technische universität dortmund

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On the design of global-in-time Newton-Pressure Schur complement solvers for incompressible flow problems

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1 Introduction

Motivation

DFG Flow around a cylinder benchmark 2D-3 from 1995 as part of the research project "Flow simulation on high-performance computers" (Schäfer et al. 1996)

Problem: Sequential time-stepping of flow solvers

- Trend of supercomputers towards increased number of cores
- Stagnating performance of each core
- ▸ Global-in-time solution strategy on massively parallel computing facilities

Related works:

- Trindade and Pereira (2004)
- Lemoine and Münch (2021)
- Danieli, Southworth, and Wathen (2022)

Contents

Definition of problem

Incompressible Navier-Stokes equations

$$
\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \mu \Delta \mathbf{v} + \operatorname{grad}(p) &= \rho \mathbf{g} && \text{in } \Omega \times (0, T), \\
\operatorname{div}(\mathbf{v}) &= 0 && \text{in } \Omega \times (0, T), \\
\mathbf{v}(\cdot, 0) &= \mathbf{v}_0 && \text{on } \Omega, \\
\mathbf{v} &= \mathbf{v}_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}} \times (0, T), \\
-p\mathbf{n} + \mu(\mathbf{n} \cdot \nabla)\mathbf{v} &= \rho \mathbf{h} && \text{on } \Gamma_{\mathrm{N}} \times (0, T)
\end{aligned}
$$

Discretization:

- θ-scheme in time ($\theta = \frac{1}{2}$ for Crank-Nicolson; $\theta = 1$ for backward Euler)
- inf-sup-stable Q_2-Q_1 Taylor-Hood FE in space
- Quadrature based mass lumping ▸ M_u is diagonal

Definition of problem

Incompressible ~~Navier-~~Stokes equations

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + \cancel{(\mathbf{v}\cdot\nabla)\mathbf{v}} - \mu\Delta\mathbf{v} + \operatorname{grad}(p) &= \rho\mathbf{g} && \text{in } \Omega\times(0,T),\\
\operatorname{div}(\mathbf{v}) &= 0 && \text{in } \Omega\times(0,T),\\
\mathbf{v}(\cdot,0) &= \mathbf{v}_0 && \text{on } \Omega,\\
\mathbf{v} &= \mathbf{v}_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}}\times(0,T),\\
-p\mathbf{n} + \mu(\mathbf{n}\cdot\nabla)\mathbf{v} &= \rho\mathbf{h} && \text{on } \Gamma_{\mathrm{N}}\times(0,T)
\end{aligned}$$

Discretization:

- θ-scheme in time ($\theta = \frac{1}{2}$ for Crank-Nicolson; $\theta = 1$ for backward Euler)
- inf-sup-stable Q_2-Q_1 Taylor-Hood FE in space
- Quadrature based mass lumping ▸ M_u is diagonal

- For simplicity: $\mu = 1$

Discretization of Stokes equations

$$\mathrm{M}_u \frac{\mathrm{u}^{(n+1)} - \mathrm{u}^{(n)}}{\delta t} + \theta \mathrm{D}_u \mathrm{u}^{(n+1)} + (1-\theta)\mathrm{D}_u \mathrm{u}^{(n)} + \mathrm{B}\mathrm{p}^{(n+1)} = \theta \mathrm{g}^{(n+1)} + (1-\theta)\mathrm{g}^{(n)},$$

$$\mathrm{B}^\top \mathrm{u}^{(n+1)} = \mathrm{f}^{(n+1)}$$

$$\mathrm{M}_u \sim \text{id}, \quad \mathrm{D}_u \sim -\Delta, \quad \mathrm{B} \sim \text{grad}, \quad \mathrm{B}^\top \sim \text{div}$$

$$\mathrm{A}_i := \mathrm{M}_u + \theta \delta t \mathrm{D}_u, \quad \mathrm{A}_e := -\mathrm{M}_u + (1-\theta)\delta t \mathrm{D}_u, \quad \tilde{\mathrm{p}}^{(n+1)} := \delta t \mathrm{p}^{(n+1)}$$

Sequential time-stepping

$$\begin{pmatrix} \mathrm{A}_i & \mathrm{B} \\ \mathrm{B}^\top & \end{pmatrix} \begin{pmatrix} \mathrm{u}^{(n+1)} \\ \tilde{\mathrm{p}}^{(n+1)} \end{pmatrix} = \begin{pmatrix} \tilde{\mathrm{g}}^{(n+1)} - \mathrm{A}_e \mathrm{u}^{(n)} \\ \mathrm{f}^{(n+1)} \end{pmatrix}, \quad n = 0, \ldots, K$$

Sequential solution strategy

$$\begin{pmatrix} \mathrm{A}_i & \mathrm{B} \\ \mathrm{B}^\top & \end{pmatrix} \begin{pmatrix} \mathrm{u}^{(n+1)} \\ \tilde{\mathrm{p}}^{(n+1)} \end{pmatrix} = \begin{pmatrix} \tilde{\mathrm{g}}^{(n+1)} - \mathrm{A}_e \mathrm{u}^{(n)} \\ \mathrm{f}^{(n+1)} \end{pmatrix}, \qquad n = 0, \ldots, K$$

Pressure Schur complement (PSC) equation:

$$\boxed{\mathrm{B}^\top \mathrm{A}_i^{-1} \mathrm{B} \tilde{\mathrm{p}}^{(n+1)} = \mathrm{B}^\top \mathrm{A}_i^{-1} (\tilde{\mathrm{g}}^{(n+1)} - \mathrm{A}_e \mathrm{u}^{(n)}) - \mathrm{f}^{(n+1)}}$$

$$\mathrm{u}^{(n+1)} = \mathrm{A}_i^{-1} (\tilde{\mathrm{g}}^{(n+1)} - \mathrm{A}_e \mathrm{u}^{(n)} - \mathrm{B} \tilde{\mathrm{p}}^{(n+1)})$$

Iterative solver

$$\tilde{\mathrm{p}}^{(n+1)} \quad \mapsto \quad \tilde{\mathrm{p}}^{(n+1)} + \mathrm{q}^{(n+1)},$$

$$\mathrm{q}^{(n+1)} = \mathrm{C}_i^{-1} (\mathrm{B}^\top \tilde{\mathrm{u}}^{(n+1)} - \mathrm{f}^{(n+1)}), \qquad \tilde{\mathrm{u}}^{(n+1)} = \mathrm{A}_i^{-1} (\tilde{\mathrm{g}}^{(n+1)} - \mathrm{A}_e \mathrm{u}^{(n)} - \mathrm{B} \tilde{\mathrm{p}}^{(n+1)})$$

using preconditioner $\mathrm{C}_i \approx \mathrm{B}^\top \mathrm{A}_i^{-1} \mathrm{B}$

Sequential solution strategy

How to define preconditioner $C_i \approx B^\top A_i^{-1} B$?

