An efficient multigrid solver for the three-dimensional incompressible Navier-Stokes equations based on discretely divergence-free finite elements

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September 02, 2025

1 Introduction

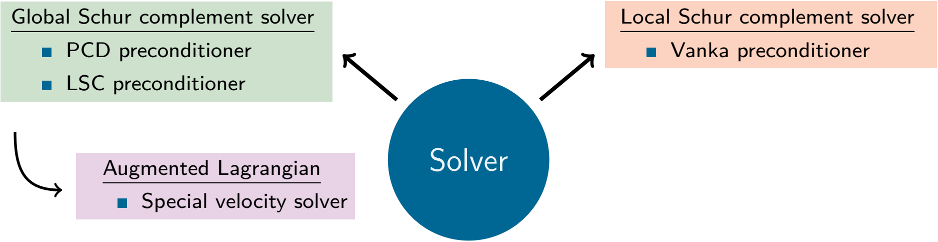
Motivation

Incompressible Navier-Stokes equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu \Delta \mathbf{v} = \mathbf{g} - \nabla p,$$
$$\nabla \cdot \mathbf{v} = 0$$

Discretization in space and time:

$$\begin{pmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{B}^\top & \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} = \begin{pmatrix} \mathrm{f} \\ 0 \end{pmatrix}$$



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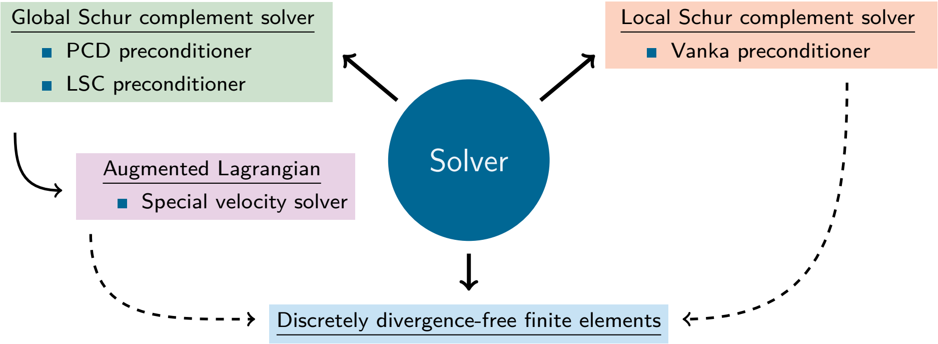

Motivation

Discretely divergence-free finite element method (DDFFEM):

- A priori elimination of pressure unknowns
- Use of discretely divergence-free FE functions
- Smaller problem sizes
- No saddle-point structure

Literature:

- 2D: Crouzeix 1976; Temam 1977, etc.
- 2D: Turek 1994a; Turek 1994b (efficient multigrid solver)
- 3D: Hecht 1981; Griffiths 1981; Fortin 1981; Thomasset 1981; Cuvelier et al. 1986, etc. (construction of basis for simple geometries)

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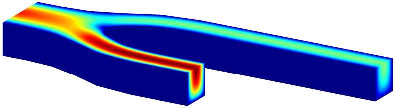
In this talk:

- Treatment of more general geometries in 3D
  - Geometries introducing bifurcations
- Efficient multigrid solver in 3D



Contents

2 Discretization

Construction of spanning set

Q_2-P_1 finite element pair

$$V_h = \{v_h \in C^0(\Omega) : v_h|_K \circ T_K \in \mathbb{Q}_2(\hat{K}) \ \forall K \in \mathcal{T}_h\},$$
$$Q_h = \{q_h \in L^2(\Omega) : q_h|_K \in \mathbb{P}_1(K) \ \forall K \in \mathcal{T}_h\},$$

Lagrange basis functions for each component of $\mathbf{v}_h$:

- vertex fcts φ_i^v
- edge fcts φ_i^e
- face fcts φ_i^f
- cell fcts φ_i^c

1. Construct functions $\zeta_i^{q\ell} = \mathbf{e}_\ell \varphi_i^q + \sum_j \mathbf{a}_{ij}^{q\ell} \varphi_j^c$, $\ell \in \{1,2,3\}$, $q \in \{v,e,f\}$ s.t.

