An Optimal-Order Multigrid Method for Quadratic Conforming Finite Elements

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Introduction and Notation

- standard 2nd order elliptic boundary problem

$$\begin{aligned} -\Delta u = f \qquad & \text{in } \Omega \subset \mathbb{R}^2 \\ u = 0 \qquad & \text{on } \partial\Omega \end{aligned}$$

- associated bilinear form

$$a(v, w) \quad := \quad (\nabla v, \nabla w)$$

- $\{V_h\}$ nested family of finite element spaces, based on triangulations of Ω

$$V_{h_{i-1}} \subset V_{h_i} \subset V := H_0^1(\Omega)$$

- $h,\ h_i \rightarrow$ grid size (on level i)

Introduction and Notation

- problem: *On finest level find* $u_h \in V_h$ *such that*

$$a(u_h, v_h) \quad = \quad (f, v_h) \qquad \forall\, v_h \in V_h$$

- $\{\varphi_h^i\} \subset V_h \;\; \rightarrow$ basis of V_h

  $\Rightarrow$ representation $\quad u_h = \sum_i u_i\, \varphi_h^i, \quad \vec{u}_h = (u_1,\, u_2,\, ...)^T$

- stiffness matrix A_h, right hand side vector $\vec{f}_h$

$\Rightarrow$ equivalent algebraic problem

$$A_h \vec{u}_h = \vec{f}_h$$

Standard Multigrid algorithm

MGM($h,\, x_0,\, b$**)** $\quad\rightarrow\quad$ to approximate the solution of $A_h x = b$

a) On coarsest grid:

- return $A_h^{-1} b$

b) On finer grids:

- $\mu \in \mathbb{N}$ presmoothing steps $\rightarrow x_\mu$
- Coarse grid correction $\rightarrow x_{\mu+1}$
- $\nu \in \mathbb{N}$ postsmoothing steps $\rightarrow x_{\mu+\nu+1}$
- return $x_{\mu+\nu+1}$

$\Rightarrow$

MGM($h_{\min},\, \vec{u}_h^0,\, \vec{f}_h$)

computes a "better" approximation to $\vec{u}_h$ than $\vec{u}_h^0$

Multigrid proof

Assumptions:

- H^2 regular problem
- approximation with linear conforming finite elements
- two-grid or multigrid W-cycle
- $\mu/0$ pre-/postsmoothing steps with damped Richardson method

$$\Rightarrow \qquad |||u_h - u_h^{\mu+1}||| \;\leq\; \frac{c}{\mu}\,|||u_h - u_h^0|||$$

$\Rightarrow$ Doubling the number of smoothing steps halfens the convergence rate

Verification

Definition (smoothing efficiency index)

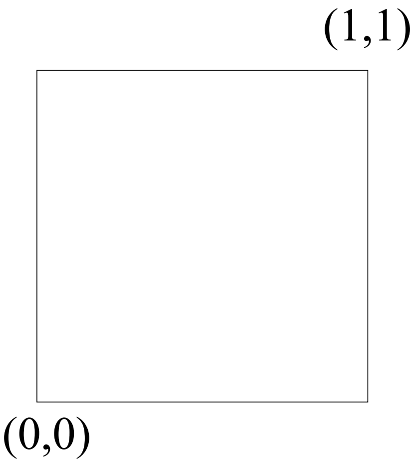
- $\rho_m \;\rightarrow\;$ convergence rate of multigrid algorithm with $m \in \mathbb{N}$ smoothing steps
- For $i \leq j \in \mathbb{N}$, $t := \log_2\left(\frac{j}{i}\right)$: $\quad G(i,j) := \left(\frac{\rho_i}{\rho_j}\right)^{\frac{1}{t}}$

$\Rightarrow$ describes approximately the improvement in the convergence rate when doubling the number of smoothing steps

$\Rightarrow$ should be ≈ 2

Verification with Q_1

Test problem:



- problem: $-\Delta u = 1$
- used element: Q_1, bilinear interpolation
- Jacobi smoother
- measurement of the asymptotic convergence rate
  $\rightarrow$ only last three steps

