Positive finite element schemes based on the flux-corrected transport procedure

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Abstract

A new approach to flux correction for finite elements is presented. The low-order convection operator is constructed from the discrete high-order operator by adding modulated dissipation so as to eliminate negative off-diagonal entries. Both explicit and implicit FEM-FCT schemes are proposed which are conservative and positivity preserving. A 2D example illustrates the performance of the algorithm.

Keywords: Finite element; Flux correction; Positivity; Unconditional stability

1. Introduction}

Flux correction has proved to be an effective way to eliminate spurious undershoots and overshoots in regions with steep gradients. However, the original FEM-FCT scheme of Löhner et al. [1] is subject to a restrictive CFL condition due to the explicit time-stepping. Moreover, it may produce nonphysical ripples in some cases. In this contribution, we follow a different approach by constructing the antidiffusive fluxes edge-by-edge and enforcing positivity of the numerical solution. In particular, this enables us to derive an implicit high-resolution finite element scheme which enjoys unconditional stability.

2. Multidimensional upwinding

To elucidate the basic ideas, we consider the pure convection equation $u_t + \mathbf{v} \cdot \nabla u = 0$. The lumped-mass Galerkin semi-discretization can be rendered positive and local extremum diminishing (LED) by adding modulated dissipation to the discrete convection operator:

$$m_i \frac{du_i}{dt} = \sum_j k_{ij}^H u_j + \sum_{j \neq i} d_{ij}(u_j - u_i) = \sum_j k_{ij}^L u_j, \qquad (1)$$

$$d_{ij} = \max\{0, -k_{ij}^H, -k_{ji}^H\}.$$

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This procedure is equivalent to upwinding in 1D and emulates it in multidimensions. Note that for linear or multilinear finite elements we have $\sum_j k_{ij}^L = \sum_j k_{ij}^H = 0$, since the sum of basis functions equals unity. If explicit time-stepping is employed, the low-order scheme (1) preserves positivity under a CFL-like condition $\Delta t \leq \min_i \left\{-m_i/k_{ii}^L \mid k_{ii}^L < 0\right\}$.

3. Explicit FEM-FCT scheme

Let the high-order scheme be a pseudo-Lax–Wendroff method, whereby the standard Galerkin formulation is stabilized by streamline diffusion inherent in a second-order time discretization. The positive end-of-step solution u^{n+1} is given by

$$\begin{aligned} m_i u_1^{n+1} &= m_i u_i^* + \sum_{j \neq i} \alpha_{ij} f_{ij}, \\ m_i u_i^* &= m_i u_i^n + \Delta t \sum_j k_{ij}^L u_j^n, \\ f_{ij} &= m_{ij}(\Delta u_i^H - \Delta u_j^H) + \Delta t\, d_{ij}(u_i^n - u_j^n) \\ &\quad - \frac{(\Delta t)^2}{2} s_{ij}(u_i^n - u_j^n), \quad f_{ij} = -f_{ij}. \end{aligned} \qquad (2)$$

Here m_{ij} and s_{ij} correspond to entries of the consistent mass matrix and of the streamline diffusion operator, respectively. The superscript H refers to the high-order solution, and Δu denotes the solution increment. The antidiffusive fluxes f_{ij} offset the error induced by the mass lumping, ‘upwinding’, and first-order time discretization. Such representation is possible because the matrices at hand have zero row- and column sums. The coefficients α_{ij} are chosen so as to preclude the arising of nonphysical extrema (see below).

4. Implicit FEM-FCT scheme

Since the fully implicit Galerkin formulation is unconditionally stable, it can be adopted as the high-order method. Hence, an implicit FCT scheme can be constructed in the form

$$m_i u_i^{n+1} - \Delta t \sum_j k_{ij}^L u_j^{n+1} = m_i u_i^* + \sum_{j \neq i} \alpha_{ij} f_{ij}, \qquad u_i^* = u_i^n,$$

$$f_{ij} = m_{ij}(\Delta u_i^H - \Delta u_j^H) + \Delta t\, d_{ij}(u_i^H - u_j^H), \qquad f_{ij} = -f_{ij}. \tag{3}$$

The low-order convection operator was designed so that it poses no threat to positivity as long as it is treated implicitly. Now we proceed to the limiting strategy for α_{ij}.

5. Flux correction

The limiting process is essentially the same for both schemes. It starts with cancelling all antidiffusive fluxes directed down the gradient, i.e. such that $f_{ij}(u_i^* - u_j^*) < 0$. This crucial prelimiting step, which was missing in [1], can be traced back to the classical finite difference FCT schemes. It was found to suppress ripples which may arise otherwise [2].

If the flux f_{ij} tries to accentuate a local extremum $u_i^* = u_i^{\substack{\max \\ \min}}$, then it also has to be cancelled. Otherwise, the right-hand side of our FCT schemes can be written as

$$RHS = m_i u_i^* + \underbrace{\frac{\sum_{j \neq i} \alpha_{ij} f_{ij}}{Q_i}}_{\geq 0} Q_i,$$

$$Q_i = \begin{cases} Q_i^+ = u_i^{\max} - u_i^* & \text{if } \sum_{j \neq i} \alpha_{ij} f_{ij} \geq 0, \\ Q_i^- = u_i^{\min} - u_i^* & \text{otherwise.} \end{cases}$$

The above representation reveals that the positivity of u^* is preserved if

$$m_i Q_i^- \leq R_i^- P_i^- \leq \sum_{j \neq i} \alpha_{ij} f_{ij} \leq R_i^+ P_i^+ \leq m_i Q^+,$$

$$P_i^{\pm} = \sum_{j \neq i} \substack{\max \\ \min} \{0, f_{ij}\}.$$

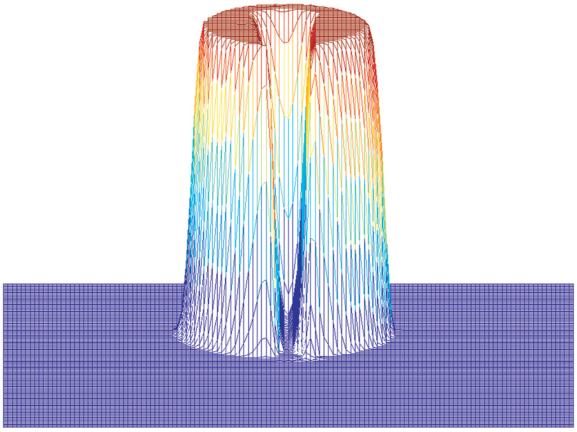

Fig. 1. Cylinder with a slot after one full revolution. 128×128 Q_1 elements.

Hence, the auxiliary quantities $R_i^{\pm}$ and correction factors α_{ij} are defined as follows:

$$R_i^{\pm} = \begin{cases} m_i Q_i^{\pm} / P_i^{\pm} & \text{if } P_i^{\pm} \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\alpha_{ij} = \begin{cases} \min\{1, R_i^+, R_j^-\} & \text{if } f_{ij} \geq 0, \\ \min\{1, R_j^+, R_i^-\} & \text{otherwise.} \end{cases}$$

6. Conclusions

A methodology for constructing and limiting edge-based antidiffusive fluxes for finite element FCT schemes was presented. The algorithm preserves positivity, conserves mass and provides a sharp resolution of discontinuities (see Fig. 1). The proposed implicit scheme is based on the backward Euler time discretization, so it is slightly more diffusive than its explicit counterpart. At the same time, it is unconditionally stable and lends itself to treatment of problems with strongly varying velocities and/or mesh sizes.

References

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