Shallow Water

Derivation and Applications

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Overview

1. The 2d Shallow Water Equations
2. Derivation from basic conservation laws
3. Solutions to SWE problems

Overview

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Basic properties

The shallow water equations (SWE) are

- are a set of PDEs, that describe fluid-flow-problems
- are derived from the physical conservation laws for the mass and momentum
- are valid for problems in which vertical dynamics can be neglected compared to horizontal effects
- are a 2d model, which is derived from a 3d model by depth averaging

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For two reasons the name shallow water is not exact

- The fluid doesn't have to be water
  For example wheather forecasting was done with a modification of the SWE
- The fluid doesn't necessarily have to be shallow
  For example a tsunami wave on the ocean (approx. 5 km deep) can be a SW wave

Is “shallow water” a good name?

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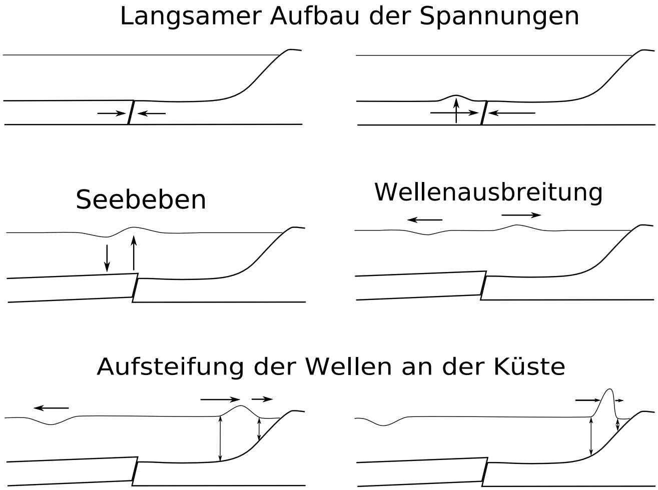

Langsamer Aufbau der Spannungen
Seebeben
Wellenausbreitung
Aufsteifung der Wellen an der Küste

The SWE have been applied to

- Tsunamis prediction
- Atmospheric flows
- Storm surges
- Flows around structures (pier)
- Planetary flows

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Counterexamples

The SWE can not be applied, when

- 3d effects become essential
- Waves become too short or too high

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Mass conservation

Conservation of mass

$$\frac{\partial}{\partial t}\varrho + \nabla \cdot (\varrho \boldsymbol{v}) = 0$$

$$\begin{aligned} \varrho &= \text{Density} \\ \boldsymbol{v} = (u, v, w) &= \text{Velocity} \end{aligned}$$

Conservation of mass

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ϱ=Density

$\boldsymbol{v} = (u, v, w) =$ Velocity

Conservation of mass

$$\nabla \cdot \boldsymbol{v} = 0$$

$$\boldsymbol{v} = (u, v, w) = \text{Velocity}$$

Simplified form

Mass continuity

$$\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v + \frac{\partial}{\partial z} w = 0$$

$$\boldsymbol{v} = (u, v, w) = \text{Velocity}$$

Conservation of momentum

$$\frac{\partial}{\partial t}(\varrho \boldsymbol{v}) + \nabla \cdot (\varrho \boldsymbol{v} \otimes \boldsymbol{v} + pI - \Pi) = \varrho \boldsymbol{g}$$

$$\begin{aligned} \varrho &= \text{Density} \\ \boldsymbol{v} = (u, v, w) &= \text{Velocity} \\ p &= \text{Pressure} \\ \Pi &= \text{Viscosity} \\ \boldsymbol{g} &= \text{Body force vector (gravitation)} \end{aligned}$$

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Simplified form

Conservation of momentum

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u = -\frac{1}{\varrho}\frac{\partial}{\partial x}p$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + w\frac{\partial}{\partial z}v = -\frac{1}{\varrho}\frac{\partial}{\partial y}p$$

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w = -\frac{1}{\varrho}\frac{\partial}{\partial z}p - g$$

Simplified form

Mass continuity

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$$

Conservation of momentum

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\end{aligned}$$

- We have obtained four PDEs for the four unknowns
- Given appropriate initial and boundary conditions, these equations can be solved
- Boundary conditions have to be applied to the surface

BUT

The position of the surface is not known a priori!

