The Use of Monolithic Multigrid Methods for the Optimal Distributed Control of the Time-Dependent Navier-Stokes Equations

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Part of the SPP1253: Optimization with PDE's

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Flow Control Model Problem

Distributed Control of the nonstationary Navier-Stokes equation with tracking-type functional for a given z:

$$J(y,u) = \frac{1}{2}||y-z||_Q^2 + \frac{\alpha}{2}||u||_Q^2 + \frac{\gamma}{2}||y(T)-z(T)||_\Omega^2 \quad \rightarrow \quad \text{min!}$$

on $Q = \Omega \times [0,T]$ such that

$$\begin{array}{rcll} y_t - \nu\Delta y + (y\nabla)y + \nabla p & = & u & \text{in } Q \\ -\nabla \cdot y & = & 0 & \text{in } Q \end{array} \quad + \text{BC}$$

No constraints (for simplicity).

Corresponding KKT-System (unconstrained case) for (y, p, λ, ξ):

$$y_t - \nu \Delta y + (y\nabla)y + \nabla p = -\frac{1}{\alpha}\lambda \quad \text{in } Q$$
$$-\lambda_t - \nu \Delta \lambda - (y\nabla)\lambda + (\nabla y)^t \lambda + \nabla \xi = (y - z) \quad \text{in } Q$$

+ incompressibility $(-\nabla \cdot y = -\nabla \cdot \lambda = 0)$
+ initial/boundary/terminal cond. $(\lambda(T) = \gamma(y(T) - z(T)))$

Aim: Solve with
$$\begin{cases} \text{costs for simulation} = O(N), \\ \text{costs for optimisation} = O(N), \\ \dfrac{\text{costs for optimisation}}{\text{costs for simulation}} \leq C \approx 10 - 50 \end{cases}$$

Corresponding KKT-System (unconstrained case) for (y, p, λ, ξ):

$$
\begin{aligned}
y_t + C(y)y \quad + \nabla p &= -\frac{1}{\alpha}\lambda && \text{in } Q \\
-\lambda_t + N^*(y)\lambda + \nabla \xi &= (y - z) && \text{in } Q
\end{aligned}
$$

+ incompressibility $\quad (-\nabla \cdot y = -\nabla \cdot \lambda = 0)$
+ initial/boundary/terminal cond. $\quad (\lambda(T) = \gamma(y(T) - z(T)))$

<u>Aim:</u> Solve with

$$
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\text{costs for simulation} = O(N), \\
\text{costs for optimisation} = O(N), \\
\dfrac{\text{costs for optimisation}}{\text{costs for simulation}} \leq C \approx 10 - 50
\end{cases}
$$

In the following:

- Short design overview
- Numerical examples

Properties of the KKT system

Observation:

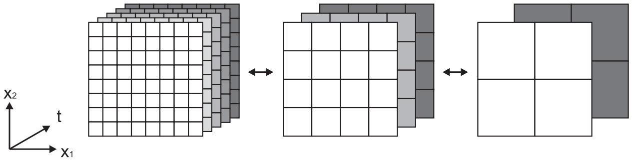
- KKT-system $\rightarrow$ *elliptic* BVP in space/time $(-y_{tt} + \Delta^2 y + \ldots)$

Idea:

- Apply 'optimal' $O(N)$ ingredients from CFD to this BVP!

  $\rightarrow$ unstructured meshes, FEM in space, implicit time-stepping
  $\rightarrow$ monolithic Multigrid + Newton solver techniques

- In particular: Solve on a space-time hierarchy



Time discretisation

Implicit Euler:

$$\frac{y_n - y_{n-1}}{\Delta t} + C_n y_n + \nabla p_n = -\frac{1}{\alpha}\lambda_n$$
$$\frac{\lambda_n - \lambda_{n+1}}{\Delta t} + N_n^* \lambda_n + \nabla \xi_n = y_n - z_n$$