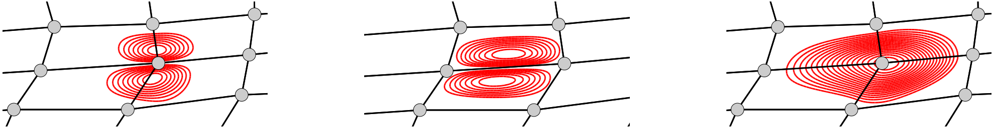
$$\int_K x_k (\nabla \cdot \zeta_i^{q\ell}) \,\mathrm{d}\mathbf{x} = 0 \quad \forall k \in \{1,2,3\}$$

2. Define functions ϕ_j as linear combination of $\zeta_i^{q\ell}$ s.t.

$$\int_K \nabla \cdot \phi_j \,\mathrm{d}\mathbf{x} = \int_{\partial K} \mathbf{n}_K \cdot \phi_j \,\mathrm{d}\mathbf{s} = 0$$

Example: $\phi_j = \sum_\ell t_\ell \zeta_j^{f\ell}$ with tangential vector $\mathbf{t} \perp \mathbf{n}_K$

Construction of spanning set



Discretely divergence-free finite element space:

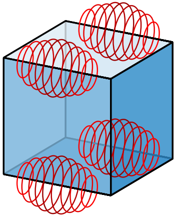
$$\mathbf{V}_h = \left\{ \mathbf{v}_h \in (V_h)^3 : \int_K q_h (\nabla \cdot \mathbf{v}_h) \, \mathrm{d}\mathbf{x} = 0 \ \forall K \in \mathcal{T}_h, \ q_h \in Q_h \right\} = \mathbf{V}_{\mathbf{t},h} + \mathbf{V}_{\mathbf{n},h}$$

- Tangential components: $\mathbf{V}_{\mathbf{t},h} = \left\{ \mathbf{v}_h \in \mathbf{V}_h : \int_F \mathbf{v}_h \cdot \mathbf{n}_F \, \mathrm{d}\mathbf{s} = 0 \ \forall \text{ faces } F \right\}$
  1. 3 functions per vertex
  2. 3 functions per edge
  3. 2 functions per face
- Normal components: $\mathbf{V}_{\mathbf{n},h} = \left\{ \mathbf{v}_h \in \mathbf{V}_h : \int_F \mathbf{v}_h \times \mathbf{n}_F \, \mathrm{d}\mathbf{s} = \mathbf{0} \ \forall \text{ faces } F \right\}$
  1. 1 function per edge

Normal contribution $\mathbf{V}_{\mathbf{n},h}$

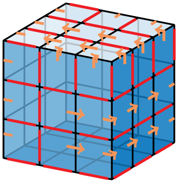
DDFFE function $\mathbf{v}_h = \sum_i v_i \phi_i + \sum_{\text{edge } E} v_E \psi_E \in \mathbf{V}_{\mathbf{t},h} + \mathbf{V}_{\mathbf{n},h}$

$$\int_F \mathbf{n}_F \cdot \mathbf{v}_h \, \mathrm{d}s = \sum_{\text{edge } E \subset \partial F} \pm v_E \quad \forall \text{ faces } F$$



Prescribing Dirichlet boundary conditions:

- Construct minimal spanning tree of face adjacency graph on Γ_{Dir}
- Iterate from leaves to roots
- Determine coefficients v_E in marching fashion



Normal contribution $\mathbf{V}_{\mathbf{n},h}$

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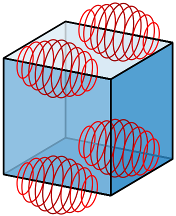
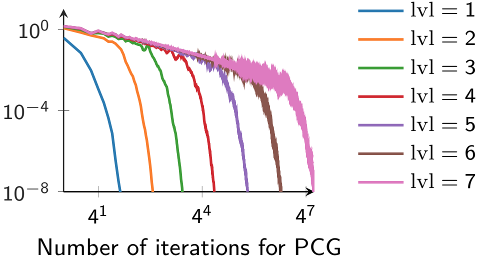


Source of all problems:

- Linear dependency
- Boundary conditions in marching fashion
- Condition number like $\mathcal{O}(h^{-4})$
- Non-standard prolongation operator
- 'Global' FE functions



Special case: Geometries introducing bifurcations

DDFFE function $\mathbf{v}_h = \sum_i v_i \phi_i + \sum_{\text{edge } E} v_E \psi_E \in \mathbf{V}_{\mathbf{t},h} + \mathbf{V}_{\mathbf{n},h}$

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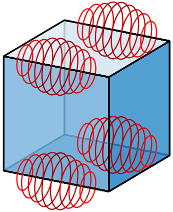
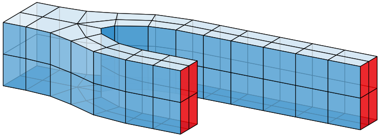
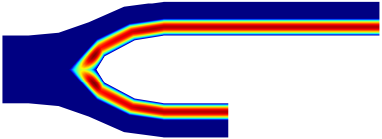


- Net flux through channel is solely defined by dofs on boundary!