1. **PCD preconditioner (Kay, Loghin, and Wathen 2002):**
Assuming $A_i^{-1} B M_p^{-1} \approx M_u^{-1} B A_{i,p}^{-1}$

$$P_i^{-1} = (B^\top A_i^{-1} B)^{-1} \approx (B^\top M_u^{-1} B A_{i,p}^{-1} M_p)^{-1} = M_p^{-1} A_{i,p} \underbrace{(B^\top M_u^{-1} B)}_{=:\hat{D}_p}{}^{-1} =: C_i^{-1}$$

$$A_{i,p} := M_p + \theta \delta t \mu \hat{D}_p \quad \Longrightarrow \quad C_i^{-1} = \hat{D}_p^{-1} + \theta \delta t \mu M_p^{-1}$$

2. **Scaled BFBt preconditioner (Elman et al. 2006):**
Using Moore-Penrose inverse $(M_u^{-1/2} B)^+ = \hat{D}_p^{-1} M_u^{-1/2} B^\top$

$$C_i^{-1} := \hat{D}_p^{-1} B^\top M_u^{-1} A_i M_u^{-1} B \hat{D}_p^{-1}$$

Sequential solution strategy

Properties of preconditioners:

	PCD preconditioner	Scaled BFBt preconditioner
C^{-1}	$\hat{\mathrm{D}}_p^{-1} + \theta\delta t \mathrm{M}_p^{-1}$	$\hat{\mathrm{D}}_p^{-1}\mathrm{B}^\top\mathrm{M}_u^{-1}\mathrm{A}_i\mathrm{M}_u^{-1}\mathrm{B}\hat{\mathrm{D}}_p^{-1}$
$\delta t \to 0$	$\hat{\mathrm{D}}_p^{-1}$	$\hat{\mathrm{D}}_p^{-1}\mathrm{B}^\top\mathrm{M}_u^{-1}\mathrm{B}\hat{\mathrm{D}}_p^{-1} = \hat{\mathrm{D}}_p^{-1}$
Effort	1×Poisson & 1×mass	2×Poisson & 2×mass

Iterative solver

$$\tilde{\mathrm{p}}^{(n+1)} \mapsto \tilde{\mathrm{p}}^{(n+1)} + \mathrm{q}^{(n+1)},$$

$$\mathrm{q}^{(n+1)} = \mathrm{C}_i^{-1}(\mathrm{B}^\top\tilde{\mathrm{u}}^{(n+1)} - \mathrm{f}^{(n+1)}), \qquad \tilde{\mathrm{u}}^{(n+1)} = \mathrm{A}_i^{-1}(\tilde{\mathrm{g}}^{(n+1)} - \mathrm{A}_e\mathrm{u}^{(n)} - \mathrm{B}\tilde{\mathrm{p}}^{(n+1)})$$

using preconditioner $\mathrm{C}_i \approx \mathrm{B}^\top\mathrm{A}_i^{-1}\mathrm{B}$ and system matrix $\mathrm{A}_i = \mathrm{M}_u + \theta\delta t\mathrm{D}_u$

2 Global-in-time Stokes solver

Sequential time-stepping:

$$\begin{pmatrix} \mathrm{A}_i & \mathrm{B} \\ \mathrm{B}^\top & \end{pmatrix} \begin{pmatrix} \mathrm{u}^{(n+1)} \\ \tilde{\mathrm{p}}^{(n+1)} \end{pmatrix} = \begin{pmatrix} \tilde{\mathrm{g}}^{(n+1)} - \mathrm{A}_e \mathrm{u}^{(n)} \\ \mathrm{f}^{(n+1)} \end{pmatrix}, \qquad n = 0, \dots, K$$

Treating K time steps simultaneously:

$$\begin{pmatrix} \mathbf{A}_K & \mathbf{B}_K \\ \mathbf{B}_K^\top & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \tilde{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{g}} \\ \mathbf{f} \end{pmatrix}$$

$$:\Longleftrightarrow \left(\begin{array}{cccc|cccc} \mathrm{A}_i & & & & \mathrm{B} & & & \\ \mathrm{A}_e & \mathrm{A}_i & & & & \mathrm{B} & & \\ & \ddots & \ddots & & & & \ddots & \\ & & \mathrm{A}_e & \mathrm{A}_i & & & & \mathrm{B} \\ \hline \mathrm{B}^\top & & & & & & & \\ & \mathrm{B}^\top & & & & & & \\ & & \ddots & & & & & \\ & & & \mathrm{B}^\top & & & & \end{array}\right) \left(\begin{array}{c} \mathrm{u}^{(1)} \\ \mathrm{u}^{(2)} \\ \vdots \\ \mathrm{u}^{(K)} \\ \hline \tilde{\mathrm{p}}^{(1)} \\ \tilde{\mathrm{p}}^{(2)} \\ \vdots \\ \tilde{\mathrm{p}}^{(K)} \end{array}\right) = \left(\begin{array}{c} \tilde{\mathrm{g}}^{(1)} - \mathrm{A}_e \mathrm{u}^{(0)} \\ \tilde{\mathrm{g}}^{(2)} \\ \vdots \\ \tilde{\mathrm{g}}^{(K)} \\ \hline \mathrm{f}^{(1)} \\ \mathrm{f}^{(2)} \\ \vdots \\ \mathrm{f}^{(K)} \end{array}\right)$$

All-at-once solution strategy

$$\begin{pmatrix} \mathbf{A}_K & \mathbf{B}_K \\ \mathbf{B}_K^\top & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \tilde{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{g}} \\ \mathbf{f} \end{pmatrix}$$

Pressure Schur complement (PSC) equation:

$$\boxed{\mathbf{B}_K^\top \mathbf{A}_K^{-1} \mathbf{B}_K \tilde{\mathbf{p}} = \mathbf{B}_K^\top \mathbf{A}_K^{-1} \tilde{\mathbf{g}} - \mathbf{f}}$$
$$\mathbf{u} = \mathbf{A}_K^{-1}(\tilde{\mathbf{g}} - \mathbf{B}_K \tilde{\mathbf{p}})$$

Iterative solver

$$\tilde{\mathbf{p}} \quad \mapsto \quad \tilde{\mathbf{p}} + \mathbf{q},$$
$$\mathbf{q} = \mathbf{C}_K^{-1}(\mathbf{B}_K^\top \tilde{\mathbf{u}} - \mathbf{f}), \qquad \tilde{\mathbf{u}} = \mathbf{A}_K^{-1}(\tilde{\mathbf{g}} - \mathbf{B}_K \tilde{\mathbf{p}})$$

using preconditioner $\mathbf{C}_K \approx \mathbf{B}_K^\top \mathbf{A}_K^{-1} \mathbf{B}_K$

All-at-once solution strategy

How to define preconditioner $\mathbf{C}_K \approx \mathbf{B}_K^\top \mathbf{A}_K^{-1} \mathbf{B}_K$?