Crank-Nicolson:

$$\frac{y_n - y_{n-1}}{\Delta t} + \frac{1}{2} C_n y_n + \frac{1}{2} C_{n-1} y_{n-1} + \nabla p_{n-\frac{1}{2}} = -\frac{1}{\alpha}\lambda_{n-\frac{1}{2}}$$
$$\frac{\lambda_{n-\frac{1}{2}} - \lambda_{n+\frac{1}{2}}}{\Delta t} + \frac{1}{2} N_n^* (\lambda_{n-\frac{1}{2}} + \lambda_{n+\frac{1}{2}}) + \nabla \xi_n = y_n - z_n$$

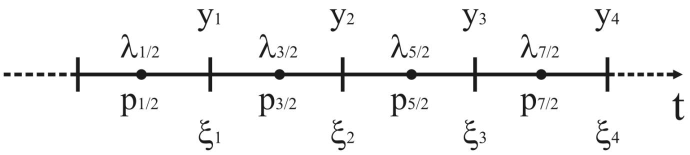
Time discretisation

Implicit Euler:

$$\frac{y_n - y_{n-1}}{\Delta t} + C_n y_n + \nabla p_n = -\frac{1}{\alpha}\lambda_n$$
$$\frac{\lambda_n - \lambda_{n+1}}{\Delta t} + N_n^* \lambda_n + \nabla \xi_n = y_n - z_n$$

Crank-Nicolson:

$$\frac{y_n - y_{n-1}}{\Delta t} + \frac{1}{2} C_n y_n + \frac{1}{2} C_{n-1} y_{n-1} + \nabla p_{n-\frac{1}{2}} = -\frac{1}{\alpha}\lambda_{n-\frac{1}{2}}$$
$$\frac{\lambda_{n-\frac{1}{2}} - \lambda_{n+\frac{1}{2}}}{\Delta t} + \frac{1}{2} N_n^* (\lambda_{n-\frac{1}{2}} + \lambda_{n+\frac{1}{2}}) + \nabla \xi_n = y_n - z_n$$



[Flaig '09]

Discretization in time (θ–scheme) leads to a system

$$F(x) = A(x)x = b$$

in the form (here e.g. for 2 timesteps, IE in time):

$$A(x)x =$$

$$\left(\begin{array}{cc|cc|cccc|cccc}
\frac{I}{\Delta t} + C & & \nabla & & & & & & & & & \\
-I & \frac{I}{\Delta t} + N^* & & \nabla & & -\frac{1}{\Delta t} & & & & & & \\
\hline
-\nabla\cdot & & 0 & & & & & & & & & \\
 & -\nabla\cdot & & 0 & & & & & & & & \\
\hline
-\frac{I}{\Delta t} & & & & \frac{I}{\Delta t} + C & \frac{I}{\alpha} & \nabla & & & & & \\
 & & & & -I & \frac{I}{\Delta t} + N^* & & \nabla & & -\frac{I}{\Delta t} & & \\
\cline{5-8}
 & & & & -\nabla\cdot & & 0 & & & & & \\
 & & & & & -\nabla\cdot & & 0 & & & & \\
\hline
 & & & & -\frac{I}{\Delta t} & & & & \frac{I}{\Delta t} + C & \frac{I}{\alpha} & \nabla & \\
 & & & & & & & & -c(\gamma, \Delta t)I & \frac{I}{\Delta t} + N^* & & \nabla \\
\cline{9-12}
 & & & & & & & & -\nabla\cdot & & 0 & \\
 & & & & & & & & & -\nabla\cdot & & 0
\end{array}\right)
\left(\begin{array}{c}
y_0 \\ \lambda_0 \\ p_0 \\ \xi_0 \\ \hline y_1 \\ \lambda_1 \\ p_1 \\ \xi_1 \\ \hline y_2 \\ \lambda_2 \\ p_2 \\ \xi_2
\end{array}\right)$$

