# Level Set Extrapolation of Immersed Boundary Data in Unfitted Finite Element Methods for Deformable Interfaces

Dmitri Kuzmin, Jan-Phillip Bäcker
TU Dortmund University, Institute of Applied Mathematics

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# State of the art

Sharp interface formulation

- XFEM, unfitted Nitsche FEM, CutFEM
- small cut cells may cause ill conditioning
- severe time step restrictions or instability

Surrogate interface formulation

- shifted boundary method
- extrapolation of boundary conditions to mesh
- complex closest-point projection algorithms

Diffuse interface formulation

- immersed boundary, phase field methods
- construction of approximate delta functions

## Level set function

Interface $\Gamma(t) = \{\mathbf{x} \in \Omega : \phi(\mathbf{x}, t) = 0\}$ determined by $\phi(\mathbf{x}, t)$ such that

- $\phi > 0$ in $\Omega_+(t)$, $\Omega_+(t)$ remains fully embedded in $\Omega$
- $\phi < 0$ in $\overline{\Omega} \backslash \overline{\Omega_+(t)}$,
- $\mathbf{n}_\pm = \mp \frac{\nabla \phi}{|\nabla \phi|}$ extended unit outward normal

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## Linear transport equation

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = 0 \quad \text{in } \Omega$$

## Conservation laws in evolving domains

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla \cdot [\mathbf{f}(u) - \kappa \nabla u] &= 0 && \text{in } \Omega_+(t), \\ [\mathbf{f}(u) - \kappa \nabla u] \cdot \mathbf{n}_+ &= g_\Gamma(u, u_\Gamma) && \text{on } \Gamma(t), \\ u &= u_0 && \text{in } \Omega_+(0) \end{aligned}$$

$u(\mathbf{x}, t) \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^d$, $d \in \{2, 3\}$, $t \geq 0$,

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$u(\mathbf{x}, t) \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^d$, $d \in \{2, 3\}$, $t \geq 0$,

$$g_\Gamma(u, u_\Gamma)(\mathbf{x}) = f_{\mathbf{n}}(u(\mathbf{x}), u_\Gamma(\mathbf{x})) - \kappa \lim_{\varepsilon \to 0} \frac{u_\Gamma(\mathbf{x}) - u(\mathbf{x} - \varepsilon \mathbf{n}_+(\mathbf{x}))}{\varepsilon}$$

# Outer condition

## Dirichlet-type condition

$$s_D(w,u) = \int_{\Omega\setminus\Omega_+(t)} \gamma_\Omega w(u-u_-)\mathrm{d}\mathbf{x}$$

$\Rightarrow$ weak imposition of $u = u_-$ in $\Omega\setminus\overline{\Omega_+(t)}$

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$\Rightarrow$ weak imposition of $u = u_-$ in $\Omega\setminus\overline{\Omega_+(t)}$

## Neumann-type condition

$$s_N(w,u) = \int_{\Omega\setminus\Omega_+(t)} \gamma_\Omega \nabla w \cdot (\nabla u - \mathbf{g}_-)\mathrm{d}\mathbf{x}$$

$\Rightarrow$ harmonic extension into $\Omega\setminus\overline{\Omega_+(t)}$

# Fictitious domain formulation

## Weak form

$$\int_{\Omega_+(t)} w \frac{\partial u}{\partial t} \mathrm{d}\mathbf{x} - \int_{\Omega_+(t)} \nabla w \cdot [\mathbf{f}(u_\Gamma) - \kappa \nabla u] \mathrm{d}\mathbf{x} + s(w, u)$$
$$+ \int_{\Gamma(t)} w[\mathbf{f}(u_\Gamma) \cdot \mathbf{n}_+ - \kappa \partial_n u] \mathrm{d}s = 0$$

# Fictitious domain formulation

## Weak form

$$\int_{\Omega_+(t)} w\frac{\partial u}{\partial t}\mathrm{d}\mathbf{x} - \int_{\Omega_+(t)} \nabla w \cdot [\mathbf{f}(u_\Gamma) - \kappa\nabla u]\mathrm{d}\mathbf{x} + s(w,u) \\ + \int_{\Gamma(t)} w[\mathbf{f}(u_\Gamma)\cdot\mathbf{n}_+ - \kappa\partial_n u]\mathrm{d}s = 0$$