$$\mathbf{A}_K = \begin{pmatrix} \mathrm{A}_i & & & \\ \mathrm{A}_e & \mathrm{A}_i & & \\ & \ddots & \ddots & \\ & & \mathrm{A}_e & \mathrm{A}_i \end{pmatrix} = \begin{pmatrix} \mathrm{M}_u & & & \\ -\mathrm{M}_u & \mathrm{M}_u & & \\ & \ddots & \ddots & \\ & & -\mathrm{M}_u & \mathrm{M}_u \end{pmatrix} + \delta t \begin{pmatrix} \theta \mathrm{D}_u & & & \\ (1-\theta)\mathrm{D}_u & \theta \mathrm{D}_u & & \\ & \ddots & \ddots & \\ & & (1-\theta)\mathrm{D}_u & \theta \mathrm{D}_u \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{pmatrix} \otimes \mathrm{M}_u + \delta t \begin{pmatrix} \theta & & & \\ 1-\theta & \theta & & \\ & \ddots & \ddots & \\ & & 1-\theta & \theta \end{pmatrix} \otimes \mathrm{D}_u$$

$$= \mathrm{U}_K \otimes \mathrm{M}_u + \delta t \mathrm{V}_K \otimes \mathrm{D}_u$$

$$\mathbf{B}_K = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix} \otimes \mathrm{B} = \mathrm{I}_K \otimes \mathrm{B}, \quad \mathbf{B}_K^\top = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix} \otimes \mathrm{B}^\top = \mathrm{I}_K \otimes \mathrm{B}^\top$$

All-at-once solution strategy

How to define preconditioner $\mathbf{C}_K \approx \mathbf{B}_K^\top \mathbf{A}_K^{-1} \mathbf{B}_K$?

1. **PCD preconditioner (cf. Danieli, Southworth, and Wathen 2022):**
Assuming $\mathbf{A}_K^{-1}\mathbf{B}_K(\mathrm{I}_K \otimes \mathrm{M}_p^{-1}) \approx (\mathrm{I}_K \otimes \mathrm{M}_u^{-1})\mathbf{B}_K\mathbf{A}_{K,p}^{-1}$

$$\begin{aligned}\mathbf{C}_K^{-1} &\coloneqq (\mathrm{I}_K \otimes \mathrm{M}_p^{-1})\mathbf{A}_{K,p}\big(\mathbf{B}_K^\top(\mathrm{I}_K \otimes \mathrm{M}_u^{-1})\mathbf{B}_K\big)^{-1} \\ &= (\mathrm{I}_K \otimes \mathrm{M}_p^{-1})\mathbf{A}_{K,p}(\mathrm{I}_K \otimes \hat{\mathrm{D}}_p^{-1}) \\ &= (\mathrm{U}_K \otimes \hat{\mathrm{D}}_p^{-1}) + \delta t \mu(\mathrm{V}_K \otimes \mathrm{M}_p^{-1})\end{aligned}$$

2. **Scaled BFBt preconditioner:**
Using Moore-Penrose inverse $(\mathrm{M}_u^{-1/2}\mathrm{B})^{+} = \hat{\mathrm{D}}_p^{-1}\mathrm{M}_u^{-1/2}\mathrm{B}^\top$

$$\mathbf{C}_K^{-1} \coloneqq \big(\mathrm{I}_K \otimes (\hat{\mathrm{D}}_p^{-1}\mathrm{B}^\top\mathrm{M}_u^{-1})\big)\mathbf{A}_K\big(\mathrm{I}_K \otimes (\mathrm{M}_u^{-1}\mathrm{B}\hat{\mathrm{D}}_p^{-1})\big)$$

Application of PCD preconditioner

$$\mathbf{r} = \mathbf{B}_K^\top \tilde{\mathbf{u}} - \mathbf{f}, \qquad \mathbf{C}_K^{-1} = (\mathrm{U}_K \otimes \hat{\mathrm{D}}_p^{-1}) + \delta t \mu (\mathrm{V}_K \otimes \mathrm{M}_p^{-1})$$
$$= (\mathrm{I}_K \otimes \hat{\mathrm{D}}_p^{-1}){\color{red}(\mathrm{U}_K \otimes \mathrm{I})} + \delta t \mu (\mathrm{I}_K \otimes \mathrm{M}_p^{-1}){\color{red}(\mathrm{V}_K \otimes \mathrm{I})}$$

- **Step 1:**

$$\tilde{\mathbf{r}}_1 = (\mathrm{U}_K \otimes \mathrm{I})\mathbf{r} = \begin{pmatrix} \mathrm{r}^{(1)} \\ \mathrm{r}^{(2)} \\ \vdots \\ \mathrm{r}^{(K)} \end{pmatrix} - \begin{pmatrix} 0 \\ \mathrm{r}^{(1)} \\ \vdots \\ \mathrm{r}^{(K-1)} \end{pmatrix},$$

$$\tilde{\mathbf{r}}_2 = (\mathrm{V}_K \otimes \mathrm{I})\mathbf{r} = \theta \begin{pmatrix} \mathrm{r}^{(1)} \\ \mathrm{r}^{(2)} \\ \vdots \\ \mathrm{r}^{(K)} \end{pmatrix} + (1-\theta) \begin{pmatrix} 0 \\ \mathrm{r}^{(1)} \\ \vdots \\ \mathrm{r}^{(K-1)} \end{pmatrix}$$

Application of PCD preconditioner

$$\mathbf{r} = \mathbf{B}_K^\top \tilde{\mathbf{u}} - \mathbf{f}, \qquad \mathbf{C}_K^{-1} = (\mathrm{U}_K \otimes \hat{\mathrm{D}}_p^{-1}) + \delta t \mu (\mathrm{V}_K \otimes \mathrm{M}_p^{-1})$$
$$= (\mathrm{I}_K \otimes \hat{\mathrm{D}}_p^{-1})(\mathrm{U}_K \otimes \mathrm{I}) + \delta t \mu (\mathrm{I}_K \otimes \mathrm{M}_p^{-1})(\mathrm{V}_K \otimes \mathrm{I})$$

- **Step 2:**

$$\begin{pmatrix} \tilde{q}_1^{(1)} \\ \vdots \\ \tilde{q}_1^{(K)} \end{pmatrix} = \begin{pmatrix} \hat{\mathrm{D}}_p^{-1} & & \\ & \ddots & \\ & & \hat{\mathrm{D}}_p^{-1} \end{pmatrix} \begin{pmatrix} \tilde{r}_1^{(1)} \\ \vdots \\ \tilde{r}_1^{(K)} \end{pmatrix}$$
$$\iff \quad \begin{pmatrix} \tilde{q}_1^{(1)} & \cdots & \tilde{q}_1^{(K)} \end{pmatrix} = \hat{\mathrm{D}}_p^{-1} \begin{pmatrix} \tilde{r}_1^{(1)} & \cdots & \tilde{r}_1^{(K)} \end{pmatrix},$$
$$\begin{pmatrix} \tilde{q}_2^{(1)} \\ \vdots \\ \tilde{q}_2^{(K)} \end{pmatrix} = \begin{pmatrix} \mathrm{M}_p^{-1} & & \\ & \ddots & \\ & & \mathrm{M}_p^{-1} \end{pmatrix} \begin{pmatrix} \tilde{r}_2^{(1)} \\ \vdots \\ \tilde{r}_2^{(K)} \end{pmatrix}$$
$$\iff \quad \begin{pmatrix} \tilde{q}_2^{(1)} & \cdots & \tilde{q}_2^{(K)} \end{pmatrix} = \mathrm{M}_p^{-1} \begin{pmatrix} \tilde{r}_2^{(1)} & \cdots & \tilde{r}_2^{(K)} \end{pmatrix}$$

