

Goal-oriented adaptivity for flux-limited Galerkin approximations to convection-dominated problems

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ENUMATH 2009, Uppsala

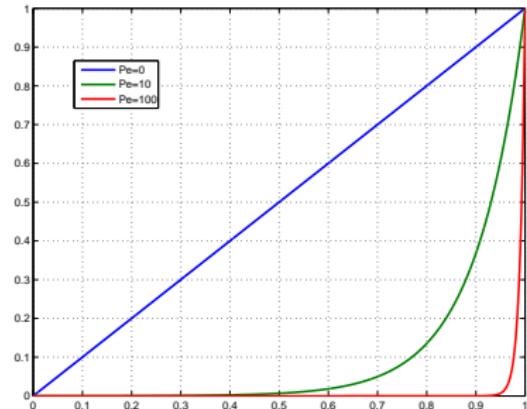
- 1 Flux-limited Galerkin schemes
 - Algebraic flux correction
- 2 Goal-oriented error estimation
 - A general review of the duality argument
 - Error estimates for steady transport problems
- 3 Mesh adaptation algorithm
 - Refinement, recoarsening, data structures
- 4 Conclusions & Outlook

Convection-diffusion in 1D

$$\begin{cases} \text{Pe} \frac{du}{dx} - \frac{d^2u}{dx^2} = 0 & \text{in } (0, 1) \\ u(0) = 0, \quad u(1) = 1 \end{cases}$$

$$\text{Pe} > 0, \quad u(x) = \frac{e^{\text{Pe} x} - 1}{e^{\text{Pe}} - 1}$$

$$x_i = ih, \quad u_i \approx u(x_i)$$

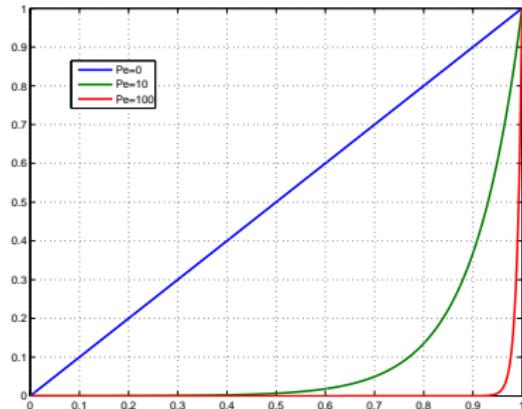


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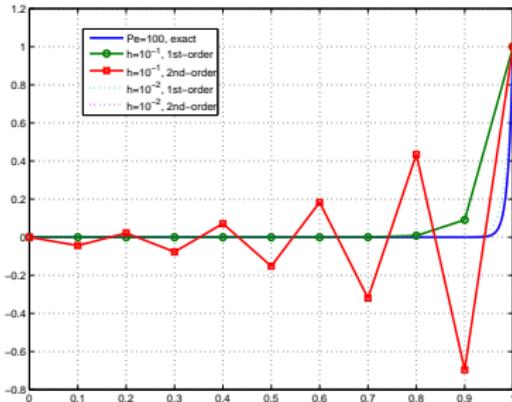
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- Exact solution: nonnegativity, no internal maxima or minima
- Numerical approximations: numerical diffusion \leftrightarrow spurious wiggles

Model problem

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{f}(u) = s$$

Roadmap

- High-order scheme

$$M_C \frac{du}{dt} = Ku, \quad \exists j \neq i : k_{ij} < 0$$

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artificial diffusion

$$L = K + D$$

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row-sum

mass-lumping

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- Residual difference

$$f = (M_L - M_C) \frac{du}{dt} - Du$$

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+

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$$M_L \frac{du}{dt} = Lu$$

Model problem

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{f}(u) = s$$

Roadmap

- High-resolution scheme

$$M_L \frac{du}{dt} = Lu + \bar{f}(u)$$

=

- Residual difference

$$f = (M_L - M_C) \frac{du}{dt} - Du$$

+ limited antidiffusion

- Low-order scheme

$$M_L \frac{du}{dt} = Lu$$

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- Flux limiting $\bar{f}_{ij} = \alpha_{ij} f_{ij} = -\bar{f}_{ji}, \quad 0 \leq \alpha_{ij} \leq 1$
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- High-resolution scheme $M_L \frac{du}{dt} = Lu + \bar{f}(u), \quad \bar{f}_i = \sum_{j \neq i} \bar{f}_{ij}$