## Fictitious domain formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega H(\phi)wu\mathrm{d}\mathbf{x} - \int_\Omega H(\phi)\nabla w \cdot [\mathbf{f}(u_\Gamma) - \kappa\nabla u]\mathrm{d}\mathbf{x} + s(w,u) \\ + \int_{\Gamma(t)} w[\mathbf{f}(u_\Gamma)\cdot\mathbf{n}_+ - v_n u_\Gamma - \kappa\partial_n u]\mathrm{d}s = 0, \quad \forall w \in V(\Omega)$$

# Fictitious domain formulation

$u_\Omega = H(\phi)u + (1 - H(\phi))u_-$

## Dirichlet-type ghost penalties

$$s_D(w, u) = \int_\Omega \gamma_\Omega w(u - u_\Omega)\mathrm{d}\mathbf{x}$$

## Fictitious domain formulation

$$u_\Omega = H(\phi)u + (1 - H(\phi))u_-$$

### Dirichlet-type ghost penalties

$$s_D(w, u) = \int_\Omega \gamma_\Omega w(u - u_\Omega)\mathrm{d}\mathbf{x}$$

$$\mathbf{g}_\Omega = H(\phi)\nabla u + (1 - H(\phi))\mathbf{g}_-$$

### Neumann-type ghost penalties

$$s_N(w, u) = \int_\Omega \gamma_\Omega \nabla w \cdot (\nabla u - \mathbf{g}_\Omega)\mathrm{d}\mathbf{x}$$

## Dirichlet penalty form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega H_\varepsilon(\phi) w u \,\mathrm{d}\mathbf{x} - \int_\Omega H_\varepsilon(\phi)\nabla w \cdot [\mathbf{f}(u) - \kappa \nabla u]\mathrm{d}\mathbf{x} + \int_\Omega \gamma_\Omega w (u - u_\Omega)\mathrm{d}\mathbf{x}$$
$$+ \int_\Omega w G(\phi, u, u_\Gamma)\delta_\varepsilon |\nabla \phi| \mathrm{d}\mathbf{x} = 0, \quad \forall w \in V(\Omega)$$

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$$+\int_\Omega wG(\phi,u,u_\Gamma)\delta_\varepsilon|\nabla\phi|\mathrm{d}\mathbf{x} = 0, \quad \forall w\in V(\Omega)$$

## Gaussian regularization

$$H_\varepsilon(\phi) = \frac{1}{2}\left(1+\operatorname{erf}\left(\frac{\pi\phi}{3\varepsilon}\right)\right), \qquad \delta_\varepsilon = H_\varepsilon'(\phi) = \frac{1}{\varepsilon}\sqrt{\frac{\pi}{9}}\exp\left(\frac{-\pi^2\phi^2}{9\varepsilon^2}\right)$$

To be defined: continuous extensions $U, \partial_n U, V$ for calculating

$$G(\phi_h, u_h, u_\Gamma) = F - \kappa \partial_n U - VU, \qquad \mathbf{v}_h = -V\mathbf{q}_h, \qquad u_{\Omega,h}(u_h, U)$$

# Extrapolation using level sets

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Main steps:

1. closest-point search
2. gradient reconstruction
3. extrapolation

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Main steps:

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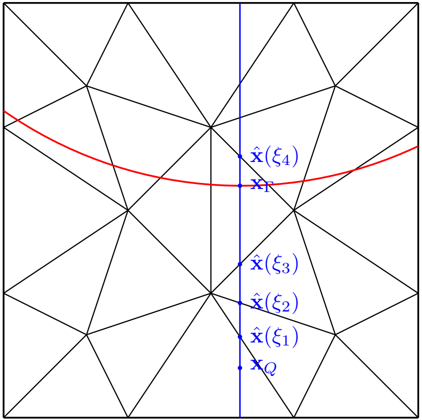
Requirements: simplicity, efficiency, accuracy