All-at-once solution strategy

Properties of preconditioners:

	PCD preconditioner	Scaled BFBt preconditioner
C^{-1}	$(\mathrm{I}_K \otimes \mathrm{M}_p^{-1})\mathbf{A}_{K,p}(\mathrm{I}_K \otimes \hat{\mathrm{D}}_p^{-1})$	$\big(\mathrm{I}_K \otimes (\hat{\mathrm{D}}_p^{-1}\mathrm{B}^\top \mathrm{M}_u^{-1})\big)\mathbf{A}_K\big(\mathrm{I}_K \otimes (\mathrm{M}_u^{-1}\mathrm{B}\hat{\mathrm{D}}_p^{-1})\big)$
$\delta t \to 0$	$\mathrm{U}_K \otimes \hat{\mathrm{D}}_p^{-1}$	$\mathrm{U}_K \otimes \hat{\mathrm{D}}_p^{-1}$
Effort	1×Poisson & 1×mass	2×Poisson & 2×mass

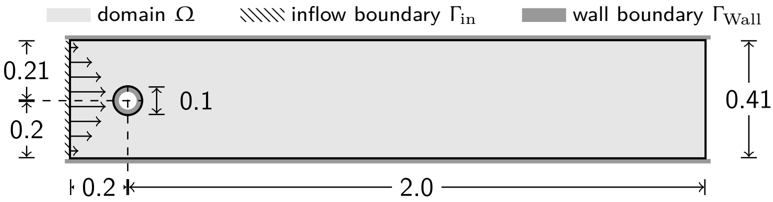
Iterative solver

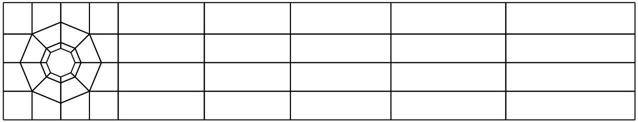
$$\tilde{\mathbf{p}} \quad \mapsto \quad \tilde{\mathbf{p}} + \mathbf{q},$$

$$\mathbf{q} = \mathbf{C}_K^{-1}(\mathbf{B}_K^\top \tilde{\mathbf{u}} - \mathbf{f}), \qquad \tilde{\mathbf{u}} = \mathbf{A}_K^{-1}(\tilde{\mathbf{g}} - \mathbf{B}_K \tilde{\mathbf{p}})$$

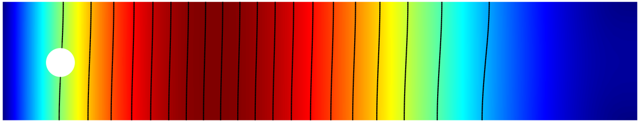
using preconditioner $\mathbf{C}_K \approx \mathbf{B}_K^\top \mathbf{A}_K^{-1} \mathbf{B}_K$ and $\mathbf{A}_K = \mathrm{U}_K \otimes \mathrm{M}_u + \delta t \mathrm{V}_K \otimes \mathrm{D}_u$

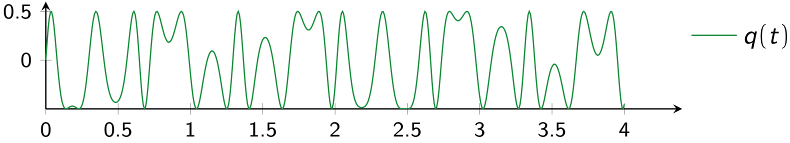
Numerical examples — Domain and triangulation





level	nodes	edges	elements	#dofs($\mathbf{v}$)	#dofs(p)	total
0	65	113	48	$452K$	$65K$	$517K$
1	226	418	192	$1672K$	$226K$	$1898K$
2	836	1604	768	$6416K$	$836K$	$7252K$
3	3208	6280	3072	$25120K$	$3208K$	$28328K$
4	12560	24848	12288	$99392K$	$12560K$	$111952K$





Manufactured solution:

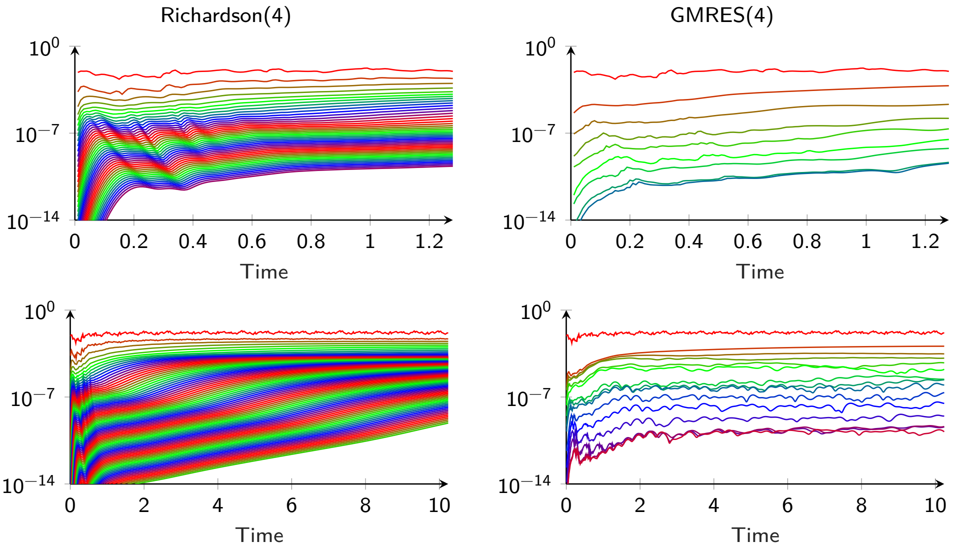
$$\mathbf{v}(\mathbf{x},t) = \begin{pmatrix} \sin(\sigma x)\sin(\sigma y)q(2t) + y(y_{\max} - y)\big(1 - 2q(t)\big) \\ \cos(\sigma x)\cos(\sigma y)q(2t) + x(x_{\max} - x)^2\big(1 + 3q(t)\big) \end{pmatrix},$$

$$p(\mathbf{x},t) = \mu\sigma\cos(\sigma x)\sin(\sigma y)q(2t) + \mu(x_{\max} - x)q(t),$$

$$q(t) = \tfrac{1}{2}\sin(4\sin(3\pi t) + 6t), \qquad \sigma = 4\pi x_{\max}^{-1}, \qquad \mu = 10^{-2}$$