Remarks on flux limiting schemes

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- No error quantification/no information about error propagation
- Violation of the Galerkin orthogonality property

Primal problem $a(w, u) = b(w), \quad \forall w \in V$

$$u_h \approx u, \quad \rho(w, u_h) = b(w) - a(w, u_h), \quad \rho(w, u) = 0$$

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■ Galerkin orthogonality $\rho(w_h, u_h) = 0, \quad \forall w_h \in V_h$

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- Linear target functional $j(u)$, mean/point value, boundary flux

Dual problem $a(z, e) = j(e), \quad \forall e \in V$

$$e = u - u_h, \quad j(e) = a(z, u) - a(z, u_h) = \rho(z, u_h)$$

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Primal problem

$$\begin{aligned}\nabla \cdot (\mathbf{v}u) &= s && \text{in } \Omega \\ u &= g && \text{on } \Gamma_-\end{aligned}$$

Dual problem

$$\begin{aligned}-\mathbf{v} \cdot \nabla z &= j && \text{in } \Omega \\ z &= h && \text{on } \Gamma_+\end{aligned}$$

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$$-\int_{\Omega} w \mathbf{v} \cdot \nabla z \, d\mathbf{x} + \int_{\Gamma_+} wz \mathbf{v} \cdot \mathbf{n} \, ds = \int_{\Omega} wh \, d\mathbf{x} + \int_{\Gamma_+} wh \mathbf{v} \cdot \mathbf{n} \, ds$$

Primal problem $\nabla \cdot (\mathbf{v}u - \epsilon \nabla u) = s$ + b.c.

$$\int_{\Omega} w \nabla \cdot (\mathbf{v}u) \, d\mathbf{x} + \int_{\Omega} \epsilon \nabla w \cdot \nabla u \, d\mathbf{x} = \int_{\Omega} ws \, d\mathbf{x}$$

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■ Residual weighted by the dual error $w = \hat{z} - z_h$

$$\rho(w, u_h) = \int_{\Omega} w (s - \nabla \cdot (\mathbf{v}u_h)) \, d\mathbf{x} - \int_{\Omega} \epsilon \nabla w \cdot \nabla u_h \, d\mathbf{x}$$

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- Approximation $\mathbf{g}_h \approx \nabla u$, $\int_{\Omega} w \nabla \cdot \mathbf{g}_h \, d\mathbf{x} + \int_{\Omega} \nabla w \cdot \mathbf{g}_h \, d\mathbf{x} = 0$

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$$\rho(w, u_h) = \int_{\Omega} w (s - \nabla \cdot (\mathbf{v}u_h - \epsilon \mathbf{g}_h)) \, d\mathbf{x} \quad \text{residual error}$$

$$+ \int_{\Omega} \epsilon \nabla w \cdot (\mathbf{g}_h - \nabla u_h) \, d\mathbf{x} \quad \text{diffusive flux error}$$

- Finite element solutions $u_h \approx u$, $z_h \approx z$, $\hat{z} \approx z$
- Error representation $j(u - u_h) \approx \rho(\hat{z} - z_h, u_h) + \rho(z_h, u_h)$

Computable error bounds $|j(u - u_h)| \leq \eta$

$$\eta = \Phi + \Psi, \quad |\rho(\hat{z} - z_h, u_h)| \leq \Phi, \quad |\rho(z_h, u_h)| \leq \Psi$$

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- Effectivity indices

$$I_{\text{eff}} = \frac{\eta}{|j(u - u_h)|}, \quad I_{\text{rel}} = \left| \frac{|j(u - u_h)| - \eta}{|j(u)|} \right|$$

Estimates for $\text{Pe} \frac{du}{dx} - \frac{d^2u}{dx^2} = 0$ in $(0, 1)$, $u(0) = 0$, $u(1) = 1$

$$u(x) = \frac{e^{\text{Pe} x} - 1}{e^{\text{Pe}} - 1}, \quad j(u) = \int_0^1 u \, dx, \quad z(x) = \frac{e^{\text{Pe}(1-x)} + x(e^{\text{Pe}} - 1) - e^{\text{Pe}}}{\text{Pe}(1 - e^{\text{Pe}})}$$