## Closest point search

### Interface pointer

$$\mathbf{n}_Q := -\mathbf{q}_h(\mathbf{x}_Q) \approx \mathbf{n}_+(\mathbf{x}_Q)$$



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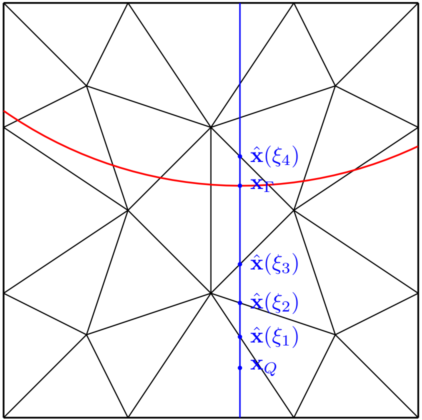
Exact distance function $\phi$

$$\mathbf{x}_\Gamma := \mathbf{x}_Q + \phi(\mathbf{x}_Q)\mathbf{n}_Q$$



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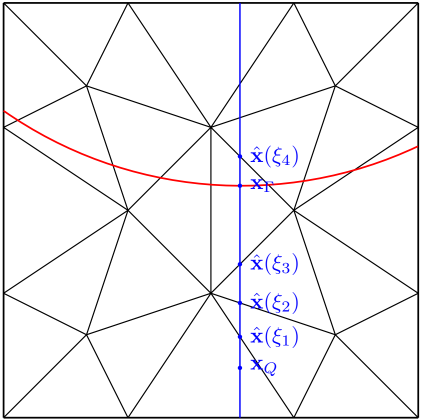
Numerical approximation $\phi_h$

$$\hat{\mathbf{x}}(\xi) = \mathbf{x}_Q + \xi \operatorname{sign}(\phi_h(\mathbf{x}_Q))\mathbf{n}_Q, \quad \xi \in \mathbb{R}$$

$$\phi_h(\mathbf{x}_\Gamma) = 0 \text{ at } \mathbf{x}_\Gamma = \hat{\mathbf{x}}(\xi_\Gamma)$$



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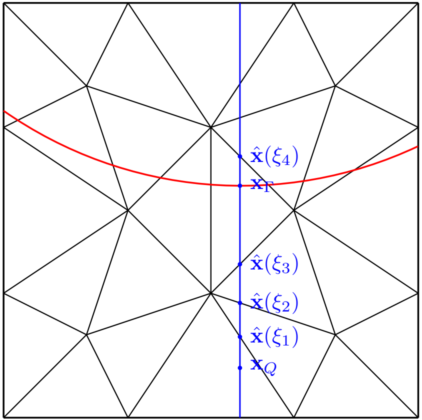
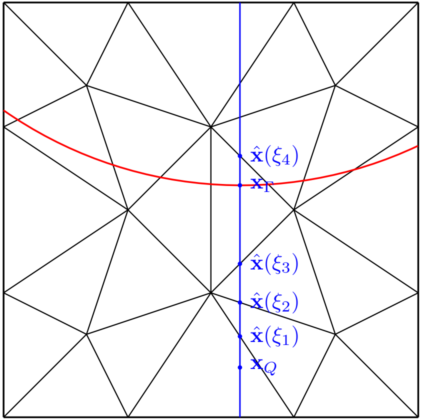
$\Rightarrow$ simple line search



# Closest point search

## Search algorithm

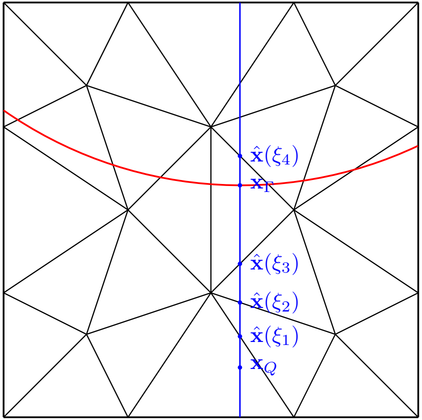
- Set $\xi_0 = 0$



# Closest point search

Search algorithm

- Set $\xi_0 = 0$
- For $i > 1$: Find next intersection $\hat{\mathbf{x}}(\xi_i)$, $\xi_i > \xi_{i-1}$ of $\hat{\mathbf{x}}(\xi)$ with boundary of a mesh cell boundary