Solveability

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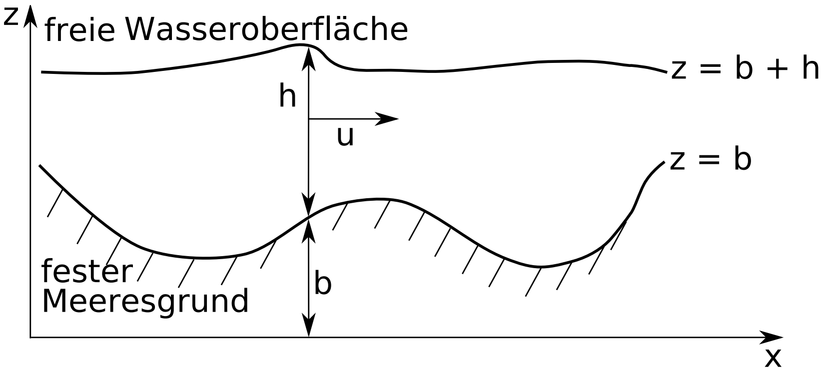
Variables



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0 = \frac{d}{dt} w &= -\frac{1}{\varrho} \frac{\partial}{\partial z} p - g
\end{aligned}$$

Main simplification

Primary assumption in SW-Theory

Horizontal scales (wave length l) are much larger, than vertical scales (undistubed water heigth h)

$$\underbrace{\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v}_{\approx \frac{U}{l}} + \underbrace{\frac{\partial}{\partial z}w}_{\approx \frac{W}{h}} = 0$$

$$\Downarrow$$

$$W \approx U\frac{h}{l}, \qquad \text{where } \frac{h}{l} \ll 1$$

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Neglegt vertical accelerations

Boundary layer assumption

$$\frac{d}{dt}w = 0$$

So you can think of the SWE as a boundary layer.
This explains the name “shallow”.

Boundary layer theory

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Hydrostatic pressure

$$\frac{\partial}{\partial z} p = -\varrho g$$

Integrate this equation

Hydrostatic pressure distribution

$$p = g \int_z^s \varrho \, \mathrm{d}z + p_a$$

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Assuming constant density along the z-axis, we obtain

$$p = \varrho g(s - z) + p_a$$

Pressure gradient

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The pressure gradient

$$\frac{\partial}{\partial x} p = \varrho g \frac{\partial}{\partial x} s$$
$$\frac{\partial}{\partial y} p = \varrho g \frac{\partial}{\partial y} s$$

- is independent of the variable z
- is responsible for horizontal accelerations

The horizontal velocities u, v

- are independent of the variable z
- can be interpreted as vertical mean velocities
- $\frac{\partial}{\partial z} u = \frac{\partial}{\partial z} v = 0$

Horizontal mean velocity

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Depth averaging

$$\int_b^s \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v + \frac{\partial}{\partial z} w \ \mathrm{d}z = 0$$

Mass continuity

$$\frac{\partial}{\partial t} h + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

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Mass continuity

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Simplified form

Mass continuity in conservative form

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Momentum equations in conservative form

$$\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}(hu^2 + \frac{1}{2}gh^2) + \frac{\partial}{\partial y}(huv) = -gh\frac{\partial}{\partial x}b$$
$$\frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}(huv) + \frac{\partial}{\partial y}(hv^2 + \frac{1}{2}gh^2) = -gh\frac{\partial}{\partial y}b$$

Final form of the equations I

SWE I

$$\frac{\partial}{\partial t}h + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

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h : Water heigth

u, v : Depth-averaged velocity in x- and y-direction

b : Function describing the bottom profile

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Final form of the equations II

SWE II

$$\frac{\partial}{\partial t}Q + \frac{\partial}{\partial x}F(Q) + \frac{\partial}{\partial y}G(Q) = S_b(Q)$$