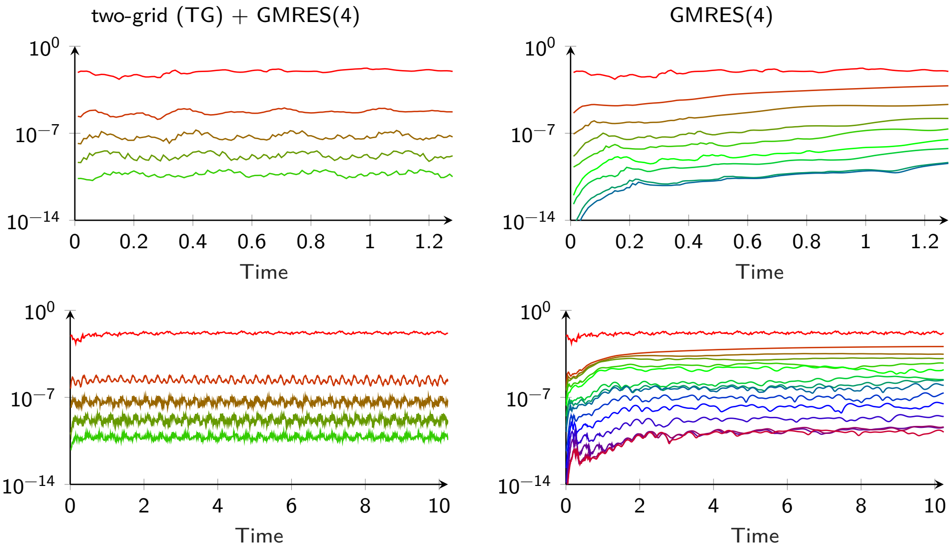
Numerical examples — Stokes equations

Norm of residual for $\mathrm{lvl} = 2$, $\delta t = \frac{1}{100}$, PCD preconditioner



Numerical examples — Stokes equations

Norm of residual for $\mathrm{lvl} = 2$, $\delta t = \frac{1}{100}$, PCD preconditioner



Multigrid acceleration

Iterative solver

$$\tilde{\mathbf{p}} \quad \mapsto \quad \tilde{\mathbf{p}} + \mathbf{q},$$

$$\mathbf{q} = \mathbf{C}_K^{-1}(\mathbf{B}_K^\top \tilde{\mathbf{u}} - \mathbf{f}), \qquad \tilde{\mathbf{u}} = \mathbf{A}_K^{-1}(\tilde{\mathbf{g}} - \mathbf{B}_K \tilde{\mathbf{p}})$$

using preconditioner $\mathbf{C}_K \approx \mathbf{B}_K^\top \mathbf{A}_K^{-1} \mathbf{B}_K$

$$\boxed{\hat{\mathbf{C}}_K^{-1} = \mathbf{P}_{\bar{K}}^{K} (\bar{\mathbf{B}}_{\bar{K}}^\top \bar{\mathbf{A}}_{\bar{K}}^{-1} \bar{\mathbf{B}}_{\bar{K}})^{-1} \mathbf{R}_K^{\bar{K}}}$$

using space-time restriction and prolongation operators $\mathbf{R}_K^{\bar{K}}$ and $\mathbf{P}_{\bar{K}}^{K}$

Possibilities:

1. space-time coarsening
   - 2^{1+d} times fewer `dofs`
   - $\frac{\delta t}{\Delta x}$ constant
2. space coarsening
   - 2^d times fewer `dofs`
   - $\frac{\delta t}{\Delta x}$ halved
3. time coarsening
   - 2 times fewer `dofs`
   - $\frac{\delta t}{\Delta x}$ doubled

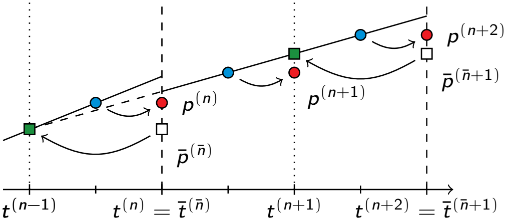
Intergrid transfer in time

$$\mathrm{M}_u \frac{\mathrm{u}^{(n+1)} - \mathrm{u}^{(n)}}{\delta t} + \theta \mathrm{D}_u \mathrm{u}^{(n+1)} + (1-\theta)\mathrm{D}_u \mathrm{u}^{(n)} + \mathrm{B}\mathrm{p}^{(n+1)} = \theta \mathrm{g}^{(n+1)} + (1-\theta)\mathrm{g}^{(n)},$$

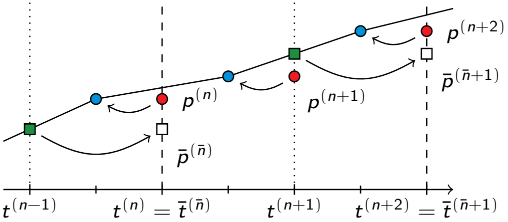
$$\mathrm{B}^\top \mathrm{u}^{(n+1)} = \mathrm{f}^{(n+1)}$$

Problem:

- pressure treated fully implicitly
- but second order accurate when $p^{(n+1)} \approx p(t^{n+1/2})$

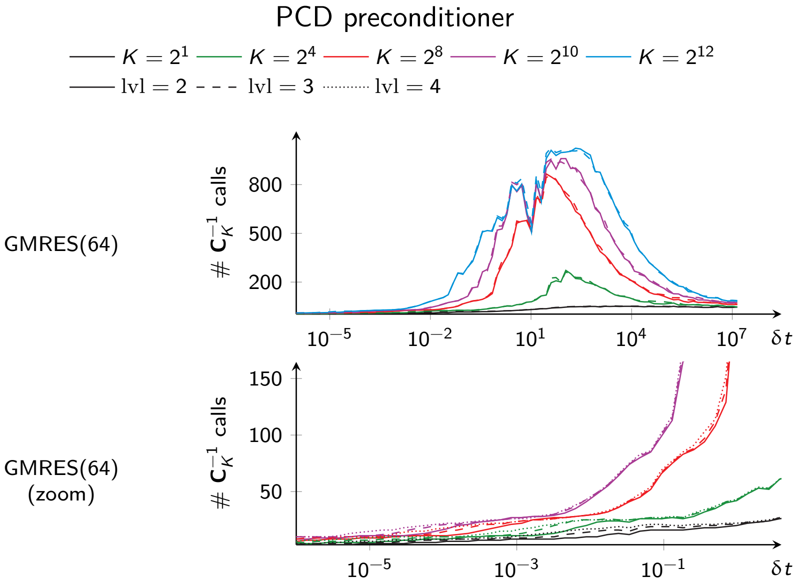


Prolongation (forward in time)



Restriction (backward in time)

Numerical examples — Stokes equations

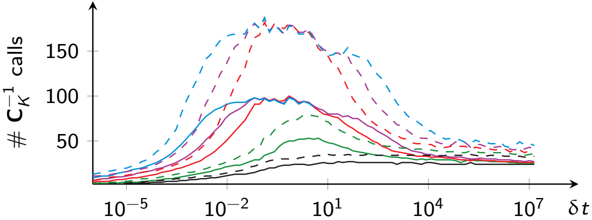


Scaled BFBt preconditioner

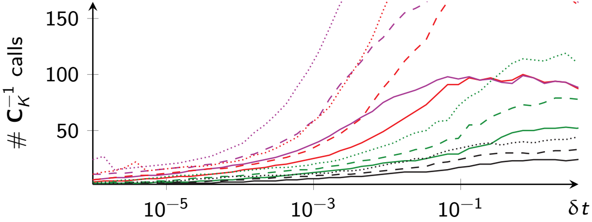
$K=2^1$ $K=2^4$ $K=2^8$ $K=2^{10}$ $K=2^{12}$