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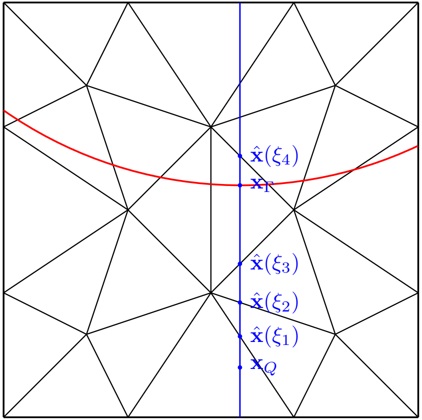

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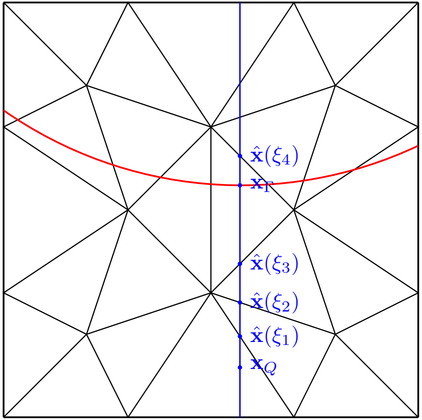
- Solve linear/quadratic equation to find root $\xi_\Gamma \in [\xi_{m-1}, \xi_m]$ of $\phi(\hat{\mathbf{x}}(\xi))$



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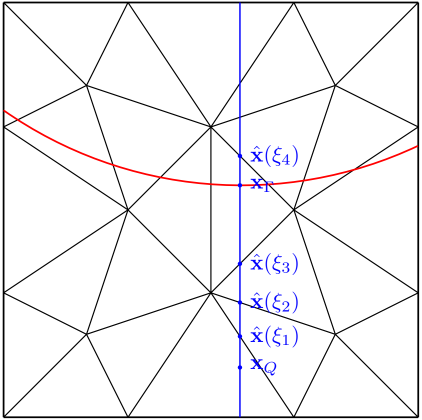
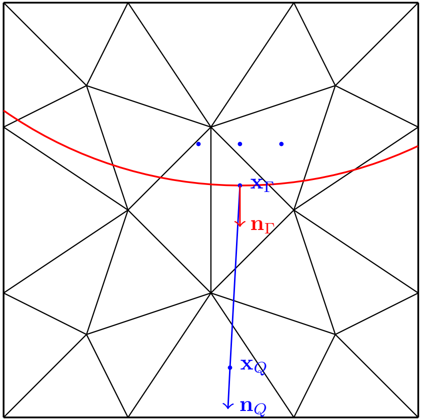
- Set $\mathbf{x}_\Gamma = \hat{\mathbf{x}}(\xi_\Gamma)$



## Unit outward normal vector

$$\mathbf{n}_\Gamma = -\frac{\mathbf{q}_h(\mathbf{x}_\Gamma)}{|\mathbf{q}_h(\mathbf{x}_\Gamma)|} \approx \mathbf{n}_+(\mathbf{x}_\Gamma)$$