Q	$=$	$(h, hu, hv)^\top$	Vector of conservered variables
$F(Q)$	$=$	$(hu, hu^2 + \frac{1}{2}gh^2, huv)^\top$	Flux in x-direction
$G(Q)$	$=$	$(hv, huv, hv^2 + \frac{1}{2}gh^2)^\top$	Flux in y-direction
S_b	$=$	$(0, -ghb_x, -ghb_y)^\top$	Source term for bottom profile

Final form of the equations II

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$$\frac{\partial}{\partial t}Q + \frac{\partial}{\partial x}F(Q) + \frac{\partial}{\partial y}G(Q) = S_b(Q)$$

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S_b & = & (0, -ghb_x, -ghb_y)^\top & \text{Source term for bottom profile}
\end{array}$$

Extensions

- Coriolis forces
- Wind shear stress
- Multilayer models
- Additional transport equations (heat, dissolved substances, sediment particles)
- The Boussinesq Equations (to model shorter and steeper waves) can be considered as an extension to the SWE

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Extensions

- Coriolis forces
- Wind shear stress
- Multilayer models
- Additional transport equations (heat, dissolved substances, sediment particles)
- The Boussinesq Equations (to model shorter and steeper waves) can be considered as an extension to the SWE

Final remarks

- The SWE are not completely 2d
- The resulting 2d velocity field is not divergence-free
- There are 3d models, but as the detph-averaging is a very important part of the SWE, they are usually just called boundary layer models

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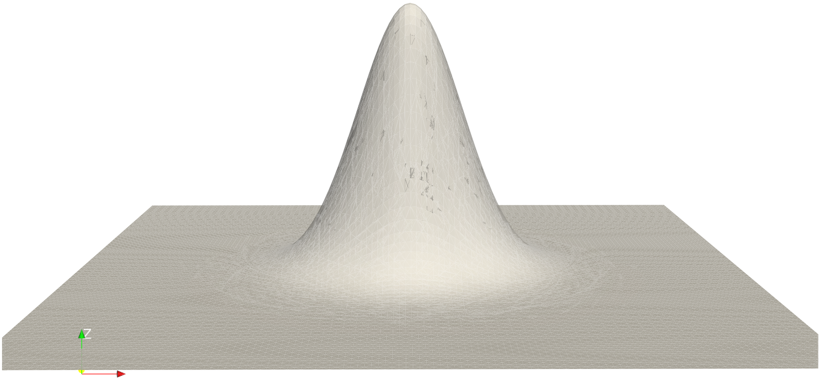
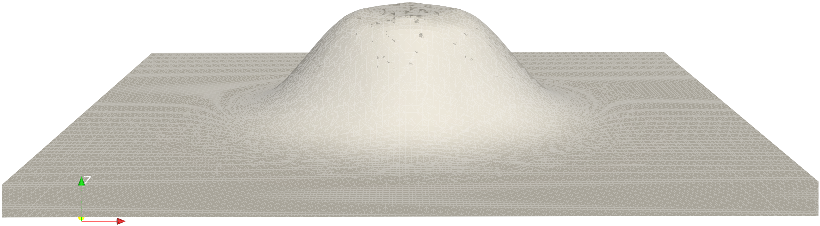
Melitta test case

Our first test case will show us, that the SWE can produce reasonable solutions

Its solution is well known from the famous tv ad of melitta :-)



Z



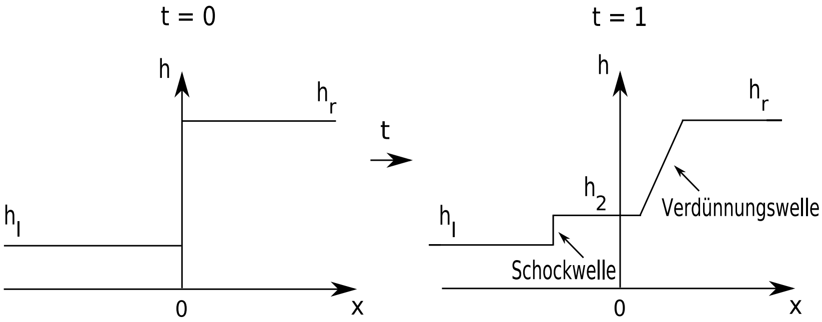
1d dambreak problem

$$\frac{\partial}{\partial t}\left[\begin{array}{c} h \\ hu \end{array}\right] + \frac{\partial}{\partial x}\left[\begin{array}{c} hu \\ hu^2 + \frac{1}{2}gh^2 \end{array}\right] = 0.$$