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Discretization: central difference scheme, $h = 1/10$

Pe	$ j(u - u_h) $	Φ	Ψ	η	I_{rel}
1	7.67e-04	7.80e-04	4.09e-16	7.80e-04	3.05e-05
10	2.84e-05	4.10e-05	3.56e-18	4.10e-05	1.25e-04
100	-	-	-	-	-

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Discretization: upwind difference scheme, $h = 1/10$

Pe	$ j(u - u_h) $	Φ	Ψ	η	I_{rel}
1	4.52e-03	7.38e-04	3.58e-03	4.32e-03	4.79e-04
10	4.91e-02	3.06e-04	4.76e-02	4.79e-02	1.21e-02
100	5.00e-02	1.59e-09	5.00e-02	5.00e-02	1.21e-08

Estimates for $\text{Pe} \frac{du}{dx} - \frac{d^2u}{dx^2} = 0$ in $(0, 1)$, $u(0) = 0$, $u(1) = 1$

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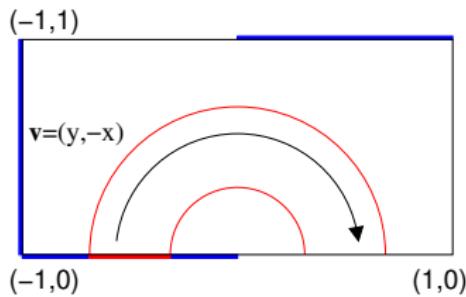
Discretization: TVD scheme, MC limiter $h = 1/10$

Pe	$ j(u - u_h) $	Φ	Ψ	η	I_{rel}
1	1.03e-03	7.74e-04	2.60e-04	1.03e-03	1.34e-05
10	1.51e-02	9.12e-05	1.50e-02	1.51e-02	3.81e-05
100	4.51e-02	4.23e-09	4.51e-02	4.51e-02	1.97e-07

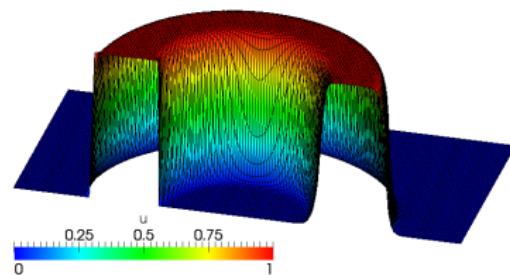
Galerkin orthogonality errors ($\hat{z} = z_h$) may be all you need *in practice*

Circular convection $\nabla \cdot (\mathbf{v}u) = 0$ in $\Omega = (-1, 1) \times (0, 1)$

$$u(x, y) = \begin{cases} 1, & 0.35 \leq r \leq 0.65 \\ 0, & \text{otherwise} \end{cases} \quad r(x, y) = \sqrt{x^2 + y^2}$$



domain and velocity

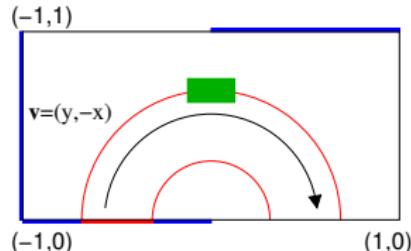


FEM-TVD, $h = 1/80$

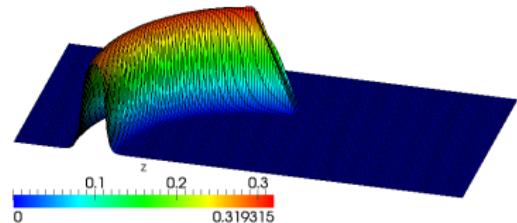
Target functional

$$j(u) = \int_{\Omega} uh \, d\mathbf{x}, \quad h(\mathbf{x}) \in \{0, 1\}$$

$h \equiv 1$ in $(-0.1, 0.1) \times (0.6, 0.7)$



h	$j(e)$	Ψ	I_{eff}	I_{rel}
1/10	4.97e-4	2.86e-3	5.75	2.49e-1
1/20	2.67e-3	2.53e-3	0.95	1.40e-2
1/40	1.91e-3	2.47e-3	1.29	5.91e-2
1/80	1.22e-3	2.19e-3	1.79	1.02e-1