$\mathrm{lvl}=2$ $\mathrm{lvl}=3$ $\mathrm{lvl}=4$

GMRES(64)



GMRES(64) (zoom)



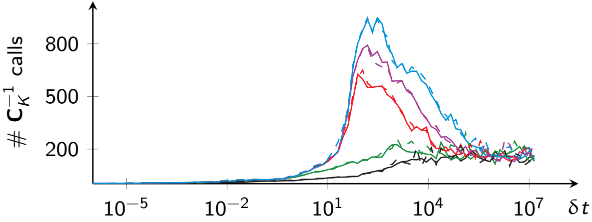
Numerical examples — Stokes equations

PCD preconditioner

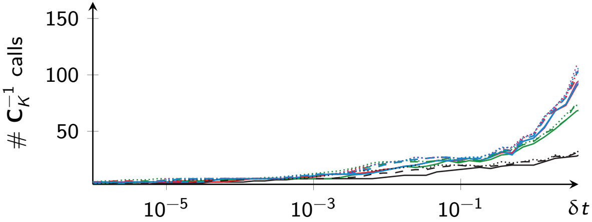
—— $K = 2^1$ —— $K = 2^4$ —— $K = 2^8$ —— $K = 2^{10}$ —— $K = 2^{12}$

—— lvl = 2 - - - lvl = 3 ······· lvl = 4

TG + GMRES(4)

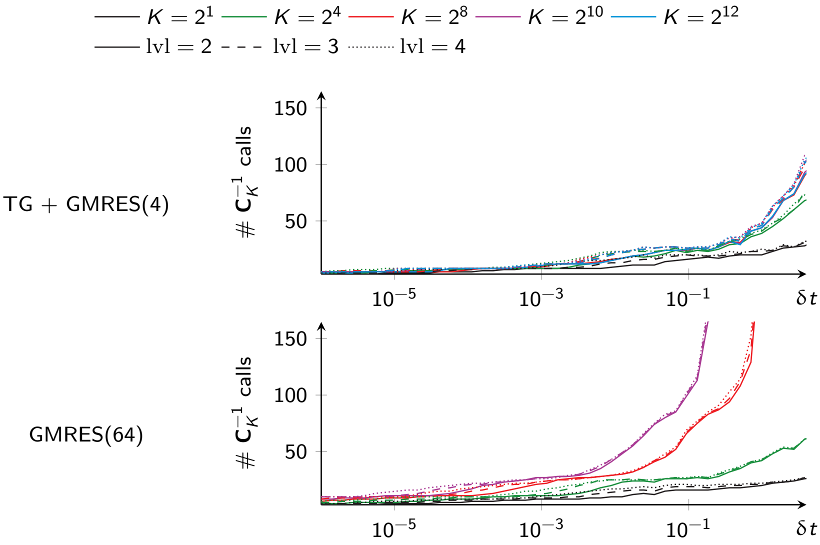


TG + GMRES(4)
(zoom)



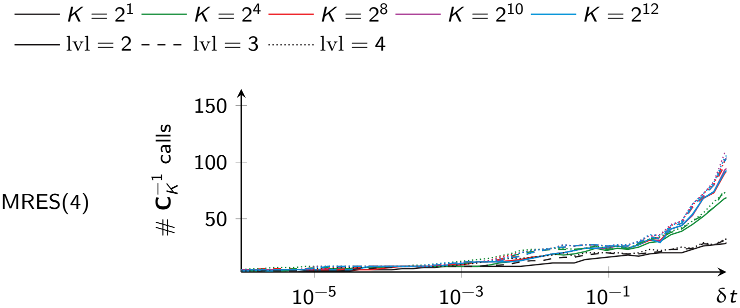
Numerical examples — Stokes equations

PCD preconditioner

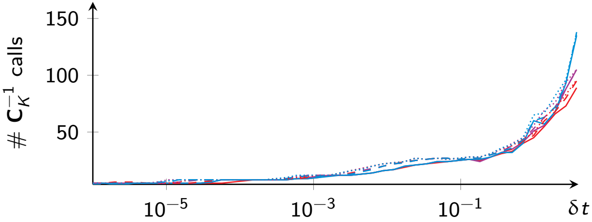


Numerical examples — Stokes equations

PCD preconditioner



TG + GMRES(4)



MG(7) + GMRES(4)

Extension to Oseen equations

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + (\tilde{\mathbf{v}} \cdot \nabla)\mathbf{v} - \mu\Delta\mathbf{v} + \operatorname{grad}(p) &= \rho\mathbf{g} && \text{in } \Omega \times (0, T),\\
\operatorname{div}(\mathbf{v}) &= 0 && \text{in } \Omega \times (0, T),\\
\mathbf{v}(\cdot, 0) &= \mathbf{v}_0 && \text{on } \Omega,\\
\mathbf{v} &= \mathbf{v}_\mathrm{D} && \text{on } \Gamma_\mathrm{D} \times (0, T),\\
-p\mathbf{n} + \mu(\mathbf{n} \cdot \nabla)\mathbf{v} &= \rho\mathbf{h} && \text{on } \Gamma_\mathrm{N} \times (0, T)
\end{aligned}$$

- **PCD preconditioner:**

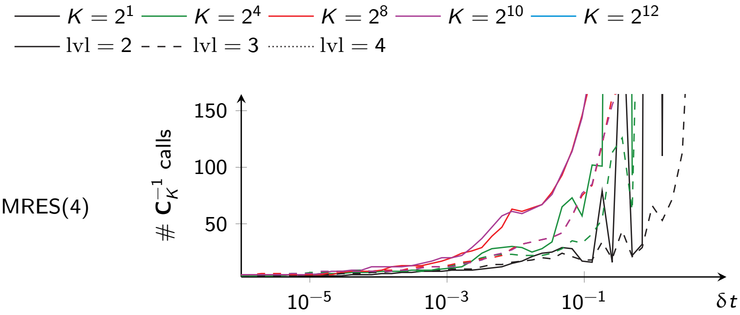
$$\begin{aligned}
\mathbf{C}_K^{-1} &= (\mathrm{I}_K \otimes \mathrm{M}_p^{-1})\mathbf{A}_{K,p}(\mathrm{I}_K \otimes \hat{\mathrm{D}}_p^{-1})\\
&= (\mathrm{U}_K \otimes \hat{\mathrm{D}}_p^{-1}) + (\mathrm{I}_K \otimes \mathrm{M}_p^{-1})\mathbf{K}_{p,K}(\mathrm{I}_K \otimes \hat{\mathrm{D}}_p^{-1}) + \delta t \mu(\mathrm{V}_K \otimes \mathrm{M}_p^{-1})
\end{aligned}$$