Unit outward normal vector

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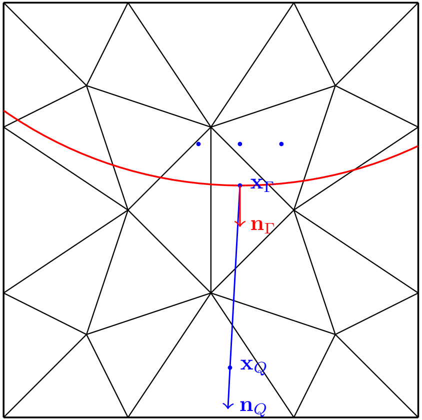
Three point interpolation stencil:

$$\mathcal{S}_{2\mathrm{D}}(\mathbf{x}_\Gamma) = \{\mathbf{x}_P - 0.5\varepsilon\boldsymbol{\tau}_\Gamma, \mathbf{x}_P, \mathbf{x}_P + 0.5\varepsilon\boldsymbol{\tau}_\Gamma\}$$

$$\mathbf{x}_P = \mathbf{x}_\Gamma - \varepsilon\mathbf{n}_\Gamma, \quad \boldsymbol{\tau}_\Gamma \perp \mathbf{n}_\Gamma$$



# Gradient reconstruction

Unit outward normal vector

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Three point interpolation stencil:

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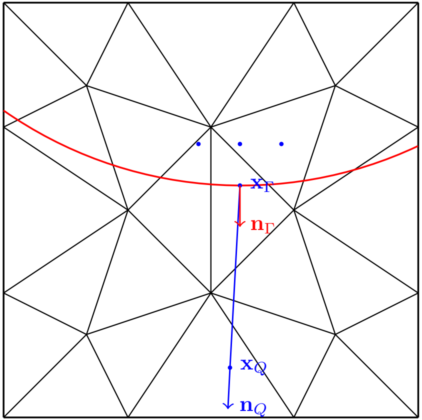

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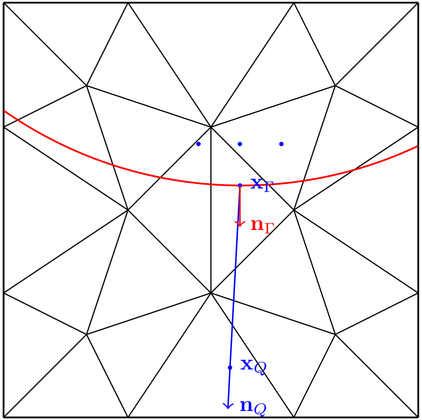
Approximate normal derivative

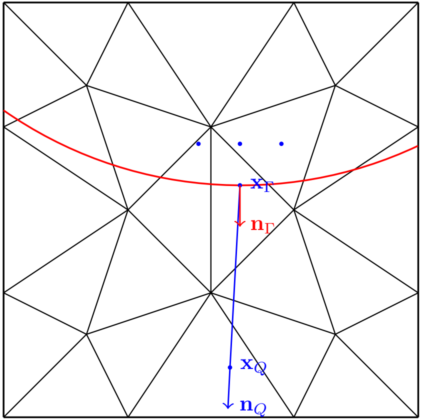
$$\partial_n U(\mathbf{x}_\Gamma) = \frac{u_\Gamma(\mathbf{x}_\Gamma) - u_h(\mathbf{x}_\Gamma - \varepsilon\mathbf{n}_\Gamma)}{\varepsilon}$$



For $\mathbb{P}_1$ or $\mathbb{Q}_1$ elements:

**Normal derivatives: constant extrapolation**

$$\partial_n U(\mathbf{x}_Q) = \partial_n U(\mathbf{x}_\Gamma)$$



# Extrapolation of interface data

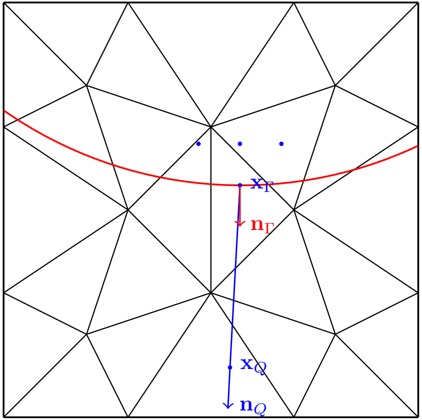
For $\mathbb{P}_1$ or $\mathbb{Q}_1$ elements:

**Normal derivatives: constant extrapolation**

$$\partial_n U(\mathbf{x}_Q) = \partial_n U(\mathbf{x}_\Gamma)$$

**Solution values: linear extrapolation**

$$U(\mathbf{x}_Q) = u_\Gamma(\mathbf{x}_\Gamma) + \nabla U(\mathbf{x}_\Gamma) \cdot (\mathbf{x}_Q - \mathbf{x}_\Gamma)$$



# Elliptic test

Fixed embedded boundary

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : (x - 0.5)^2 + (y - 0.5)^2 = 0.0625\} = \partial\Omega_+$$

## Elliptic test

$$\begin{aligned} \Delta u = 0 \quad & \text{in } \Omega_+, \\ u = u_\Gamma \quad & \text{on } \Gamma \end{aligned}$$

$$u_\Gamma(x, y, t) = u_{ex}(x, y, t) = (x - 0.5)^2 - (y - 0.5)^2$$

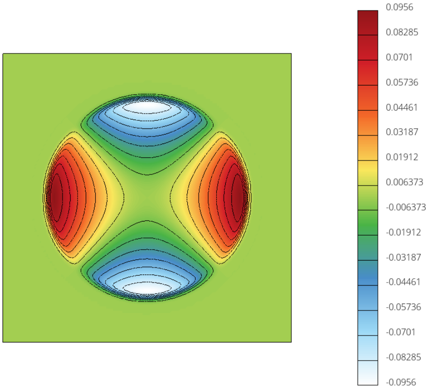
- Dirichlet ghost penalties with $\gamma_{\Omega,D} = h^{-1}$
- Neumann ghost penalties with $\gamma_{\Omega,N} = h$

# Elliptic test

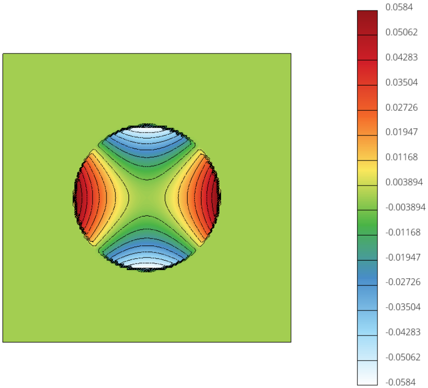
| $h^{-1}$ | full NGP | EOC | damped NGP | EOC | full DGP | EOC | damped DGP | EOC |
|---|---|---|---|---|---|---|---|---|
| 16 | 3.61e-03 | | 3.61e-03 | | 2.28e-03 | | 2.28e-03 | |
| 32 | 1.05e-03 | 1.78 | 1.06e-03 | 1.77 | 6.87e-04 | 1.73 | 6.87e-04 | 1.73 |
| 64 | 2.45e-04 | 2.10 | 2.61e-04 | 2.02 | 1.72e-04 | 2.00 | 1.72e-04 | 2.00 |
| 128 | 5.50e-05 | 2.15 | 6.03e-05 | 2.11 | 3.97e-05 | 2.11 | 3.97e-05 | 2.11 |
| 256 | 1.25e-05 | 2.14 | 1.39e-05 | 2.12 | 8.70e-06 | 2.19 | 8.70e-06 | 2.19 |
| 512 | 3.06e-06 | 2.03 | 3.42e-06 | 2.03 | 1.95e-06 | 2.15 | 1.95e-06 | 2.15 |
| 1024 | 7.37e-07 | 2.05 | 8.46e-07 | 2.01 | 4.21e-07 | 2.22 | 4.21e-07 | 2.22 |

# Elliptic test

(a) $u_h$, damped DGP




(b) $H_\epsilon(\phi)u_h$, damped DGP



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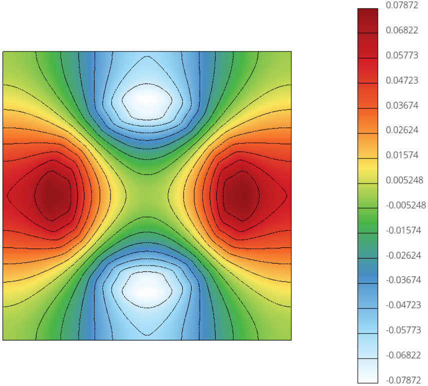
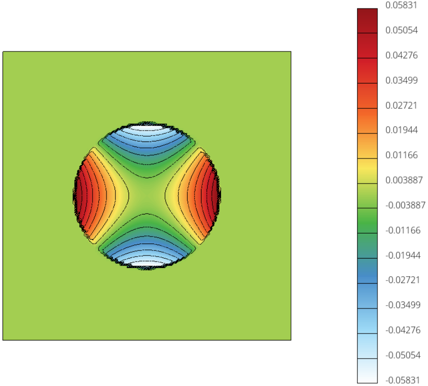