+Space-discr. $\Rightarrow$ **Sparse, (block) tridiagonal system**

Design of an efficient optimal control solver

- Nonlinearity: Newton method for quadratic convergence

$$x^{i+1} = x^i + F'^{-1}(x^i)(b - A(x^i)x^i)$$

- Linear subproblems: space-time MG solver (complexity $O(N)$)

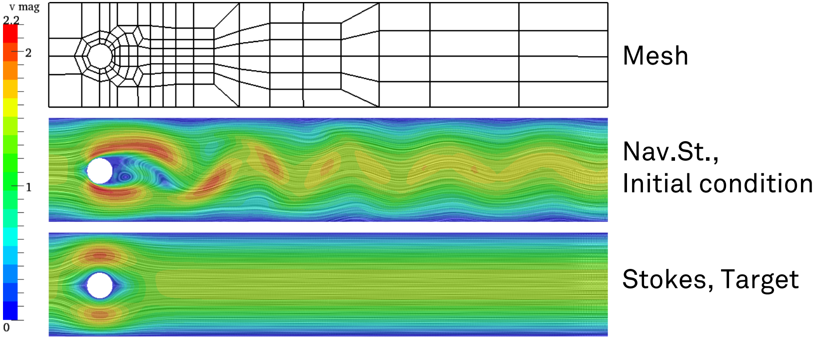
  $\rightarrow$ using Block-Jacobi/Block-SOR smoothing techniques

- Linear subproblems in space: Monolithic Multigrid solver

  $\rightarrow$ using 'local Pressure-Schur complement' techniques in each timestep for generalised Navier–Stokes subproblems

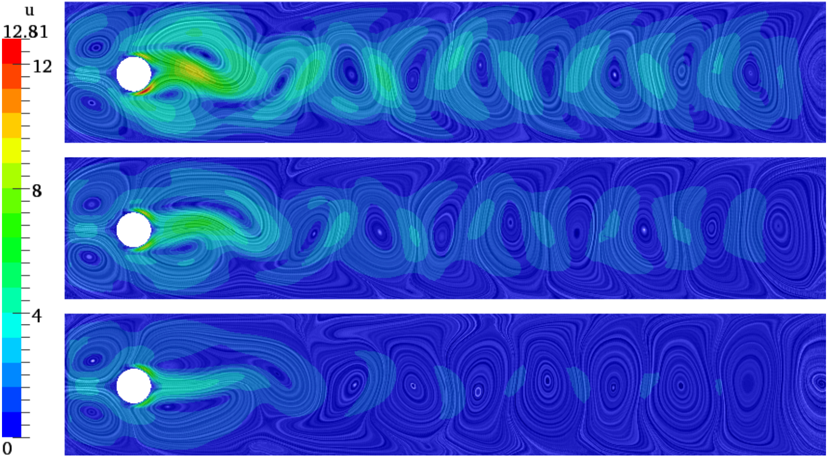
Software Package: FEATFLOW

Flow-around-cylinder (based on DFG benchmark BENCH2)



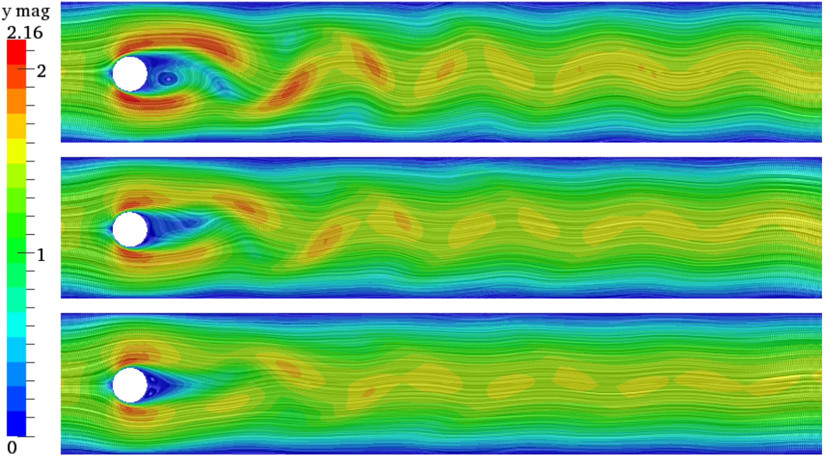
- Problem/Init. Cond.: Navier–Stokes, $Re = 100$, $t \in [0, 0.35]$
- Target flow z: Stationary Stokes flow
- Coarse mesh: Standard DFG benchmark
  $\rightarrow$ 1.404 DOF's in space, 35 timesteps, $\times 8$ per level