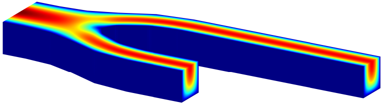
Geometry with outflow boundary parts (red)



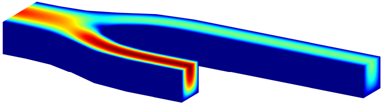
'Global' FE function:



Flux boundary condition:



Neumann boundary condition:



Linear system of equations

- Unfiltered system matrix $\mathrm{A} \in \mathbb{R}^{N \times N}$ assembled with DDFFE
- Right-hand side vector $\mathrm{b} \in \mathbb{R}^{N}$ assembled with DDFFE
- s 'global' FE function(s) $\mathrm{G} \in \mathbb{R}^{N \times s}$
- Discretely divergence-free solution $\tilde{\mathrm{x}} = \mathrm{x} + \mathrm{G}\mathrm{v} \in \mathbb{R}^{N}$

Linear system of equations to be solved (using filtered quantities A_1, A_0, b_1):

$$\begin{pmatrix} \mathrm{A}_1 & \mathrm{A}_0\mathrm{G} \\ \mathrm{G}^\top\mathrm{A} & \mathrm{G}^\top\mathrm{A}\mathrm{G} \end{pmatrix} \begin{pmatrix} \mathrm{x} \\ \mathrm{v} \end{pmatrix} = \begin{pmatrix} \mathrm{b}_1 \\ \mathrm{G}^\top\mathrm{b} \end{pmatrix}$$

Schur complement iteration (using $\mathrm{S} = \mathrm{G}^\top\mathrm{A}\mathrm{A}_1^{-1}\mathrm{G} \quad \approx \quad \mathrm{G}^\top\mathrm{A}\mathrm{G}$):

$$\begin{pmatrix} \mathrm{x} \\ \mathrm{v} \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \mathrm{x} \\ \mathrm{v} \end{pmatrix} + \begin{pmatrix} \mathrm{A}_1 & \mathrm{A}_0\mathrm{G} \\ 0 & \mathrm{S} \end{pmatrix}^{-1} \left[\begin{pmatrix} \mathrm{b}_1 \\ \mathrm{G}^\top\mathrm{b} \end{pmatrix} - \begin{pmatrix} \mathrm{A}_1 & \mathrm{A}_0\mathrm{G} \\ \mathrm{G}^\top\mathrm{A} & \mathrm{G}^\top\mathrm{A}\mathrm{G} \end{pmatrix} \begin{pmatrix} \mathrm{x} \\ \mathrm{v} \end{pmatrix} \right]$$

$$\Longrightarrow \qquad \boxed{\tilde{\mathrm{x}} \quad \mapsto \quad \tilde{\mathrm{x}} + \mathrm{A}_1^{-1}(\mathrm{r}_\mathrm{x} + \mathrm{G}\mathrm{S}^{-1}\mathrm{r}_\mathrm{v})} \qquad \mathrm{r}_\mathrm{x} = (\mathrm{b} - \mathrm{A}\tilde{\mathrm{x}})_0, \quad \mathrm{r}_\mathrm{v} = \mathrm{G}^\top(\mathrm{b} - \mathrm{A}\tilde{\mathrm{x}})$$

3 Multigrid solver

Multigrid solver

$\mathbf{MG}_\gamma(w_h, b_h)$ with γ-cycle

if $\mathcal{T}_h$ is coarsest triangulation **then**

0 *Solve coarse grid problem*

$w_h \leftarrow A_h^{-1} b_h$

else

1 *Perform pre-smoothing*

$w_h \leftarrow S_h(w_h, b_h)$

2 *Restrict residual*

$r_{2h} \leftarrow P_h^\top (b_h - A_h w_h)$

$w_{2h} \leftarrow 0$

3 *Call MG-cycle γ times on $\mathcal{T}_{2h}$*

$w_{2h} \leftarrow \mathbf{MG}_\gamma(w_{2h}, r_{2h})$

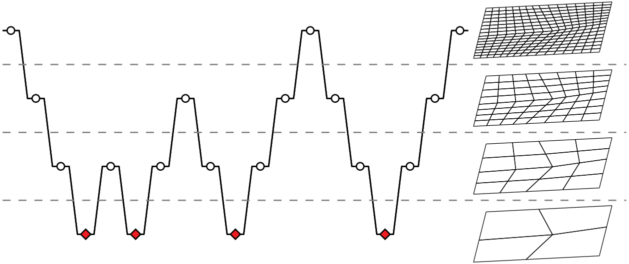
4 *Correct solution*

$w_h \leftarrow w_h + P_h w_{2h}$

5 *Perform post-smoothing*

$w_h \leftarrow S_h(w_h, b_h)$

end if



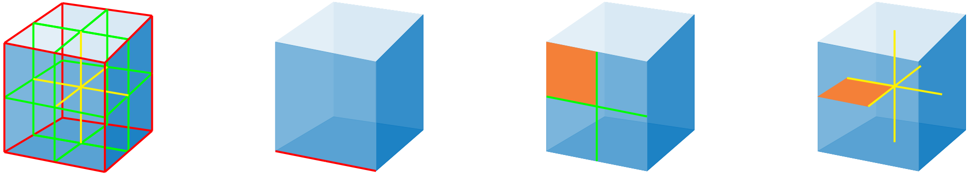
Technical details:

- GMRES smoother
  (with different preconditioners)
- Basis only used on coarsest grid

Open questions:

- Definition of prolongation matrix

Intergrid transfer operators



Prolongation of coarse grid solution $\mathbf{v}_{2h} \in \mathbb{V}_{2h}$:

1. Compute interpolation $\tilde{\mathbf{v}}_h \in (V_h)^3$ of coarse grid function $\mathbf{v}_{2h} \in \mathbf{V}_{2h}$.
2. Determine $\mathbf{v}_h^{\mathbf{t}} \in \mathbf{V}_{\mathbf{t},h}$ using tangential components of $\tilde{\mathbf{v}}_h$
3. Calculate FE function $\mathbf{v}_h^{\mathbf{n}} \in \mathbf{V}_{\mathbf{n},h}$ using normal fluxes of $\tilde{\mathbf{v}}_h$ through faces:

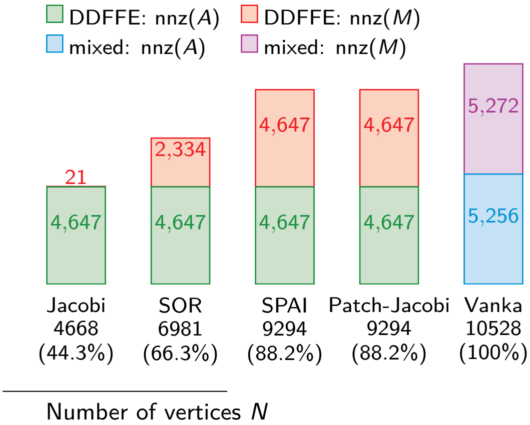
   a) edges of $\mathcal{T}_{2h}$, b) faces of $\mathcal{T}_{2h}$, c) cells of $\mathcal{T}_{2h}$

