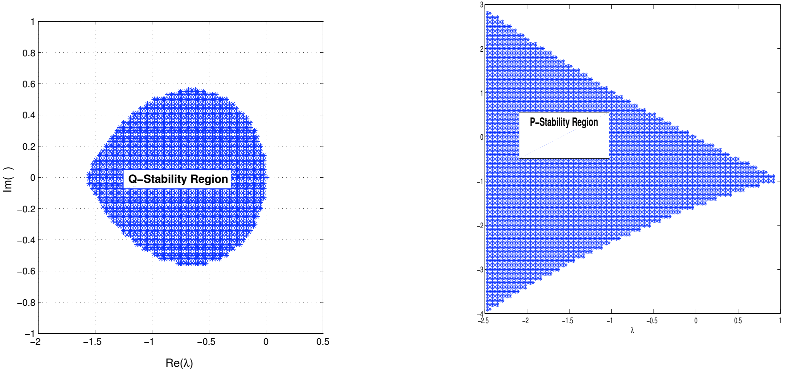


Figure: Left banner shows bounded Q-stability region for the test problem (20) with $\tau = 1$, using the third order scheme ($s = 3$) and $\beta_{21} = 0$. The right banner shows the unbounded P-stability region, with

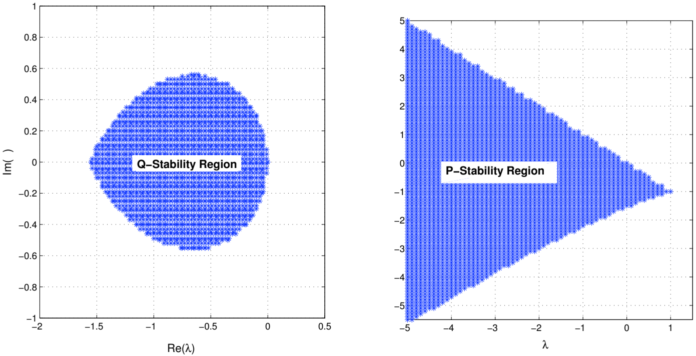
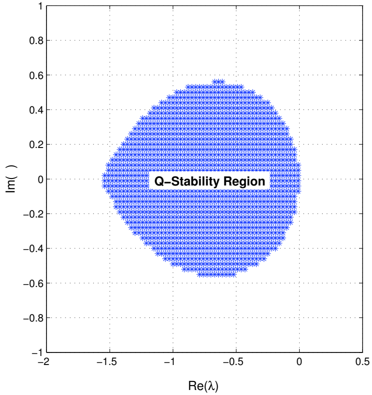


Figure: Left banner shows bounded Q-stability region for the test problem (20) with $\tau = 1$, using the fourth order scheme ($s = 4$) and $\gamma_{32} = 0.5$, $\gamma_{20} = 0$. However, the right banner shows unbounded P-stability region.



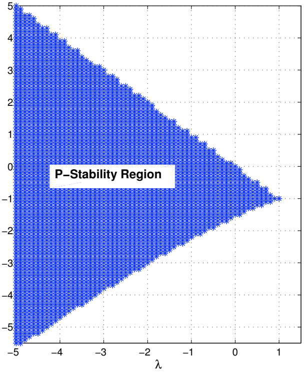


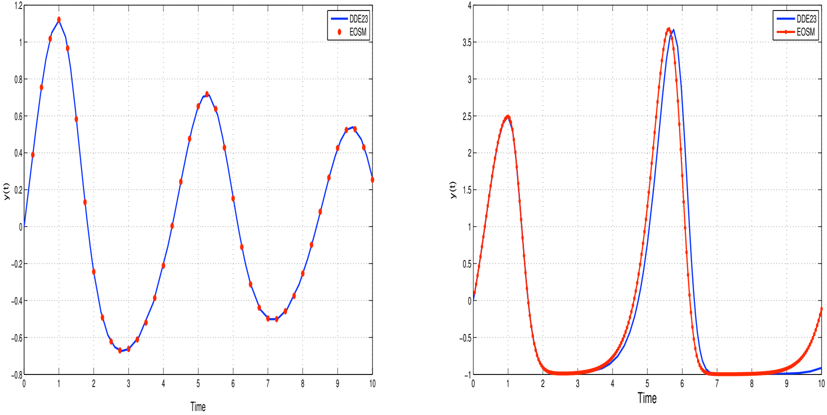
Figure: Left banner shows bounded Q-stability region for the test problem(20) with $\tau = 1$, using the fifth order scheme $(s = 5)$ and $\gamma_{43} = 2/19$, $\gamma_{32} = 0$. However, the right banner shows unbounded P-stability region.

