

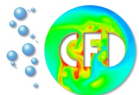
# A Space-Time Multigrid Solver Methodology for the Optimal Control of Time-Dependent Fluid Flow

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Distributed Control of the nonstationary Navier-Stokes equation with tracking-type functional for a given  $z$ :

$$J(y, u) = \frac{1}{2} \|y - z\|_Q^2 + \frac{\alpha}{2} \|u\|_Q^2 + \frac{\gamma}{2} \|y(T) - z(T)\|_\Omega^2 \rightarrow \min!$$

on  $Q = \Omega \times [0, T]$  such that

$$\begin{aligned} y_t - \nu \Delta y + (y \nabla) y + \nabla p &= u \quad \text{in } Q \\ -\nabla \cdot y &= 0 \quad \text{in } Q \end{aligned} + \text{BC}$$

No constraints (for simplicity).

Aim: Moderate performance measure; for  $C$  not too large ( $\approx 10$ ):

$$\frac{\text{costs for optimization}}{\text{costs for simulation}} \leq C$$

By modern numerical CFD techniques  
( $\rightarrow$  special FEM on solution adapted grids, fast MG+Newton solvers)

$$\text{costs for simulation} = O(N)$$

Aim: costs for optimization  $\stackrel{!}{=} O(N)$

Corresponding KKT-System (unconstrained case):

$$\begin{aligned} y_t - \nu \Delta y + (y \nabla) y + \nabla p &= u && \text{in } Q \\ -\nabla \cdot y &= 0 && \text{in } Q \end{aligned}$$

$$\begin{aligned} -\lambda_t - \nu \Delta \lambda - (y \nabla) \lambda + (\nabla y)^t \lambda + \nabla \xi &= (y - z) && \text{in } Q \\ -\nabla \cdot \lambda &= 0 && \text{in } Q \end{aligned}$$

$$u = -\frac{1}{\alpha} \lambda \quad \text{in } Q$$

+ boundary conditions

+ initial condition

+ terminal condition  $\lambda(T) = \gamma(y(T) - z(T))$  in  $\Omega$

Corresponding KKT-System (unconstrained case):

$$\begin{aligned}y_t + N(y)y + \nabla p + \frac{1}{\alpha}\lambda &= 0 && \text{in } Q \\-\nabla \cdot y &= 0 && \text{in } Q\end{aligned}$$

$$\begin{aligned}-\lambda_t + N^*(y)\lambda + \nabla \xi - y &= -z && \text{in } Q \\-\nabla \cdot \lambda &= 0 && \text{in } Q\end{aligned}$$

+ boundary conditions

+ initial condition

+ terminal condition  $\lambda(T) = \gamma(y(T) - z(T))$  in  $\Omega$

Observation:

- KKT-system  $\rightarrow$  *elliptic* BVP in space/time

Idea:

- Apply highly efficient ingredients from CFD (Multigrid + Newton) to this BVP!

Feasible?

KKT-System:

$$\begin{pmatrix} y_t \\ -\lambda_t \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} N(y) & \frac{1}{\alpha} \nabla & \nabla \\ -I & \tilde{N}^*(y) & \nabla \\ -\nabla \cdot & -\nabla \cdot & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} y \\ \lambda \\ p \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ -z \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} w_t \\ 0 \end{pmatrix} + \begin{pmatrix} N(w) & \nabla \\ -\nabla \cdot & 0 \end{pmatrix} \begin{pmatrix} w \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$\Rightarrow$  similar to a generalised Navier–Stokes equation





- Nonlinearity: Newton method for quadratic convergence

$$x^{i+1} = x^i + A'^{-1}(x^i)(b - A(x^i)x^i)$$

- Linear subproblems: space-time MG solver (for  $O(N)$  complexity)  
→ using Block-Jacobi/Block-SOR smoothing techniques
- Linear subproblems in space: Monolithic Multigrid solver  
→ using 'local Pressure-Schur complement' techniques  
in each timestep for the coupled Navier–Stokes subproblems

## Essential multigrid components:

- Mesh hierarchy for a space-time cylinder  $Q = \Omega \times [0, T]$ :  
Choose arbitrary space-time coarse mesh and refine!
- Prolongation/Restriction in space + time.  
Combination of FE in space & FD in time.
- An efficient smoother! E.g. with  $\tilde{A} := A'(x^i)$ :

$$v^{j+1} = v^j + \omega P^{-1}(b - \tilde{A}v^j), \quad j = 1, \dots, \text{NSM}$$

What to use as preconditioner  $P$ ?