(a) $u_h$, damped NGP




(b) $H_\epsilon(\phi)u_h$, damped NGP

Moving embedded boundary

$$\Gamma(t) = \{(x, y) \in \mathbb{R}^2 : (x - 0.5)^2 + (y - 0.5)^2 = (0.25 + 0.15t)^2\} = \partial\Omega_+(t)$$

**Parabolic test**

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u = 0 &\quad \text{in } \Omega_+(t), \\ u = u_\Gamma &\quad \text{on } \Gamma(t), \\ u = u_0 &\quad \text{in } \Omega_+(0) \end{aligned}$$

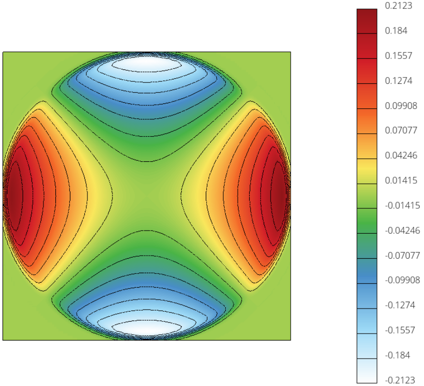
$$u_\Gamma(x, y, t) = u_0(x, y) = u_{ex}(x, y, t) = (x - 0.5)^2 - (y - 0.5)^2$$

- Dirichlet ghost penalties with $\gamma_{\Omega,D} = h^{-1}$
- Crank-Nicolson scheme

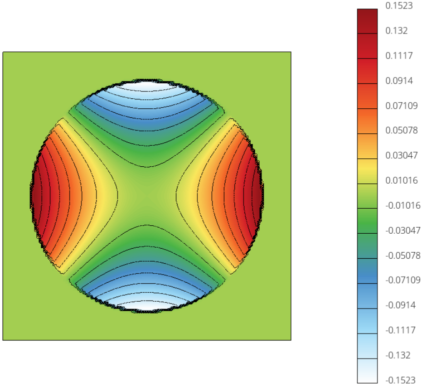
# Parabolic test

| $h^{-1}$ | $(\Delta t)^{-1}$ | full DGP | EOC | damped DGP | EOC |
|---|---|---|---|---|---|
| 16 | 5 | 6.30e-03 | | 6.30e-03 | |
| 32 | 10 | 1.16e-03 | 2.44 | 1.16e-03 | 2.44 |
| 64 | 20 | 2.86e-04 | 2.02 | 2.86e-04 | 2.02 |
| 128 | 40 | 6.95e-05 | 2.04 | 6.95e-05 | 2.04 |
| 256 | 80 | 1.67e-05 | 2.06 | 1.67e-05 | 2.06 |
| 512 | 160 | 4.24e-06 | 1.98 | 4.24e-06 | 1.98 |
| 1024 | 320 | 1.10e-06 | 1.95 | 1.10e-06 | 1.95 |