Part 4: Numerical results

Example

Consider the logistic DDE

$$\begin{aligned} y'(t) &= -\lambda y(t-1)(1+y(t)), \quad && t \geq 0, \\ \phi(t) &= 1, && t \leq 0. \end{aligned} \tag{23}$$



Solution for the logistic DDE for $(\lambda = 1.5)$ (left) and for $(\lambda = 2)$ (right) by EOSMs versus DDE23

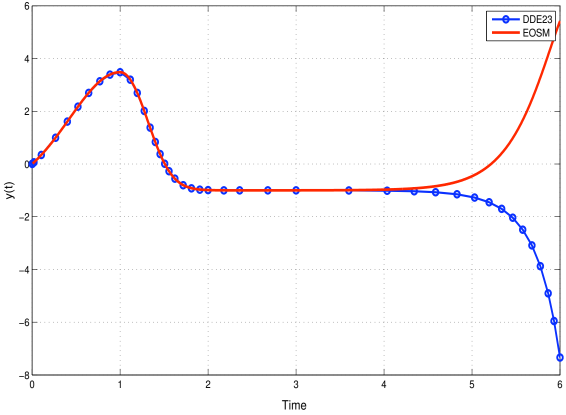


Figure: Solution for (23) $(\lambda = 3)$ by EOSMs versus DDE23.

Example

We extend our analysis to solve a system of DDEs of the form

$$\begin{aligned} y\prime_1(t) &= y_2(t) \\ y\prime_2(t) &= 1 - y_2(t-1) - y_1(t) \end{aligned} \qquad t \geq 0, \qquad (24)$$

with

$$y(0) = [y_1(0), y_2(0)]^T = [0,0]^T, \qquad t \leq 0$$

and analytical solution

$$y_1(t) = \begin{cases} 1 - \cos(t), & 0 \leq t \leq 1; \\ 1 - \cos(t) + \frac{1}{2}(t-1) + \cos(t-1)\frac{1}{2}\sin(t-1), & 1 \leq t \leq 2. \end{cases}$$

$$y_2(t) = \begin{cases} \sin(t), & 0 \leq t \leq 1; \\ \sin(t) + \frac{1}{2}(1-t)\sin(t-1), & 1 \leq t \leq 2. \end{cases}$$

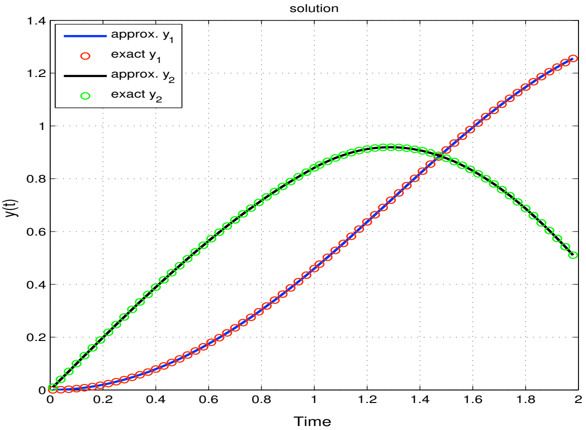


Figure: Solution for the system of DDEs (24) using EOSMs versus exact solution.

Example

Consider the varying-delay scalar DDE of the form

$$\begin{aligned} y'(t) &= 1 - y(1 - \texttt{exp}(1 - \frac{1}{t})), && t \geq 1, \\ \phi(t) &= \texttt{ln}(t), && 0 < t \leq 1, \end{aligned} \tag{25}$$

and analytical solution

$$y(t) = \ln(t), \quad t > 0.$$

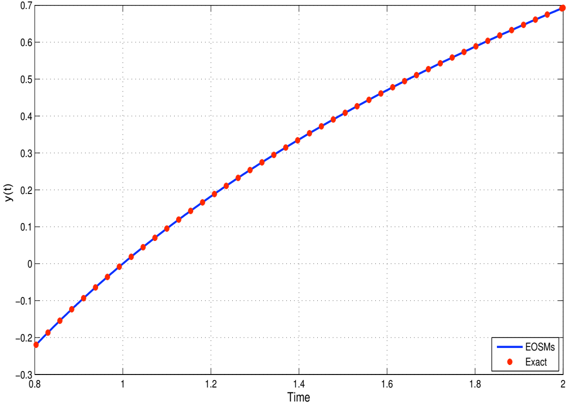


Figure: Solution for variable delay DDE (25) using EOSMs versus exact solution.