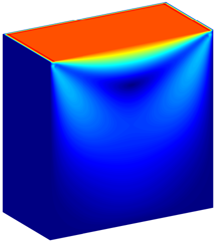
cf. Turek 1994a; Turek 1994b

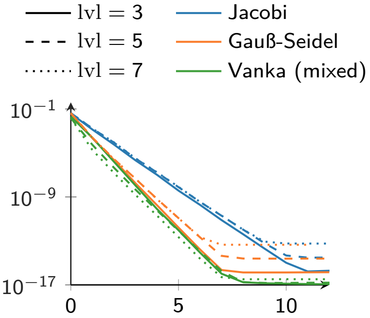
4 Numerical examples

Lid Driven Cavity

	DDFFE		mixed	
	dofs	nnz	dofs	nnz
Q_2-P_1	$21N$	$4647N$	$28N$	$5256N$

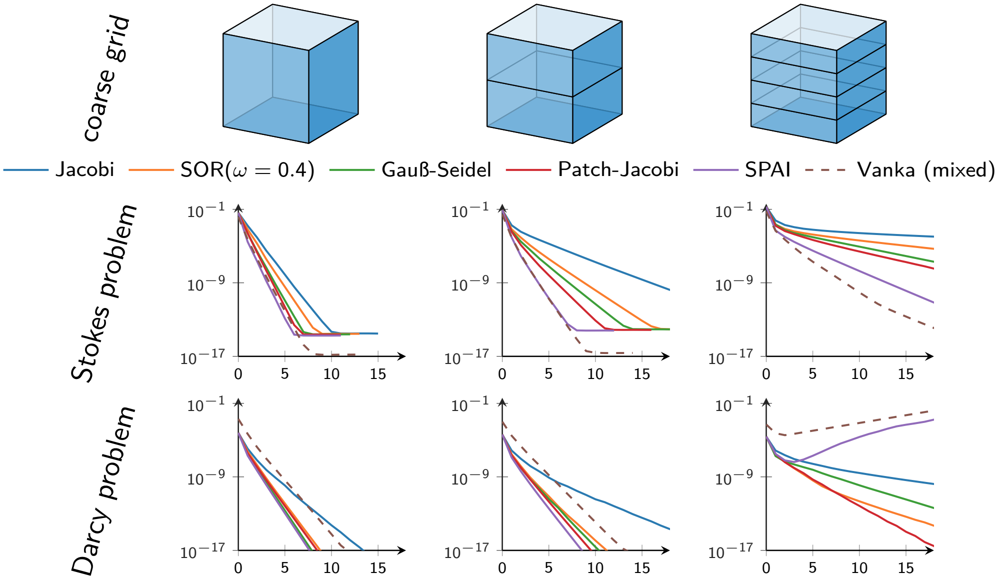




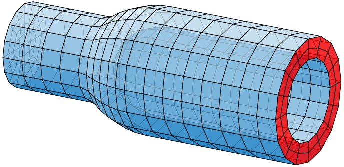


Stokes problem: $4 + 4$ smoothing steps.

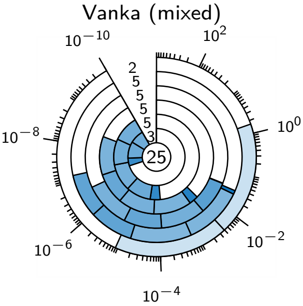
Lid Driven Cavity

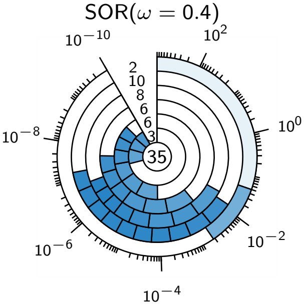


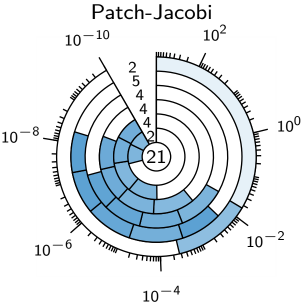
Flow Through Extrusion Die

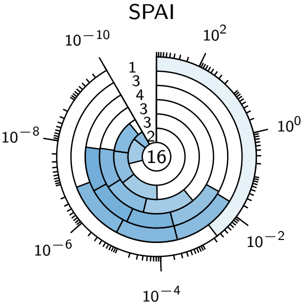


Navier-Stokes problem on $\mathrm{lvl} = 3$: 16 + 16 smoothing steps.

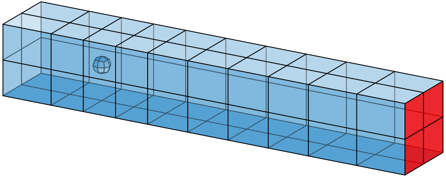






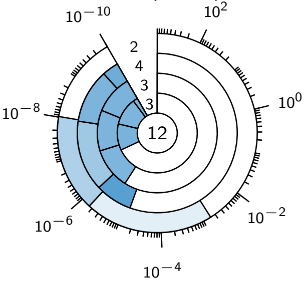


Flow Around Sphere

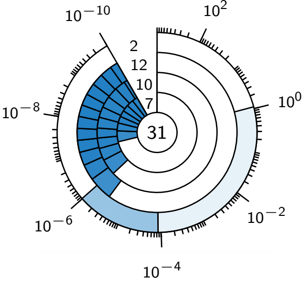


Navier-Stokes problem on $\mathrm{lvl} = 4$: $16 + 16$ smoothing steps.

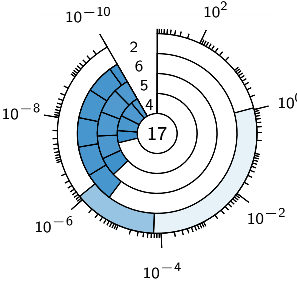
Vanka (mixed)



SOR($\omega = 0.4$)



Patch-Jacobi



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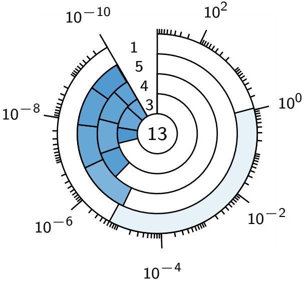

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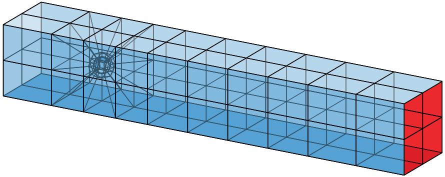

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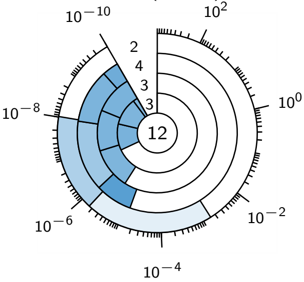