- **Scaled BFBt preconditioner:**

$$\mathbf{C}_K^{-1} = \big(\mathrm{I}_K \otimes (\hat{\mathrm{D}}_p^{-1}\mathrm{B}^\top\mathrm{M}_u^{-1})\big)\mathbf{A}_K\big(\mathrm{I}_K \otimes (\mathrm{M}_u^{-1}\mathrm{B}\hat{\mathrm{D}}_p^{-1})\big)$$

Numerical examples — Oseen equations

PCD preconditioner



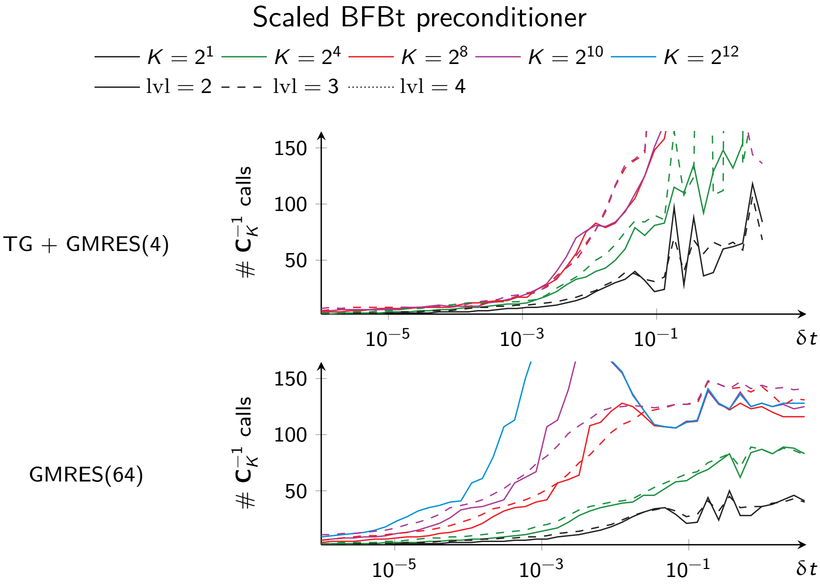
TG + GMRES(4)

GMRES(64)

$K = 2^1$ $K = 2^4$ $K = 2^8$ $K = 2^{10}$ $K = 2^{12}$

lvl = 2 lvl = 3 lvl = 4

Numerical examples — Oseen equations



3 Global-in-time Navier-Stokes solver

Treating nonlinearities

Incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \mu\Delta\mathbf{v} + \mathrm{grad}(p) = \rho\mathbf{g}, \qquad \mathrm{div}(\mathbf{v}) = 0$$

Residual of momentum equation

$$\mathbf{r}(\mathbf{v}, p) \coloneqq \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \mu\Delta\mathbf{v} + \mathrm{grad}(p) - \rho\mathbf{g}.$$

Solution update:

$$(\mathbf{v}, p) \quad \mapsto \quad (\mathbf{v} - \bar{\mathbf{v}}, p - \bar{p})$$

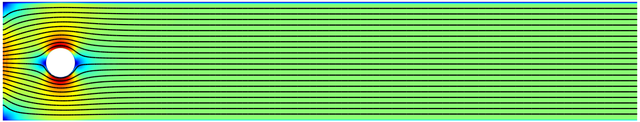
- Picard iteration

$$\frac{\partial \bar{\mathbf{v}}}{\partial t} + (\mathbf{v}\cdot\nabla)\bar{\mathbf{v}} - \mu\Delta\bar{\mathbf{v}} + \mathrm{grad}(\bar{p}) = \mathbf{r}(\mathbf{v}, p), \qquad \mathrm{div}(\bar{\mathbf{v}}) = \mathrm{div}(\mathbf{v})$$

- Newton's method

$$\frac{\partial \bar{\mathbf{v}}}{\partial t} + (\mathbf{v}\cdot\nabla)\bar{\mathbf{v}} + (\bar{\mathbf{v}}\cdot\nabla)\mathbf{v} - \mu\Delta\bar{\mathbf{v}} + \mathrm{grad}(\bar{p}) = \mathbf{r}(\mathbf{v}, p), \qquad \mathrm{div}(\bar{\mathbf{v}}) = \mathrm{div}(\mathbf{v})$$

Numerical examples — FAC Bench 2D-1, laminar case

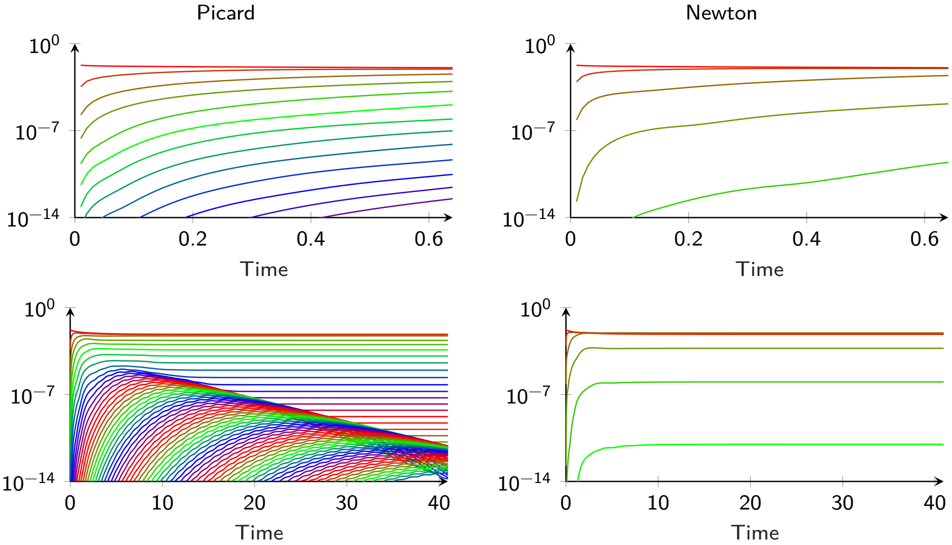


$\nu = 10^{-3}$, $\mathrm{Re} = 20$
Inflow boundary condition

$$\mathbf{v}_{\mathrm{D}} = 0.3 \frac{4y(0.41 - y)}{0.41^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Numerical examples — FAC Bench 2D-1, laminar case

Picard iteration vs. Newton's method for $\delta t = \frac{1}{100}$, $\mathrm{lvl} = 2$, and different T



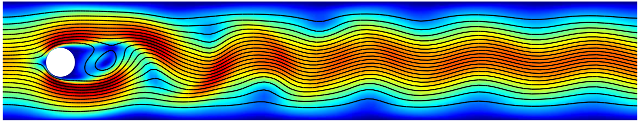
Numerical examples — FAC Bench 2D-1, laminar case

Newton's method, zero initial guess, TG, $\mathrm{tol}_{\mathrm{rel}} = 10^{-4}$, 4 smooth. steps, PCD precon.