FEM-TVD, $h = 1/80$

- Error estimation based on $\eta = \Psi(z_h, u_h)$ neglecting $\Phi(\hat{z} - z_h, u_h)$

- Error representation $j(u - u_h) \approx \rho(\hat{z} - z_h, u_h) + \rho(z_h, u_h)$

$$|\rho(\hat{z} - z_h, u_h)| \leq \Phi = \sum_i \Phi_i, \quad |\rho(z_h, u_h)| \leq \Psi = \sum_i \Psi_i$$

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- Nodal localization $z_h = \sum_i z_i \varphi_i, \quad \Psi_i = |\rho(z_i \varphi_i, u_h)|$

$$\hat{z} - z_h = \sum_i w_i, \quad w_i = \varphi_i(\hat{z} - z_h), \quad \Phi_i = |\rho(w_i, u_h)|$$

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$$|\rho(\hat{z} - z_h, u_h)| \leq \Phi = \sum_i \Phi_i, \quad |\rho(z_h, u_h)| \leq \Psi = \sum_i \Psi_i$$

- Nodal localization $z_h = \sum_i z_i \varphi_i, \quad \Psi_i = |\rho(z_i \varphi_i, u_h)|$

$$\hat{z} - z_h = \sum_i w_i, \quad w_i = \varphi_i(\hat{z} - z_h), \quad \Phi_i = |\rho(w_i, u_h)|$$

- Conversion to element contributions $\eta = \sum_K \eta_K = \Phi + \Psi$

$$\xi = \sum_i \xi_i \varphi_i, \quad \xi_i = \frac{\Phi_i + \Psi_i}{\int_{\Omega} \varphi_i \, d\mathbf{x}}, \quad \eta_K = \int_K \xi \, d\mathbf{x}$$

- Error representation $j(u - u_h) \approx \rho(\hat{z} - z_h, u_h) + \rho(z_h, u_h)$

$$|\rho(\hat{z} - z_h, u_h)| \leq \Phi = \sum_i \Phi_i, \quad |\rho(z_h, u_h)| \leq \Psi = \sum_i \Psi_i$$

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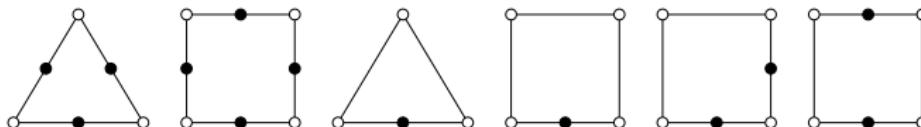
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Conformal refinement algorithm

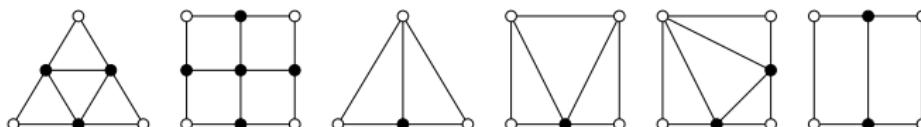
Bank, et al. '83

- 1 subdivide marked elements regularly (red rule)
- 2 eliminate 'hanging nodes' by transition cells (green rule)