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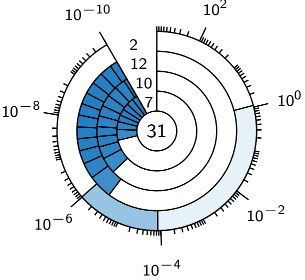

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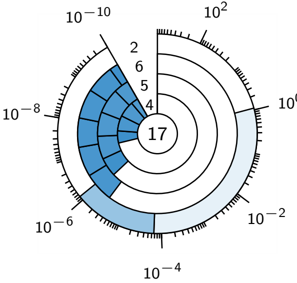

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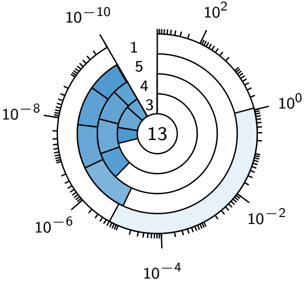
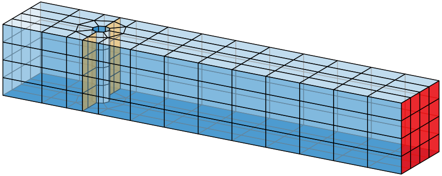


Flow Around Cylinder



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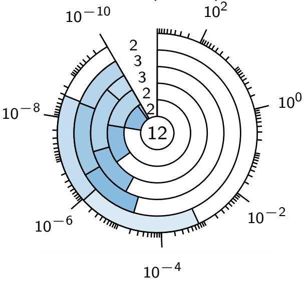

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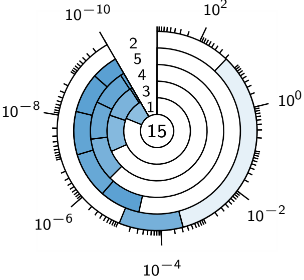

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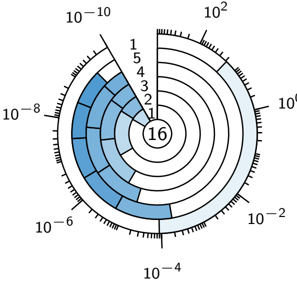

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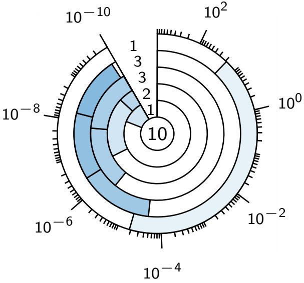
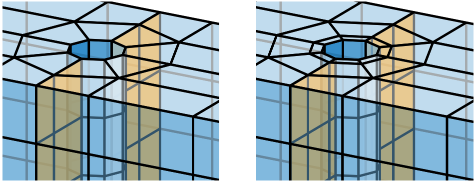


Flow Around Cylinder



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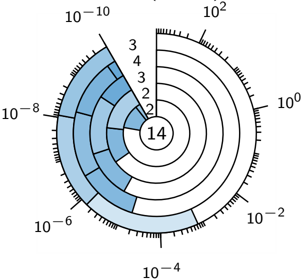

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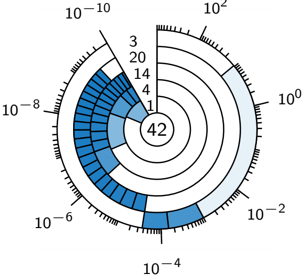

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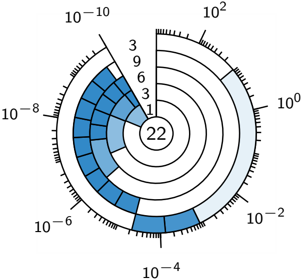

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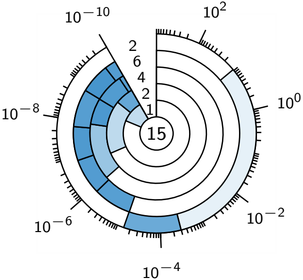
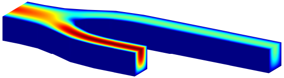
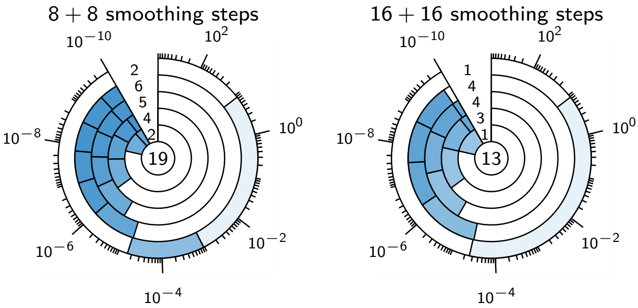
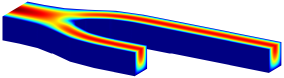


