technische universität dortmund

Mathematical Models for Bio-Medical Sciences

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Level Set Extrapolation of Immersed Boundary Data in Unfitted Finite Element Methods for Deformable Interfaces

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Motivation

Problems with moving boundaries:
- tumor growth
- two phase flows
- moving objects

⇒ mesh adaption or unfitted meshes

Types of boundary motion:
- translation
- deformation
- expansion

Problem statement

Interface description: $\Gamma(t) = \{\mathbf{x} \in \Omega : \phi(\mathbf{x}, t) = 0\}$

$$\phi > 0 \text{ in } \Omega_+(t), \quad \overline{\Omega_+(t)} \subset \Omega \, \forall t$$

Level set advection

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = 0 \quad \text{in } \Omega$$

Conservation laws in evolving domains

$$\frac{\partial u}{\partial t} + \nabla \cdot [\mathbf{f}(u) - \kappa \nabla u] = 0 \qquad \text{in } \Omega_+(t),$$

$$[\mathbf{f}(u) - \kappa \nabla u] \cdot \mathbf{n}_+ = g_\Gamma(u, u_\Gamma) \qquad \text{on } \Gamma(t),$$

$$u = u_0 \qquad \text{in } \Omega_+(0)$$

Ghost penalites: extension into $\Omega \backslash \overline{\Omega_+(t)}$

Dirichlet penalization

$$s_D(w, u) = \int_{\Omega \backslash \Omega_+(t)} \gamma_\Omega w (u - u_-) \mathrm{d}\mathbf{x}$$

Neumann penalization

$$s_N(w, u) = \int_{\Omega \backslash \Omega_+(t)} \gamma_\Omega \nabla w \cdot (\nabla u - \mathbf{g}_-) \mathrm{d}\mathbf{x}$$

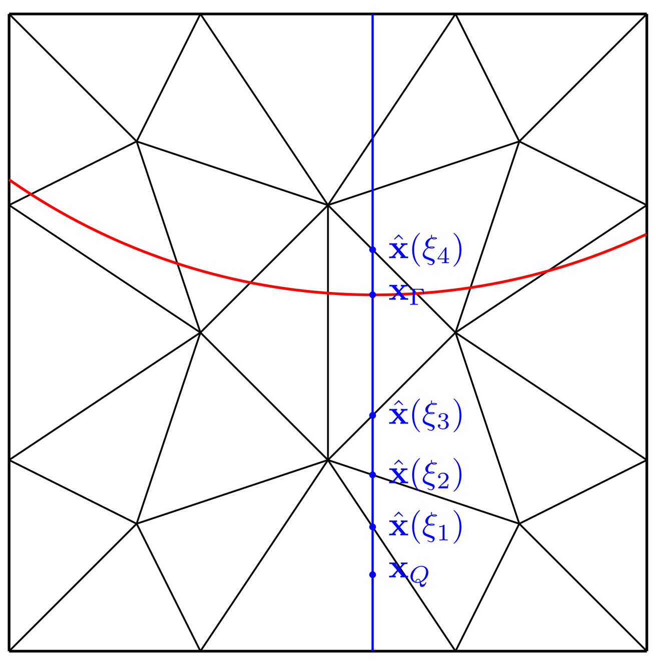
Fictitious domain formulation

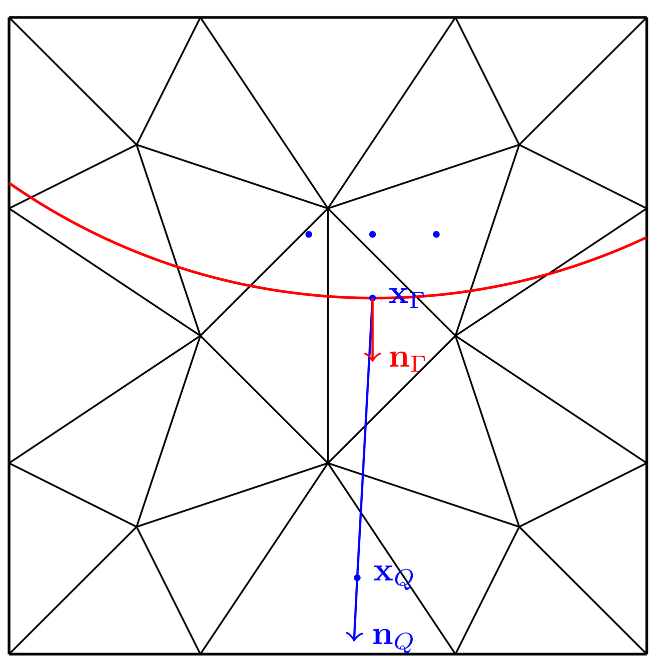
$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega H_\varepsilon(\phi) w u \mathrm{d}\mathbf{x} - \int_\Omega \nabla H_\varepsilon(\phi) w \cdot [\mathbf{f}(u) - \kappa \nabla u] \mathrm{d}\mathbf{x} + s(w, u)$$