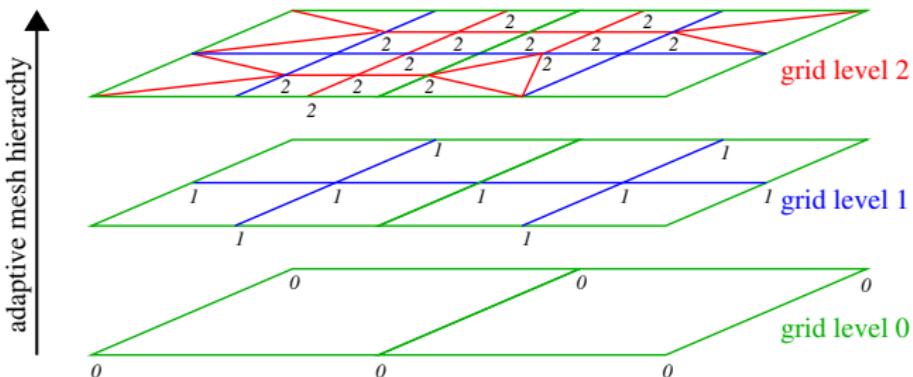
red refinement/recoarsening

green refinement/recoarsening



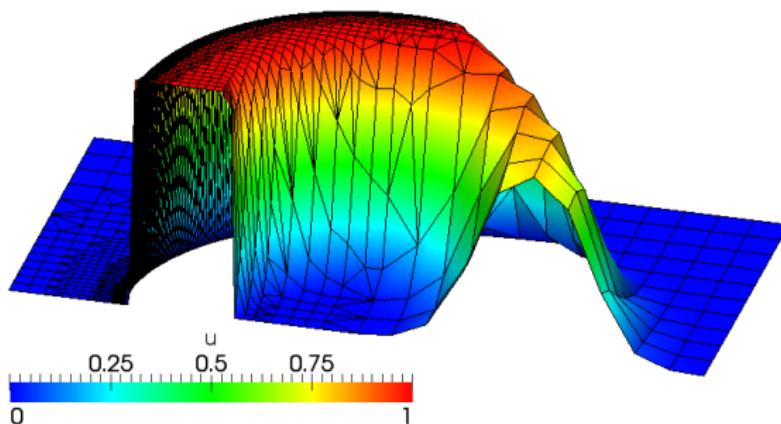
- Vertex-locking algorithm is used to reverse mesh refinement
- Nodal generation function provides all necessary information:
element type, inter-element relationship, refinement level, ...

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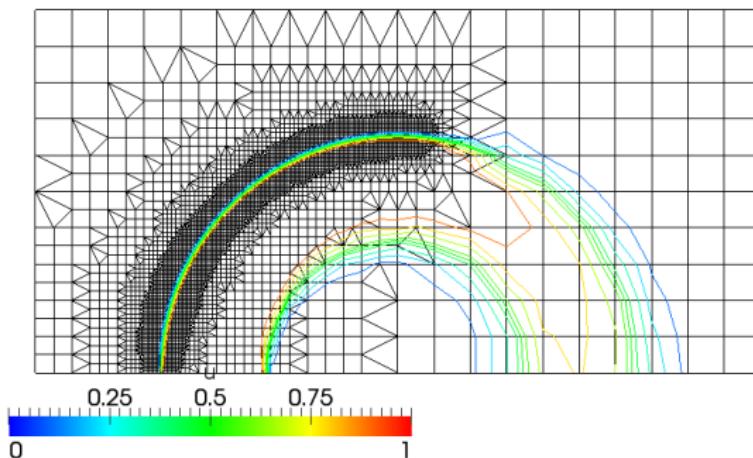
Adaptive mesh refinement $tol = 1e-8$, $h_{\min} = 1/160$

$$j(e) = 6.08e-4, \quad \Psi = 1.47e-3, \quad I_{\text{eff}} = 2.42, \quad I_{\text{rel}} = 9.16e-2$$



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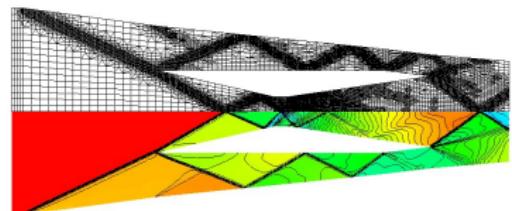
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- Flux-limited Galerkin discretizations of AFC type
 - high resolution, violation of the Galerkin orthogonality
 - flux correction provides feedback for mesh adaptation
- Goal-oriented error estimation for transport problems
 - no dubious constants, control of local and transmitted errors
 - nodal decomposition of the error in the target functional
 - Galerkin orthogonality error is a handy criterion for refinement
 - Stabilization of diffusive fluxes using gradient recovery

- Time-dependent transport problems
 - high-resolution schemes of FCT type ✓
 - dynamic mesh adaptation algorithm ✓
 - goal-oriented error estimation in space *and* time

- Nonlinear (systems of) equations
 - compressible Euler equations ✓
 - characteristic flux limiting ✓
 - control of target quantities



- Time-dependent transport problems
 - high-resolution schemes of FCT type ✓
 - dynamic mesh adaptation algorithm ✓
 - goal-oriented error estimation in space *and* time
- Nonlinear (systems of) equations