Flow Through Y-Pipe

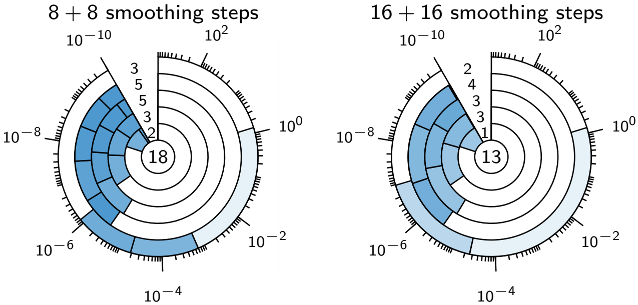


Navier-Stokes problem with
Neumann boundary on $\mathrm{lvl} = 4$:
SOR($\omega = 0.4$) preconditioner.

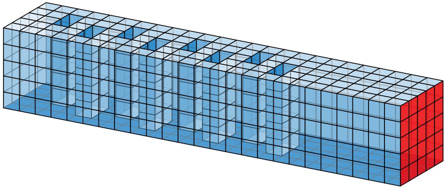




Navier-Stokes problem with
Flux boundary on $\mathrm{lvl} = 4$:
SOR($\omega = 0.4$) preconditioner.



Flow Around Multiple Obstacles



Stokes problem on $\mathrm{lvl} = 3$: Jacobi preconditioner.

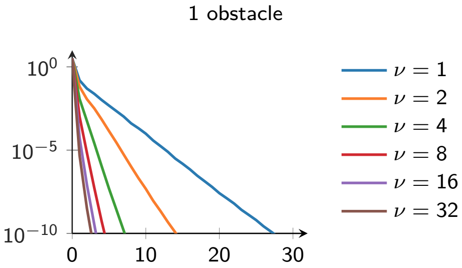
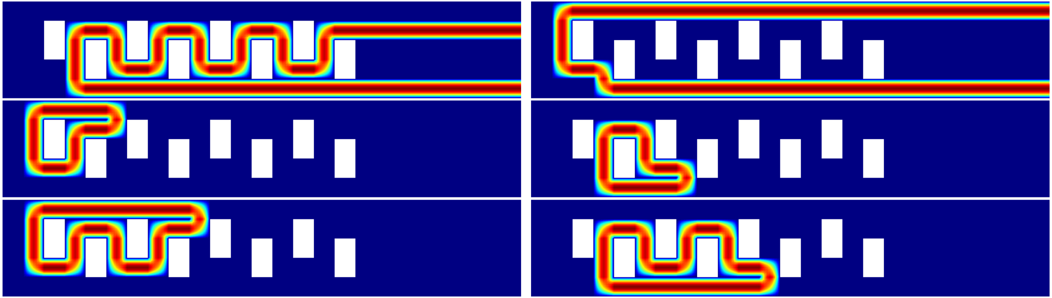
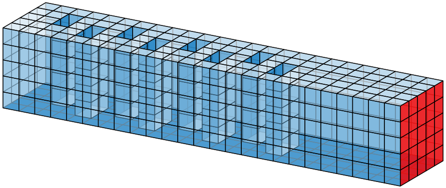


Illustration of 'global' FE functions on $\mathrm{lvl} = 0$:



Flow Around Multiple Obstacles



Stokes problem on $\mathrm{lvl} = 3$: Jacobi preconditioner.

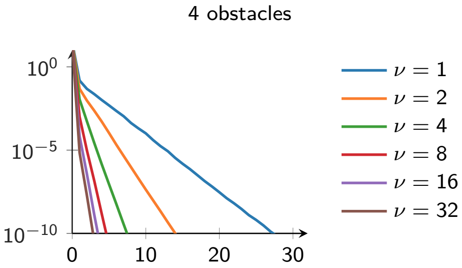
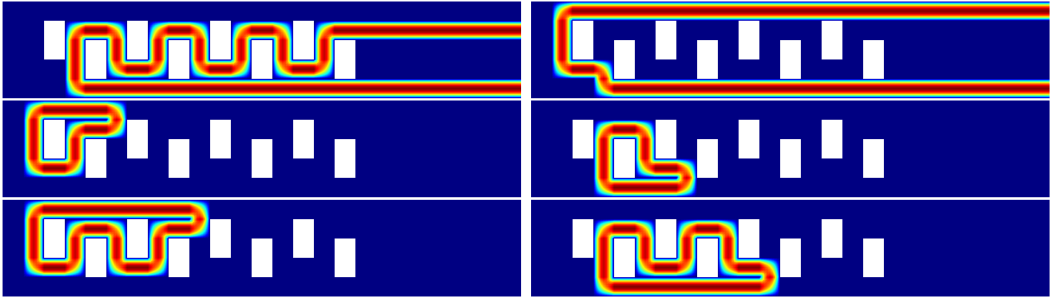
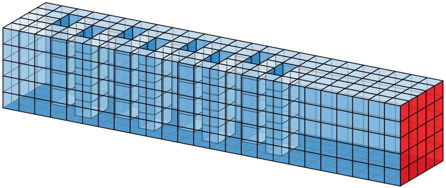


Illustration of 'global' FE functions on $\mathrm{lvl} = 0$:



Flow Around Multiple Obstacles



Stokes problem on $\mathrm{lvl} = 3$: Jacobi preconditioner.