$$h_0 := \begin{cases} h_l & x < 0 \\ h_r & x > 0 \end{cases}$$

$$u_0 := 0$$

1d dambreak



There are 2d versions of the dambreak problem.

First we will consider a water revervoir break, then we will observe the solution of an asymmetric dambreak problem.

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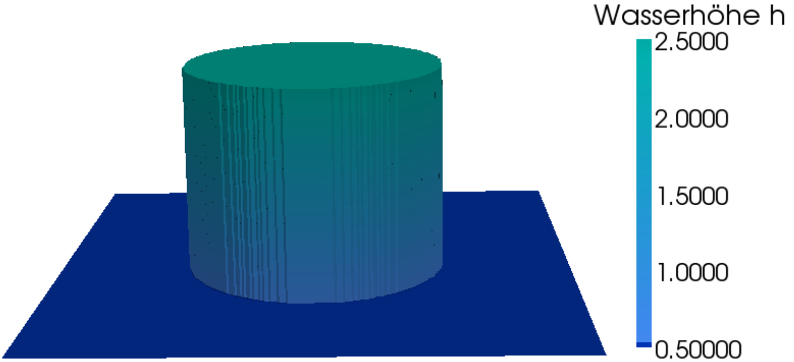
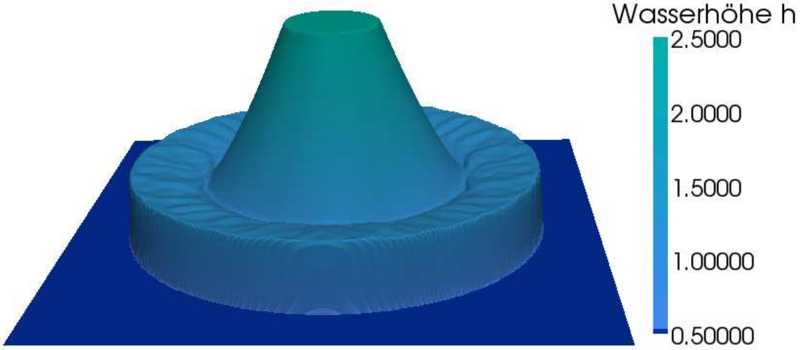
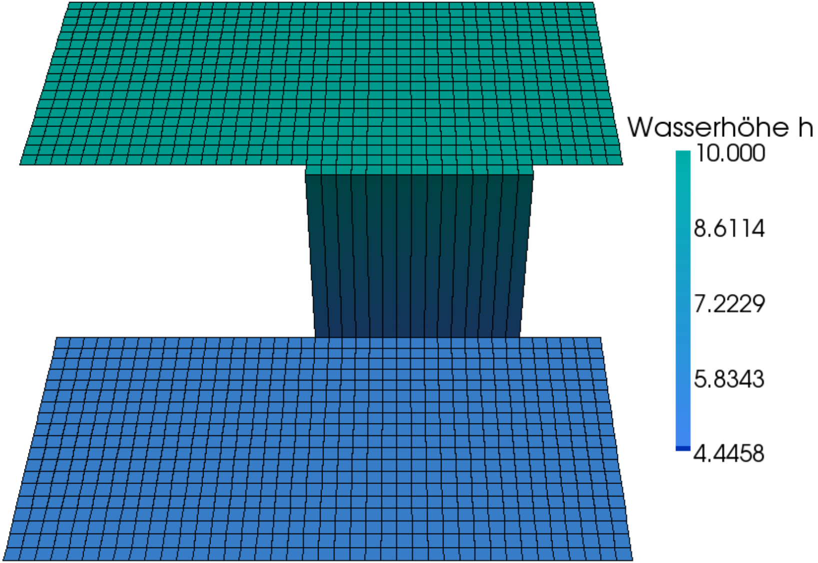
2d circular dambreak