$$+ \int_\Omega w G(\phi, u, u_\Gamma) \delta_\varepsilon(\phi) |\nabla \phi| \mathrm{d}s = 0, \quad \forall w \in V(\Omega)$$

Calculation of interface data

Search algorithm
- Set $\xi_0 = 0$
- For $i > 1$: Find next intersection $\hat{\mathbf{x}}(\xi_i)$, $\xi_i > \xi_{i-1}$ of $\hat{\mathbf{x}}(\xi)$ with a mesh cell boundary
- If $\phi(\hat{\mathbf{x}}(\xi_i))\phi(\hat{\mathbf{x}}(\xi_{i-1})) < 0$ for $i = m$ exit loop
- Solve linear/quadratic equation to find the root $\xi_\Gamma \in [\xi_{m-1}, \xi_m]$ of $\phi(\hat{\mathbf{x}}(\xi))$
- Set $\mathbf{x}_\Gamma = \hat{\mathbf{x}}(\xi_\Gamma)$





Gradient Reconstruction

Three point interpolation stencil:

$$\mathcal{S}_{2D}(\mathbf{x}_\Gamma) = \{\mathbf{x}_P - 0.5\varepsilon \boldsymbol{\tau}_\Gamma, \mathbf{x}_P, \mathbf{x}_P + 0.5\varepsilon \boldsymbol{\tau}_\Gamma\}$$

$$\mathbf{x}_P = \mathbf{x}_\Gamma - \varepsilon \mathbf{n}_\Gamma, \quad \boldsymbol{\tau}_\Gamma \perp \mathbf{n}_\Gamma$$

Approximate normal derivative

$$\partial_n U(\mathbf{x}_\Gamma) = \frac{u_\Gamma(\mathbf{x}_\Gamma) - u_h(\mathbf{x}_\Gamma - \varepsilon \mathbf{n}_\Gamma)}{\varepsilon}$$

Extrapolation of interface data

To be defined: continous extensions $U, \partial_n U, V$ for calculating

$$G(\phi_h, u_h, u_\Gamma) = F - \kappa \partial_n U - VU, \qquad \mathbf{v}_h = -V\mathbf{q}_h, \qquad u_{\Omega,h}(u_h, U)$$

Solution values

$$U(\mathbf{x}_Q) = u_\Gamma(\mathbf{x}_\Gamma) + \nabla U(\mathbf{x}_\Gamma) \cdot (\mathbf{x}_Q - \mathbf{x}_\Gamma)$$

Normal derivatives

$$\partial_n U(\mathbf{x}_Q) = \partial_n U(\mathbf{x}_\Gamma)$$

Ghost penalties

Dirichlet version:

$$u_{\Omega,h}(\mathbf{x}_Q) = H_\varepsilon(\phi_h(\mathbf{x}_Q)) u_h(\mathbf{x}_Q) + (1 - H_\varepsilon(\phi_h(\mathbf{x}_Q))) U(\mathbf{x}_Q), \quad \gamma_{\Omega,D} = \mathcal{O}(h^{-1})$$

Neumann version:

$$\mathbf{g}_{\Omega,h}(\mathbf{x}_Q) = H_\varepsilon(\phi_h(\mathbf{x}_Q)) \nabla u_h(\mathbf{x}_Q) + (1 - H_\varepsilon(\phi_h(\mathbf{x}_Q))) \nabla U(\mathbf{x}_\Gamma), \quad \gamma_{\Omega,N} = \mathcal{O}(h)$$

Damping function

$$D_\varepsilon(\phi) = H_\varepsilon(\phi + m\varepsilon) - H_\varepsilon(\phi - m\varepsilon), \qquad m \geq 2$$