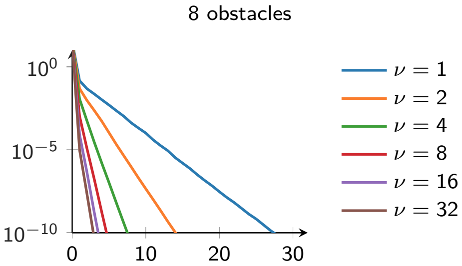
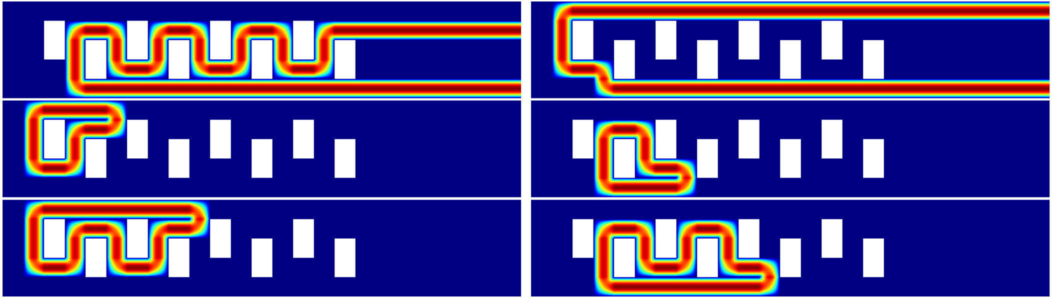
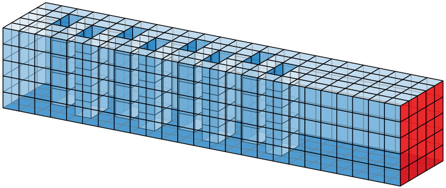


Illustration of 'global' FE functions on $\mathrm{lvl} = 0$:



Flow Around Multiple Obstacles



Stokes problem on $\mathrm{lvl} = 3$: Jacobi preconditioner.

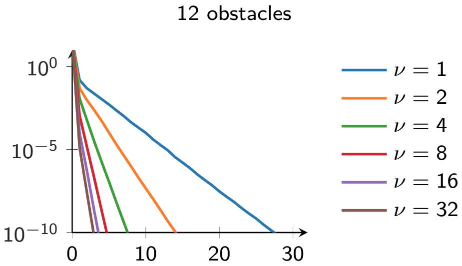
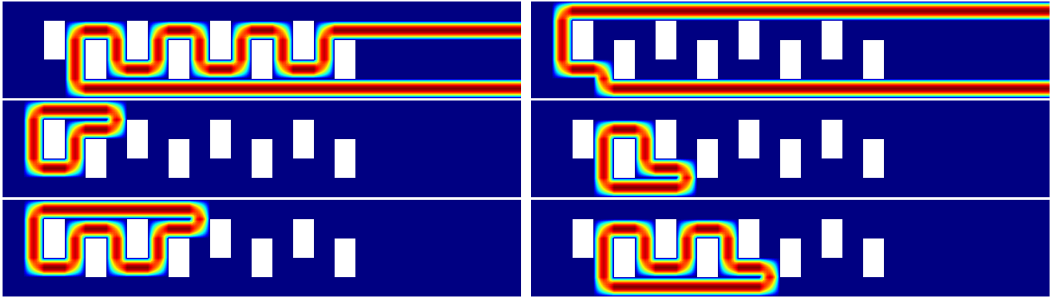


Illustration of ‘global’ FE functions on $\mathrm{lvl} = 0$:



5 Conclusions

DDFFEM in a nutshell

Benefits:

- No saddle-point structure
- No Schur complement solver
- More flexibility in preconditioning
- Smaller problem sizes
- Flux boundary condition

Drawbacks:

- Non-standard intergrid transfer
- 'Global' FE function
- Condition number $\mathcal{O}(h^{-4})$
- (Singular system matrix)

▸ Competitive convergence behavior especially for 'isotropic' meshes

Future work:

- Recovering pressure in a marching fashion
- Design of robust and efficient preconditioners
- Investigation of other solution strategies

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