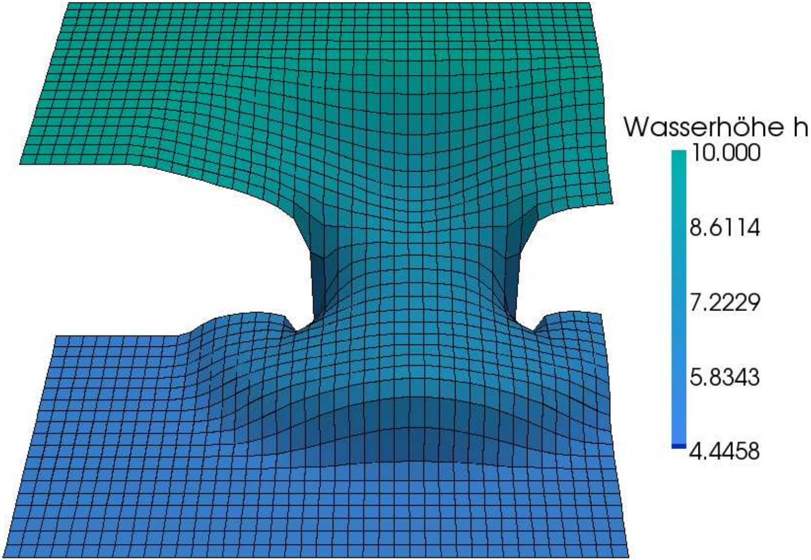
2d circular dambreak



2d asymmetric dambreak



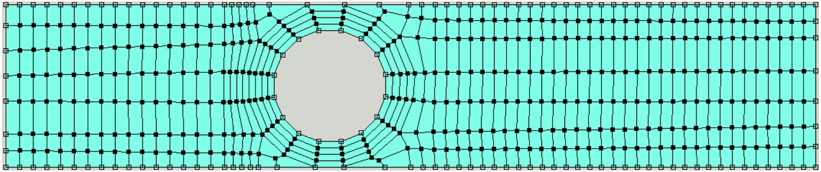
2d asymmetric dambreak



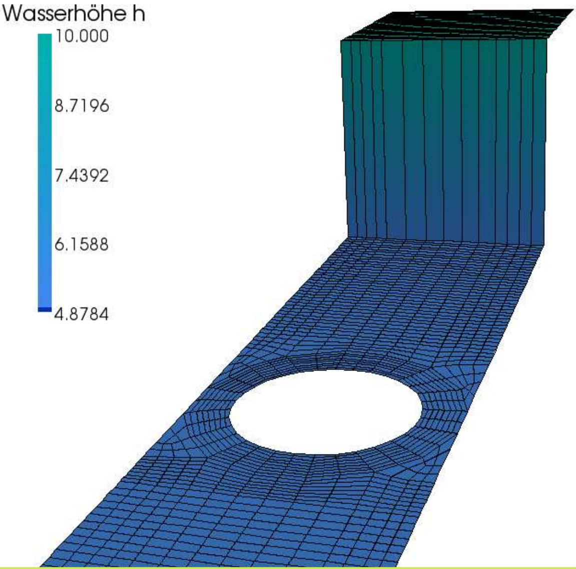
Flow around a pillar

Floodwave flowing around a pillar

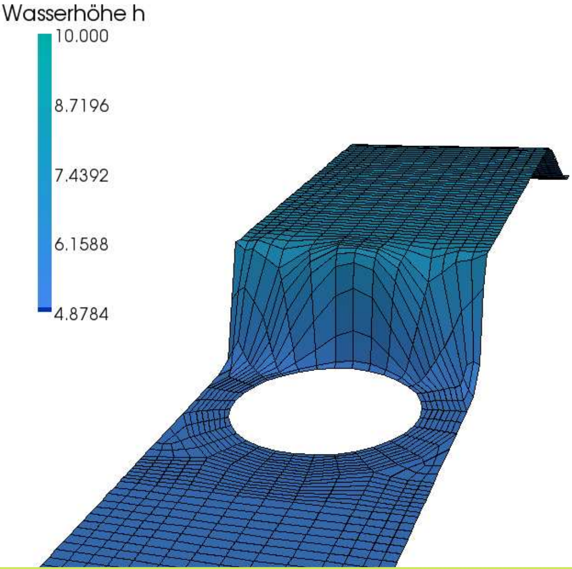