Localized ghost penalties:

$$u_{\Omega,h} = H_\varepsilon(\phi_h) u_h + (1 - H_\varepsilon(\phi_h)) D_\varepsilon(\phi_h) U,$$

$$\mathbf{g}_{\Omega,h} = H_\varepsilon(\phi_h) \nabla u_h + (1 - H_\varepsilon(\phi_h)) D_\varepsilon(\phi_h) \nabla U$$

⇒ efficient narrow band implementation

Numerical examples

(A) Elliptic test - fixed boundary

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : (x - 0.5)^2 + (y - 0.5)^2 = 0.0625\} = \partial\Omega_+$$

$$u_\Gamma(x, y, t) = (x - 0.5)^2 - (y - 0.5)^2 = u(x, y, t)$$

⇒ second order convergence

(B) Parabolic test - moving boundary

$$\Gamma(t) = \{(x, y) \in \mathbb{R}^2 : (x - 0.5)^2 + (y - 0.5)^2 = (0.25 + 0.15t)^2\} = \partial\Omega_+(t)$$

$$u_\Gamma(x, y, t) = (x - 0.5)^2 - (y - 0.5)^2 = u(x, y, t)$$

⇒ second order convergence

(C) Hyperbolic test - fixed boundary

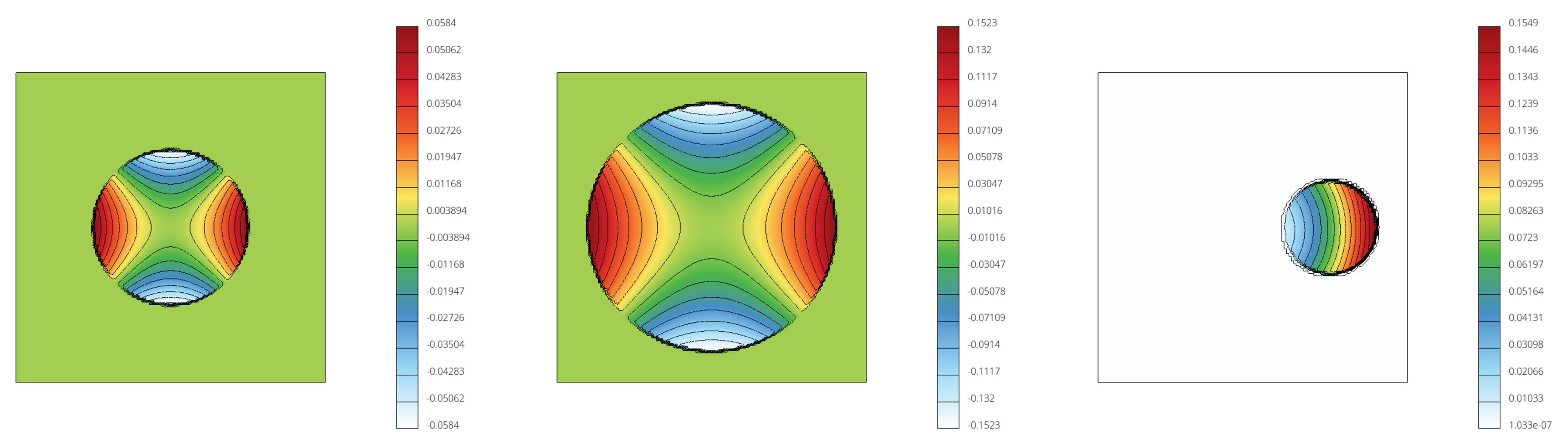
$$\Gamma = \{(x, y) \in \mathbb{R}^2 : (x - 0.75)^2 + (y - 0.5)^2 = 0.0225\} = \partial\Omega_+$$

$$u_\Gamma(x, y, t) = u_0(x, y, t) = (x - 0.5)^2 + (y - 0.5)^2 = u(x, y, t)$$

$$\mathbf{v}(x, y) = (0.5 - y, x - 0.5)^T$$

⇒ first order convergence

(A) (B) (C)



Summary of new features

- approximation of surface integrals by volume integrals
- fast closest-point search algorithm
- new way to define and calculate compact-support extensions
- narrow-band integration of terms involving extended fluxes, ghost penalties and velocity fields

D. Kuzmin and J.-P. Bäcker. An unfitted finite element method using level set functions for extrapolation into deformable diffuse interfaces. *J. Comput. Phys.*, 461:111218, 2022.