Example

Consider the more general state-dependent DDE of the form

$$\begin{aligned} y\prime(t) &= \frac{y(y(t)-\sqrt{2}+1)}{2\sqrt{t}}, & t \geq 1, \\ \phi(t) &= 1, & 0 < t \leq 1. \end{aligned} \tag{26}$$

and analytical solution:

$$y(t) = \sqrt{t}, \qquad 1 \leq t \leq 2.$$

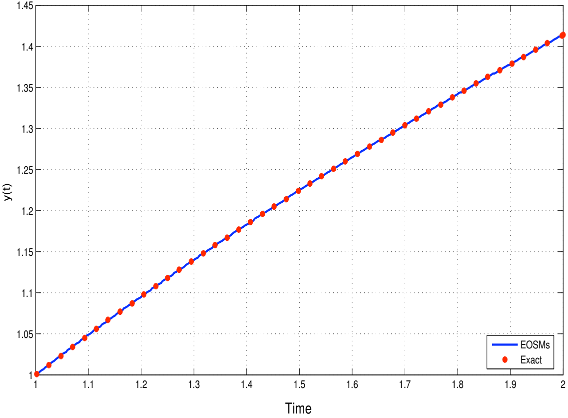


Figure: Solution for state dependent DDE (26) using EOSMs versus exact solution

Example

Consider the stiff DDE of the form

$$
\begin{aligned}
& y\prime(t) = -1000y(t) + q\ y(t-1) + c, && t \geq 0, \\
S1: \quad & q = 997e^{-3}, \quad c = 1000 - q \\
& y(t) = 1 + e^{(-3t)}, && t \geq 0, \text{(exact solution)} \\
S1: \quad & q = 999e^{-1}, \quad c = 1000 - q \\
& y(t) = 1 + e^{(-t)}, && t \geq 0, \text{(exact solution)} \\
S1: \quad & q = 999.99e^{-0.01}, \quad c = 1000 - q \\
& y(t) = 1 + e^{(-0.01t)}, && t \geq 0, \text{(exact solution)}.
\end{aligned}
\tag{27}
$$

The choice of parameters produces a stiff DDE. comparing the exact solution with the numerical results obtained by EOSMs for different values for the parameter q

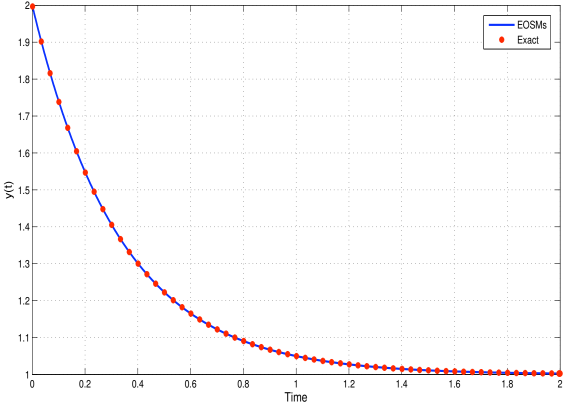


Figure: Solution for (27) ($q = 997e^{-3}$) using EOSMs versus exact solution.

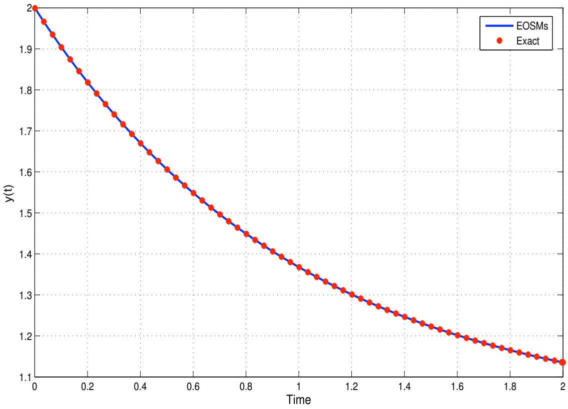


Figure: Solution for (27) ($q = 999e^{-1}$) using EOSMs versus exact solution.

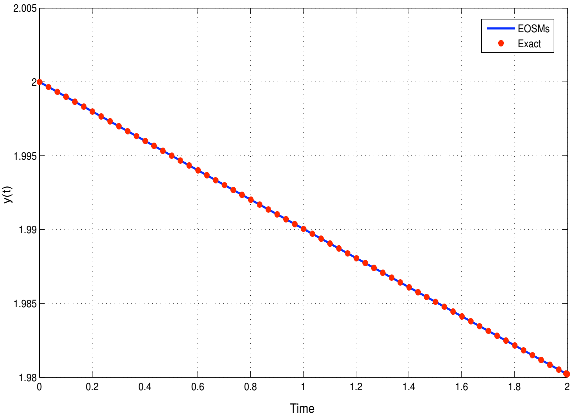


Figure: Solution for (27) ($q = 999.99e^{-0.01}$) using EOSMs versus exact solution.

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Thanks for your attention!
Any questions?
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