Computational grid



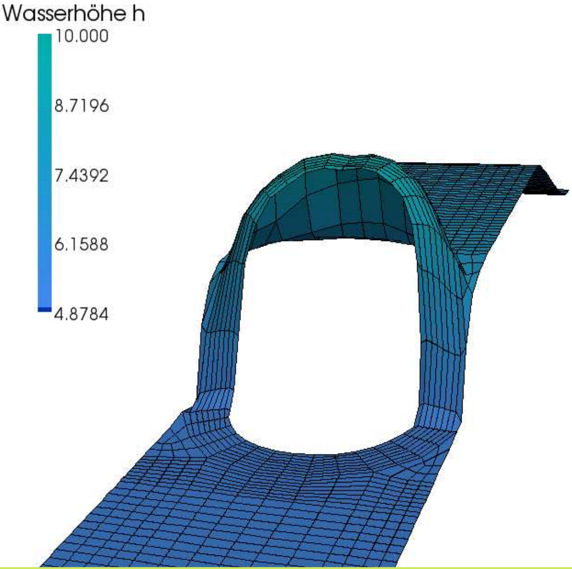
Flow around pillar



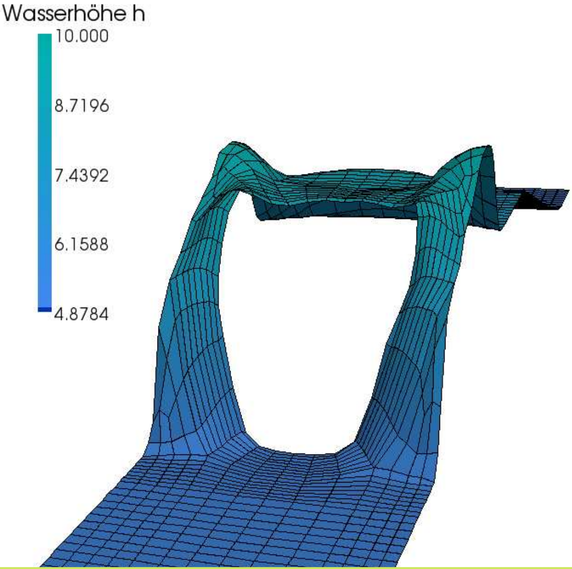
Flow around pillar



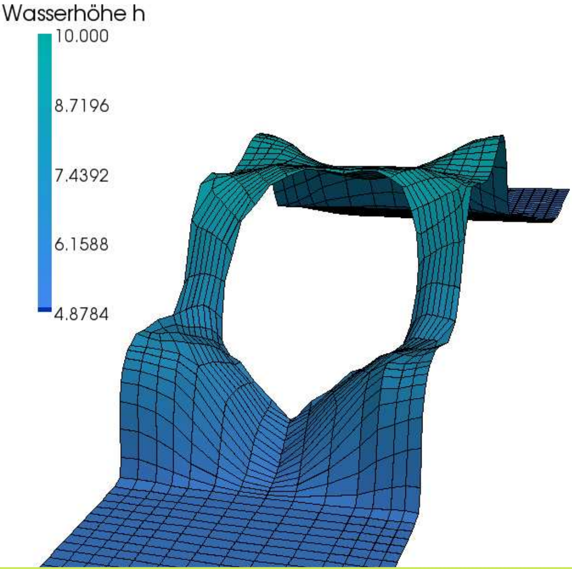
Flow around pillar



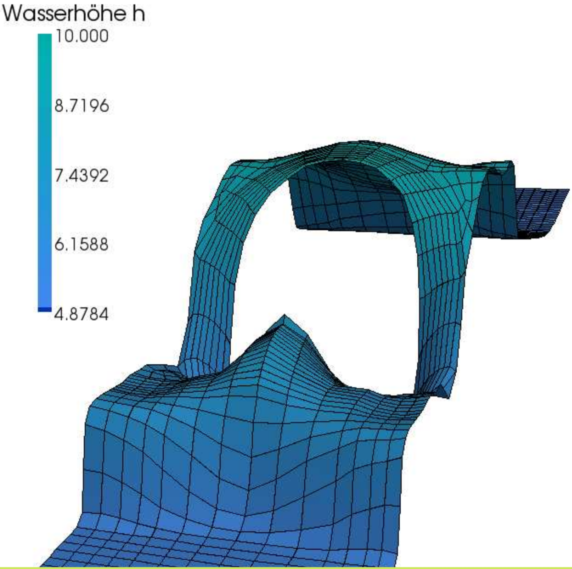
Flow around pillar



Flow around pillar