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# Space-time discretisation and preconditioner

$\tilde{A}$  in compressed form (omitting  $B$  and  $B^T$  here):

$$\begin{pmatrix} M & & & & \\ -M & NST^* & & & \\ -\frac{M}{\Delta t} & & -\frac{M}{\Delta t} & & \\ & NST & \frac{1}{\alpha} M & & \\ & -M & NST^* & & \\ & -\frac{M}{\Delta t} & & -\frac{M}{\Delta t} & \\ & & NST & \frac{1}{\alpha} M & \\ & & -M & NST^* & -\frac{M}{\Delta t} \\ & & & \dots & \dots \end{pmatrix}$$

$\Rightarrow$  Block-Jacobi preconditioner:

$$P := P_{Jac} := \begin{pmatrix} M & & & & \\ -M & NST^* & & & \\ & NST & \frac{1}{\alpha} M & & \\ & -M & NST^* & & \\ & & & NST & \frac{1}{\alpha} M \\ & & & -M & NST^* \\ & & & & \dots \end{pmatrix}$$

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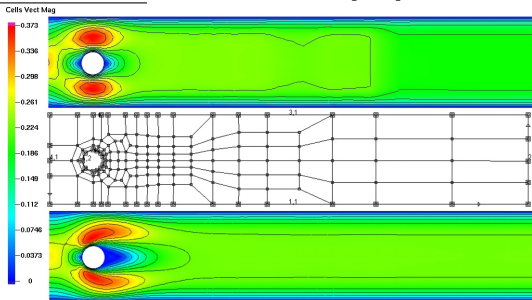
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## Flow-around-cylinder

- Target flow  $z$ : Stokes flow,  $t \in [0, 1]$ , starting from rest



Stokes at  $t = 1$

Mesh

uncontrolled Nav.St.

- Optimal control problem: Navier–Stokes,  $Re = 20$
- Coarse mesh: Standard DFG benchmark  
→ 1404 DOF's in space, 5 timesteps,  $\Delta t := 0.2$   
⇒ 8424 DOF's,  $\times 8$  per level

## Convergence of the Newton solver

		Simulation				Optimisation	
$\Delta t$	Space-Lv.	#NL	#MG	○#NL	○#MG	#NL	#MG
1/20	3	63	312	3	16	4	47
1/40	4	123	709	3	18	4	14
1/80	5	246	1589	3	20	4	9

- Nonlinear solver gained 5 digits
- Space-time MG gained 2 digits per step
- Space-preconditioner gained 2 digits per step



## Convergence of the Newton solver

$\Delta t$	Space-Lv.	$T_{\text{sim}}$	$T_{\text{opt}}$	$\frac{T_{\text{opt}}}{T_{\text{sim}}}$
1/20	3	27.0	1384.42	51.3
1/40	4	209.6	3895.59	18.6
1/80	5	2227.1	22882.87	10.3

$\Rightarrow C \approx 10 - 20$  on reasonable refinement levels.



## Convergence of the solver for different $\nu$

		$\nu = 1/1000$		$\nu = 1/250$	
$\Delta t$	Space-Lv.	#NL	#MG	#NL	#MG
1/10	2	4	36	4	13
1/20	3	4	24	3	8
1/40	4	4	14	3	7
1/80	5	4	13	3	8

$\Delta t = 1/40$ , Space-level 4

⇒ Stabilisation necessary for higher RE numbers

e.g. EO-stabilisation (consistent, only in space)

$$j(u, v) = \sum_{\text{edge} E} \gamma |E|^2 \int_E [\nabla u][\nabla v] d\sigma$$

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Shown:

- Concept and realisation of an optimal control flow solver with linear complexity ( $C \approx 10 - 20$ )

Key points:

- linear complexity (due to MG techniques)
- flexible (due to FEM approach, 3D possible)
- robust (FEM-stabilisation possible, e.g. EO-FEM/interior penalty)
- generalisation to more complex problems possible  
(Non-Newtonian + Non-isothermal flow,  
boundary control, constraint control)