(a) $u_h$, damped DGP




(b) $H_\epsilon(\phi)u_h$, damped DGP



Fixed embedded boundary

$$\Gamma = \{(x,y) \in \mathbb{R}^2 : (x-0.75)^2 + (y-0.5)^2 = 0.0225\} = \partial\Omega_+$$

$$\mathbf{v}(x,y) = (0.5-y, x-0.5)^T$$

### Hyperbolic test

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u) = 0 \quad & \text{in } \Omega_+, \\ u = u_\Gamma \quad & \text{on } \Gamma_{\text{in}}, \\ u = u_0 \quad & \text{in } \Omega_+ \end{aligned}$$

$$u_\Gamma(x,y,t) = u_0(x,y,t) = u_{ex}(x,y,t) = (x-0.5)^2 + (y-0.5)^2$$

# Hyperbolic test

- Extended upwind flux

$$F_Q(t) = V_Q \hat{U}(x_Q, y_Q, t)$$

$$V_Q = \mathbf{v}(x_\Gamma, y_\Gamma) \cdot \mathbf{n}_\Gamma$$

- Dirichlet ghost penalties with $\gamma_{\Omega,D} = h^{-1}$ and $\hat{U}(x_Q, y_Q, t)$ linear extrapolation of

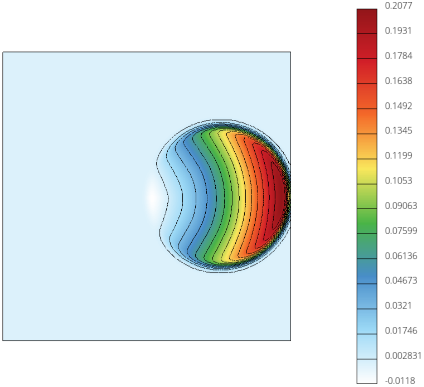
$$\hat{u}(x_\Gamma, y_\Gamma, t) = \begin{cases} u_\Gamma(x_\Gamma, y_\Gamma, t) & \text{if } \mathbf{v}(x_\Gamma, y_\Gamma) \cdot \mathbf{n}_\Gamma < 0, \\ u_h(x_\Gamma, y_\Gamma, t) & \text{if } \mathbf{v}(x_\Gamma, y_\Gamma) \cdot \mathbf{n}_\Gamma \geq 0. \end{cases}$$

- Heun's method with $\Delta t = 0.2h$

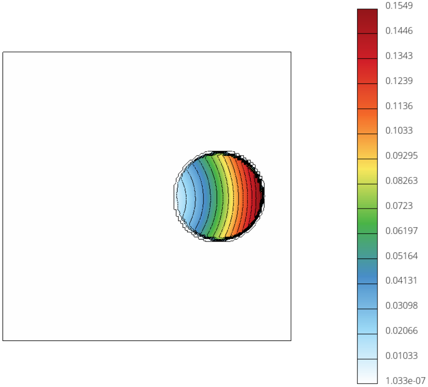
# Hyperbolic test

| $h^{-1}$ | $(\Delta t)^{-1}$ | full DGP | EOC | damped DGP | EOC |
|---|---|---|---|---|---|
| 16 | 32 | 4.11e-03 | | 4.11e-03 | |
| 32 | 64 | 2.35e-03 | 0.81 | 2.35e-03 | 0.81 |
| 64 | 128 | 1.24e-03 | 0.92 | 1.24e-03 | 0.92 |
| 128 | 256 | 6.45e-04 | 0.95 | 6.45e-04 | 0.95 |
| 256 | 512 | 3.30e-04 | 0.97 | 3.30e-04 | 0.97 |
| 512 | 1024 | 1.67e-04 | 0.98 | 1.67e-04 | 0.98 |
| 1024 | 2048 | 8.40e-05 | 0.99 | 8.40e-05 | 0.99 |

(a) $u_h$, damped DGP




(b) $H_\epsilon(\phi)u_h$, damped DGP



## Summary

- approximation of surface integrals by volume integrals
- fast closest-point search algorithm
- new way to define and calculate compact-support extensions
- narrow-band integration of terms involving extended fluxes, ghost penalties and velocity fields

# Conclusions

## Summary

- approximation of surface integrals by volume integrals
- fast closest-point search algorithm
- new way to define and calculate compact-support extensions
- narrow-band integration of terms involving extended fluxes, ghost penalties and velocity fields

## Outlook

- theoretical studies of proposed approach
- new approximate delta functions with compact support
- application to interface problems and PDE systems

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# Thank you for your attention!

jan-phillip.baecker@tu-dortmund.de