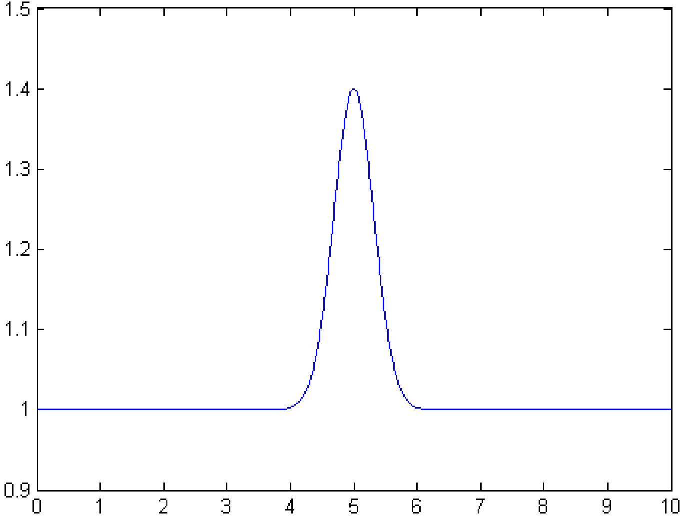
Flow around pillar

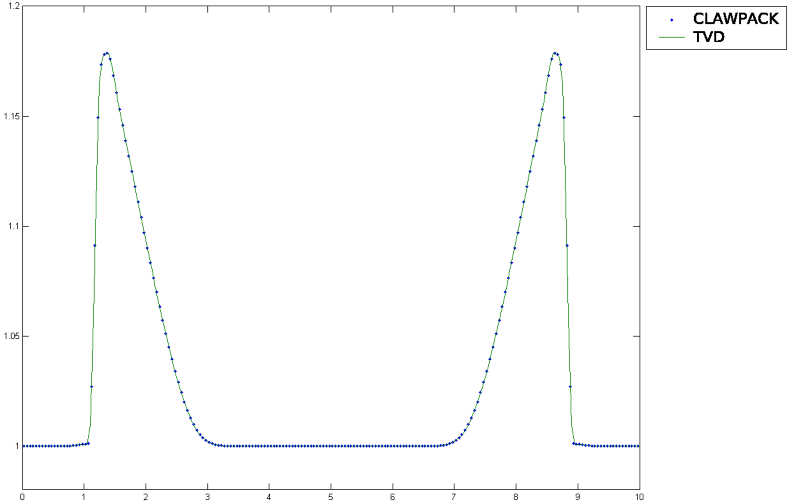


Nonlinear hyperbolic charakter

In our last test case we observe the nonlinear hyperbolic character of the SWE.

Nonlinear hyperbolic charakter




CLAWPACK
TVD
1.2
1.15
1.1
1.05
1
0
1
2
3
4
5
6
7
8
9
10

The end