Flow-around-cylinder (Velocity/Control visualisation)



Control u at $t = 0.0175, 0.0875$ and 0.175, $\alpha = 0.02$, $\gamma = 0.0$

Flow-around-cylinder (Velocity/Control visualisation)



Velocity y at $t = 0.0175, 0.0875$ and 0.175, $\alpha = 0.02$, $\gamma = 0.0$

Flow-around-cylinder

(Solver behaviour)

Discretisation with $\tilde{Q}_1/Q_0/\text{IE}$

Space-lv.	#steps	#NL	#MG	T_{opt}	T_{sim}	$\frac{T_{opt}}{T_{sim}}$
3	140	7	18	5.879	303	19.4
4	280	6	15	39.193	1.571	24.9

Discretisation with $\tilde{Q}_1/Q_0/\text{CN}$

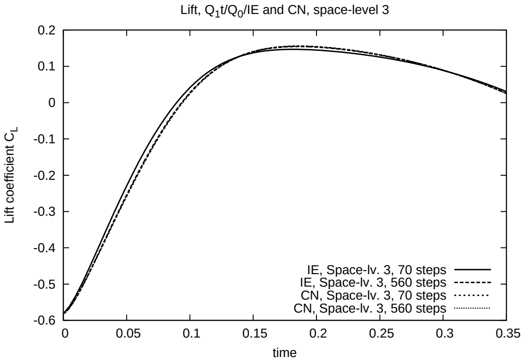
Space-lv.	#steps	#NL	#MG	T_{opt}	T_{sim}	$\frac{T_{opt}}{T_{sim}}$
3	140	7	17	6.849	283	24.2
4	280	7	17	50.505	1.324	38.1

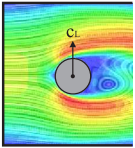
Discretisation with $Q_2/P_1^{\text{disc}}/\text{CN}$

Space-lv.	#steps	#NL	#MG	T_{opt}	T_{sim}	$\frac{T_{opt}}{T_{sim}}$
3	140	7	12	21.628	5.034	4.2
4	280	7	13	200.580	13.655	14.6

Flow-around-cylinder

(Lift over time)





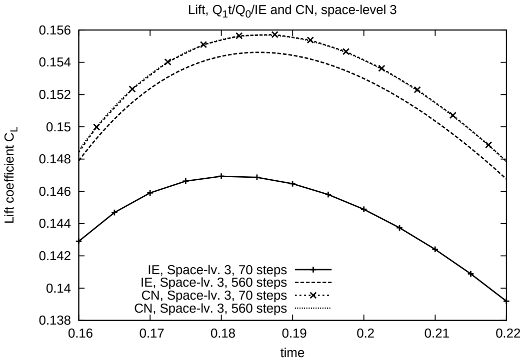
Lift coefficient:

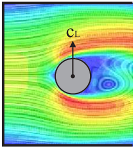
$\alpha = 0.02, \gamma = 0.0$

$$C_L = \frac{2}{(\frac{2}{3} U_{\max})^2 \cdot 0.1} \int_\Gamma (\sigma n) n_y \, d\Gamma$$

Flow-around-cylinder

(Lift over time, zoomed)





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Flow-around-cylinder

(Solver behaviour)

Discretisation with $\tilde{Q}_1/Q_0/\text{IE}$

Space-lv.	#steps	#NL	#MG	T_{opt}
3	70	7	16	2.610
3	560	7	18	19.373

Discretisation with $\tilde{Q}_1/Q_0/\text{CN}$

Space-lv.	#steps	#NL	#MG	T_{opt}
3	70	7	18	3.539
3	560	7	17	23.378

$\Rightarrow$ Similar Accuracy!