Verification with Q_1

m	ITE	ρ	G(m/2,m)	G(m/4,m)	G(m/8,m)
4	25	7,707E-02			
6	22	5,279E-02			
8	21	4,027E-02	1,914		
12	19	2,723E-02	1,938		
16	17	2,044E-02	1,970	1,942	
24	16	1,340E-02	2,032	1,985	
32	15	1,030E-02	1,984	1,977	1,956
48	14	6,539E-03	2,049	2,041	2,006
64	13	5,026E-03	2,049	2,017	2,001
96	12	2,766E-03	2,364	2,201	2,143
128	12	2,204E-03	2,280	2,162	2,101
192	11	1,464E-03	1,889	2,113	2,092

$\Rightarrow$ Factor ≈ 2 as expected

Verification with $\tilde{Q}_1$, Q_2

Same test problem, with different elements:

- used element: $\tilde{Q}_1$ (nonconforming rotated bilinear)
- used grid transfer: appropriate linear interpolation

$\Rightarrow$ **Roughly the same results**

- used element: Q_2 (conforming biquadratic)
- used grid transfer:

bilinear interpolation

$\leftrightarrow$

fully biquadratic interpolation

Q_2 with the linear interpolation

m	ITE	ρ	G(m/2,m)	G(m/4,m)	G(m/8,m)
4	30	2,931E-01			
6	30	2,258E-01			
8	30	1,808E-01	1,621		
12	30	1,287E-01	1,754		
16	28	1,032E-01	1,752	1,685	
24	25	7,368E-02	1,747	1,751	
32	23	5,702E-02	1,810	1,781	1,726
48	20	3,916E-02	1,881	1,813	1,793
64	19	2,983E-02	1,911	1,860	1,823
96	17	2,021E-02	1,938	1,910	1,854
128	16	1,503E-02	1,985	1,948	1,901
192	15	1,032E-02	1,958	1,948	1,926

$\Rightarrow$ Factor ≈ 2

Q_2 *with biquadratic interpolation*

m	ITE	ρ	G(m/2,m)	G(m/4,m)	G(m/8,m)
4	30	1,426E-01			
6	26	8,112E-02			
8	23	6,391E-02	2,232		
12	20	4,032E-02	2,012		
16	18	2,554E-02	2,502	2,363	
24	14	1,011E-02	3,988	2,833	
32	13	4,971E-03	5,138	3,586	3,062
48	11	1,990E-03	5,079	4,501	3,442
64	10	1,167E-03	4,259	4,678	3,797
96	9	5,515E-04	3,609	4,281	4,181
128	9	3,370E-04	3,463	3,841	4,232
192	8	1,576E-04	3,499	3,553	4,003

$\Rightarrow$ Factor ≈ 4

Conclusion

$\Rightarrow$ Q_2 shows a smoothing efficiency index of ≈ 4 with biquadratic interpolation

$\Rightarrow$ this indicates:

Doubling the number of smoothing steps quarters the convergence rate!!!

Lower order interpolation destroys this property!

$\Rightarrow$ First statement can be proven!

Multigrid proof for quadratic FE

Assumptions:

- H^3 regular problem, quadratic conforming finite elements
- L_2 projection as grid transfer operator
  ($\sim$ *quadratic interpolation*)
- two-grid or W-cycle multigrid
- $\mu/0$ pre-/postsmoothing steps

$\Rightarrow$ **Main result:**

$$|||u_h - u_h^{\mu+1}|||_{-1} \leq \frac{c}{\mu^2} |||u_h - u_h^0|||_{-1}$$

Multigrid proof for quadratic FE

Further assumptions and notations:

- Scale the FEM-basis $\{\varphi_h^i\}$ appropriately so that:

$\Rightarrow$ Eigenvalues $\{\lambda\}$ of stiffness matrix A_h: $\qquad c \leq \lambda \leq c' \, h^{-2}$

$\Rightarrow$ Eigenvalues $\{\mu\}$ of mass matrix M_h: $\qquad c \leq \mu \leq c'$

- **Scale of norms:**
  For $\quad u_h = \sum_j x_j \varphi_h^j, \quad x = (x_1, x_2, ...), \quad s \in \mathbb{R} \quad$ define

$$|||u_h|||_s := \|A_h^{s/2} x\|_E$$

$\Rightarrow$ Norm equivalence on V_h:

$$|||\cdot|||_1 \;\sim\; \|\cdot\|_1$$

$$|||\cdot|||_0 \;\sim\; \|\cdot\|_0$$

Multigrid proof for quadratic FE

Multigrid proof in 2 steps:

With the error:

$$e_m := u_h - u_h^m$$

a) Smoothing property:

$$|||e_\mu|||_3 \le \frac{c}{\mu^2} h^{-4} |||e_0|||_{-1}$$

b) Approximation property:

$$|||e_{\mu+1}|||_{-1} \le c\, h^4 |||e_\mu|||_3$$

Multigrid proof for quadratic FE

Smoothing via damped Richardson scheme

Algebraic system to be solved: $A_h x = b$

$$\Rightarrow \quad Sx \;:=\; x - \frac{1}{\lambda_{\max}}(A_h x - b)$$

Then: $s > t$ arbitrary, μ smoothing steps:

$$\Rightarrow \quad |||e_\mu|||_s \le \frac{c}{\mu^{\frac{s-t}{2}}}\, h^{-(s-t)}\, |||e_0|||_t$$

$\Rightarrow$ smoothing property for $s = 3$, $t = -1$
$\Rightarrow$ independent of the degree of the local polynomials
$\Rightarrow$ independent of grid transfer operators

Multigrid proof for quadratic FE

Approximation property via dual problems

Continuous dual problem: Find $z_h \in V_h$ such that

$$a(z_h, \phi_h) \quad = \quad (e_{\mu+1}, \phi_h) \qquad \forall\, \phi_h \in V_h$$

Discrete dual problem: Find $\eta_h \in V_h$ such that

$$A_h \vec{\eta}_h \quad = \quad \vec{e}_{\mu+1}$$

$$\Rightarrow \qquad a(\eta_h, \phi_h) = < \vec{e}_{\mu+1}, \vec{\phi}_h > \qquad \forall\, \phi_h \in V_h$$

$$\Rightarrow \qquad z_h \neq \eta_h$$

Multigrid proof for quadratic FE

To show: $|||e_{\mu+1}|||_{-1} \;\leq\; c\,h^4\,|||e_\mu|||_3$

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a) *Shift theorem* (Braess): With $\mathcal{A}_h\eta_h = e_{\mu+1}$:

$$|||e_{\mu+1}|||_{-1} \;=\; |||\mathcal{A}_h\eta_h|||_{-1} \;=\; |||\eta_h|||_1 \;\leq\; c\,\|\eta_h\|_1$$

b) Connection between the two solutions:

$$\vec{\eta}_h \;=\; A_h^{-1}M_h^{-1}A_h\,\vec{z}_h$$

$$\Rightarrow \qquad \|\eta_h\|_1 \;\leq\; c\,\|M_h^{-1}\|_1\,\|z_h\|_1 \;\leq\; c\,\|z_h\|_1$$

c) Connection to $\|\cdot\|_{-1}$ norm:

$$\|z_h\|_1 \;\leq\; c\,\|e_{\mu+1}\|_{-1}$$

Multigrid proof for quadratic FE

To show: $|||e_{\mu+1}|||_{-1} \;\leq\; c\,h^4\,|||e_\mu|||_3$

We have: $|||e_{\mu+1}|||_{-1} \;\leq\; c\,\|e_{\mu+1}\|_{-1}$

d) Duality argument for quadratic finite elements:

$$\begin{aligned} \|e_{\mu+1}\|_{-1} &\leq c\,h^2\,\|e_{\mu+1}\|_1 \\ &\leq c\,h^2\,|||e_{\mu+1}|||_1 \end{aligned}$$

yields:

$$|||e_{\mu+1}|||_{-1} \;\leq\; c\,h^2\,|||e_{\mu+1}|||_1$$

Multigrid proof for quadratic FE

To show: $|||e_{\mu+1}|||_{-1} \le c\,h^4\,|||e_\mu|||_3$

We have: $|||e_{\mu+1}|||_{-1} \le c\,\|e_{\mu+1}\|_{-1} \le c\,h^2\,|||e_{\mu+1}|||_1$

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e) Last ingredient: *Logarithmic convexity* (Braess):

$$|||e_{\mu+1}|||_1^2 \le |||e_{\mu+1}|||_{1-s}\,|||e_\mu|||_{1+s}, \qquad s > 0$$

Take $s = 2$:

$$\begin{aligned} |||e_{\mu+1}|||_{-1}^2 &\le c\,h^4\,|||e_{\mu+1}|||_1^2 \\ &\le c\,h^4\,|||e_{\mu+1}|||_{-1}\,|||e_\mu|||_3 \end{aligned}$$