Space level 3
560 timesteps $\Rightarrow$ 21.216 DOF's
$\Delta t = 0.000625$

Flow-around-cylinder

(Solver behaviour)

Discretisation with $\tilde{Q}_1/Q_0/$IE

Space-lv.	#steps	#NL	#MG	T_{opt}
3	70	7	16	2.610
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Discretisation with $\tilde{Q}_1/Q_0/$CN

Space-lv.	#steps	#NL	#MG	T_{opt}
3	70	7	18	3.539
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$\Rightarrow$ Similar Accuracy!

Space level 3 / 560 timesteps $\Rightarrow$ 21.216 DOF's / $\Delta t = 0.000625$

Summary and Outlook

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Shown:

- Concept and realisation of an optimal control flow solver with optimal complexity

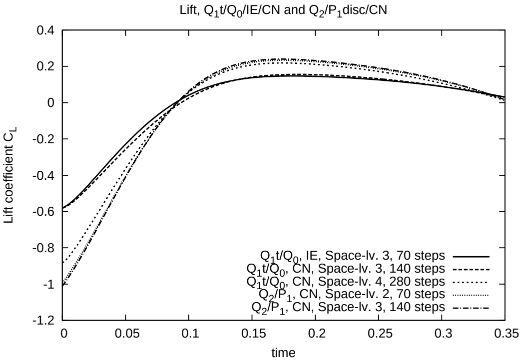
Key points:

- linear complexity (due to MG techniques)
- flexible (due to FEM approach, higher order + 3D possible)
- robust (FEM-stab. possible, e.g. EO-FEM/interior penalty)
- extension to more complex problems possible

  (Non-Newtonian + Non-isothermal flow,
  boundary control, constraint control)

Flow-around-cylinder

(Lift over time)

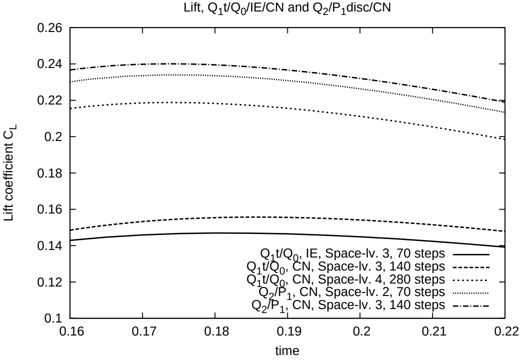


Lift coefficient: $\alpha = 0.02, \gamma = 0.0$

$$C_L = \frac{2}{(\frac{2}{3}U_{\max})^2 \cdot 0.1} \int_\Gamma (\sigma n) n_y \, d\Gamma$$

Flow-around-cylinder

(Lift over time, zoomed)



Lift coefficient: $\alpha = 0.02, \gamma = 0.0$

$$C_L = \frac{2}{(\frac{2}{3}U_{\max})^2 \cdot 0.1} \int_\Gamma (\sigma n) n_y \, d\Gamma$$

Size of the systems

$\tilde{Q}_1/Q_0$/IE or CN

Space-lv.	#steps	DOF space	DOF total
1	35	1.404	50.544
2	70	5.408	383.968
3	140	21.216	2.991.456
4	280	84.032	15.182.992
5	560	334.464	187.634.304
3	70	21.216	1.506.336
3	560	21.216	11.902.176

Q_2/P_1^{disc}/CN

Space-lv.	#steps	DOF space	DOF total
1	35	3.068	110.448
2	70	11.856	841.776
3	140	46.592	6.569.472
4	280	184.704	51.901.824