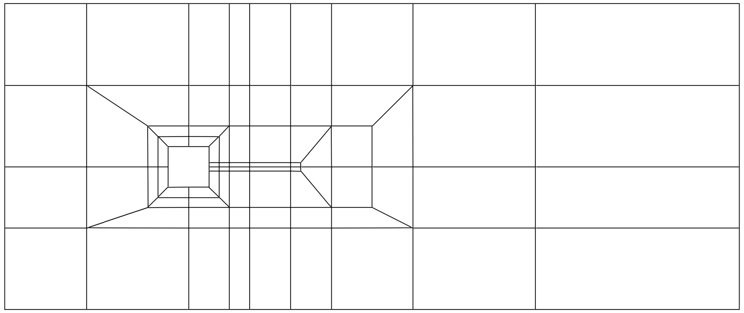
Cancel out $|||e_{\mu+1}|||_{-1} \quad \Rightarrow$ proof complete.

Q_2 *in a practical comparison to* Q_1

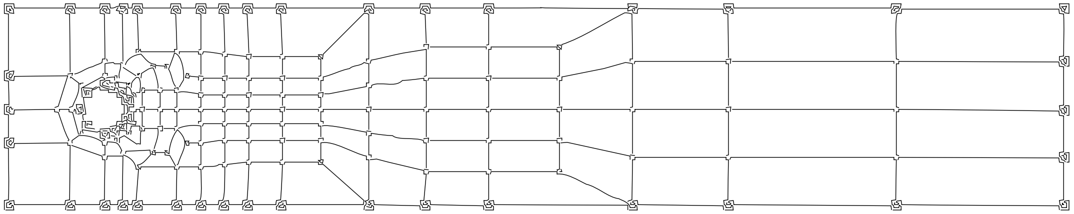
Multigrid test problems:

- Grids:

anisos3



bench1



- problem: $-\Delta u = 1, \quad u_{|\partial\Omega} = 0$
- used element: Q_1, Q_2, full interpolation
- smoothing steps: 0/2, 0/4 $\rightarrow$ BiCGStab (ILU(k))

Q_2 in a practical comparison to Q_1

grid	elem.	sms.\ lv.	4	5	6	7
anisos3	Q_1	0/2	0,008	0,015	0,028	0,049
(k=1)		0/4	0,003	0,003	0,006	0,008
	Q_2	0/2	0,003	0,005	0,006	0,009
		0/4	0,003	0,003	0,006	0,002
bench1	Q_1	0/2	0,010	0,014	0,018	0,021
(k=0)		0/4	0,002	0,003	0,004	0,005
	Q_2	0/2	0,006	0,008	0,007	0,008
		0/4	0,003	0,003	0,002	0,002

$\Rightarrow Q_2$ (at least) of same quality as Q_1

Open questions

Key inequalities for multigrid proofs:

(besides FEM duality argument)

$\rightarrow$ linear FE: $|||v_h|||_0 \;\le\; c\,\|v_h\|_0$

$\rightarrow$ quadratic FE: $|||v_h|||_{-1} \;\le\; c\,\|v_h\|_{-1}$

$\rightarrow$ using Q_{s+1}:

$$??? \quad \boxed{|||v_h|||_{-s} \;\le\; c\,\|v_h\|_{-s}} \quad ???$$

In that case: If problem is H^{s+1} regular,

$$\boxed{|||e_{\mu+1}|||_{1-s} \;\le\; \frac{c}{\mu^s}\,|||e_0|||_{1-s}}$$

Open questions

Key inequalities for multigrid proofs:

(besides FEM duality argument)

$\rightarrow$ linear FE: $\quad |||v_h|||_0 \;\leq\; c\,\|v_h\|_0$
$\rightarrow$ quadratic FE: $\quad |||v_h|||_{-1} \;\leq\; c\,\|v_h\|_{-1}$
$\rightarrow$ using Q_{s+1}:

$$??? \qquad \boxed{|||v_h|||_{-s} \;\leq\; c\,\|v_h\|_{-s}} \qquad ???$$

In that case: If problem is H^{s+1} regular,

$$\boxed{|||e_{\mu+1}|||_{1-s} \;\leq\; \frac{c}{\mu^s}\,|||e_0|||_{1-s}}$$

$\Rightarrow$ Higher order FEM can be solved much faster!?