$\delta t = 1/400$, TG	$K = 2^8$			$K = 2^{10}$			$K = 2^{12}$		
	nonlinear	linear	smother	nonlinear	linear	smother	nonlinear	linear	smother
$\mathrm{lvl} = 1$	5	10	33	6	11	43	6	16	46
$\mathrm{lvl} = 2$	5	10	28	6	12	39	6	14	47
$\mathrm{lvl} = 3$	5	8	27	6	10	39	6	13	49

$\mathrm{lvl} = 2$, TG	$K = 2^8$			$K = 2^{10}$			$K = 2^{12}$		
	nonlinear	linear	smother	nonlinear	linear	smother	nonlinear	linear	smother
$\delta t = 1/25$	7	28	99	6	28	105	6	28	95
$\delta t = 1/100$	6	13	46	6	18	57	6	17	56
$\delta t = 1/400$	5	10	28	6	12	39	6	14	47

$K = 2^{10}$	$\delta t = 1/100$, $\mathrm{lvl} = 2$			$\delta t = 1/400$, $\mathrm{lvl} = 2$			$\delta t = 1/400$, $\mathrm{lvl} = 3$		
	nonlinear	linear	smother	nonlinear	linear	smother	nonlinear	linear	smother
TG	6	18	57	6	12	39	6	10	39
F-cycle(5)	6	20	63	6	12	39	6	11	39
V-cycle(5)	6	39	150	6	13	47	6	10	39

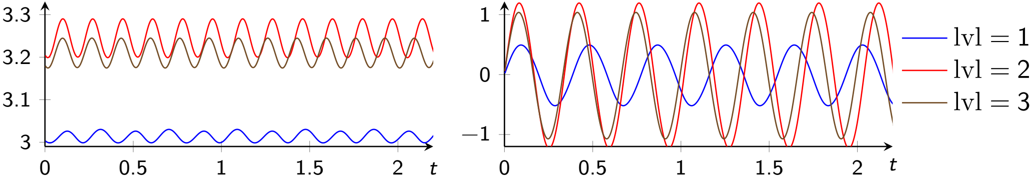


$\nu = 10^{-3}$, $\mathrm{Re} = 100$

Inflow boundary condition

$$\mathbf{v}_{\mathrm{D}} = 1.5 \frac{4y(0.41 - y)}{0.41^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Using different hierarchical starting procedures



Drag Lift

Numerical examples — FAC Bench 2D-2, time-periodic case

Newton's method, TG, $\mathrm{tol}_{\mathrm{rel}} = 10^{-4}$, 4 smoothing steps, PCD preconditioner

hier. in time, $\mathrm{lvl} = 2$	$K = 2^8$			$K = 2^9$			$K = 2^{10}$		
	nonlinear	linear	smother	nonlinear	linear	smother	nonlinear	linear	smother
$\delta t = 1/200$	5	85	338	6	109	443	—	—	—
$\delta t = 1/400$	4	26	98	4	27	101	5	34	130
$\delta t = 1/800$	4	15	48	4	15	54	4	16	57

hier. in time	$K = 2^7$			$K = 2^8$			$K = 2^9$		
	nonlinear	linear	smother	nonlinear	linear	smother	nonlinear	linear	smother
$\delta t = \frac{1}{200}$, $\mathrm{lvl} = 2$	4	57	221	5	85	338	6	109	443
$\delta t = \frac{1}{400}$, $\mathrm{lvl} = 2$	4	24	87	4	26	98	4	27	101
$\delta t = \frac{1}{400}$, $\mathrm{lvl} = 3$	4	45	172	4	52	204	4	52	203

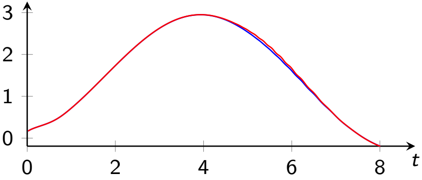
$\delta t = 1/400$, $\mathrm{lvl} = 3$	$K = 2^7$			$K = 2^8$			$K = 2^9$		
	nonlinear	linear	smother	nonlinear	linear	smother	nonlinear	linear	smother
hier. in time	4	45	172	4	52	204	4	52	203
hier. in space	4	48	186	6	56	218	7	59	232
hier. in s.-t.	6	49	191	6	58	226	7	61	241

$\nu = 10^{-3}$, $\mathrm{Re} = 100$, $t \in [0, 8]$

Inflow boundary condition

$$\mathbf{v}_{\mathrm{D}} = 1.5 \sin(0.125\pi t) \frac{4y(0.41 - y)}{0.41^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Using Stokes solution as initial guess



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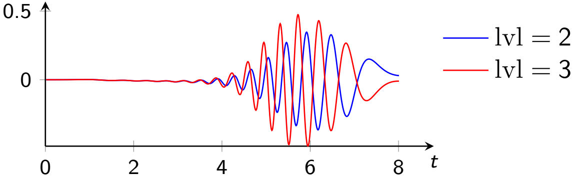
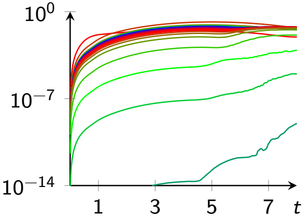


Lift

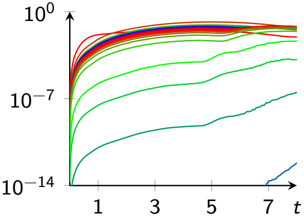
Numerical examples — FAC Bench 2D-3, fixed time interval

Newton's method using adaptive damping strategy

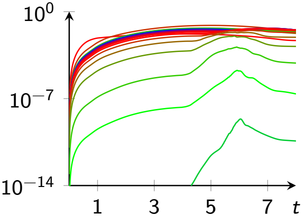
exact (top) vs. TG, $\mathrm{tol}_{\mathrm{rel}} = 10^{-4}$, 4 smoothing steps, PCD preconditioner (bottom)



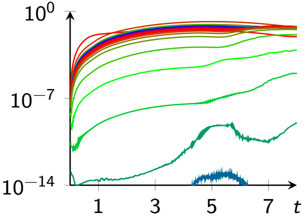
$\mathrm{lvl} = 2$, $\delta t = 1/100$ (24)



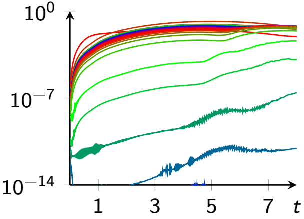
$\mathrm{lvl} = 2$, $\delta t = 1/200$ (22)



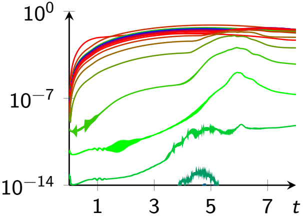
$\mathrm{lvl} = 3$, $\delta t = 1/200$ (23)



$\mathrm{lvl} = 2$, $\delta t = 1/100$ (24, 205, 801)



$\mathrm{lvl} = 2$, $\delta t = 1/200$ (25, 101, 381)



$\mathrm{lvl} = 3$, $\delta t = 1/200$ (24, 186, 719)

Conclusions

Summary

- Candidate for global-in-time and K-independent flow solver
  1. Preconditioner
  2. Multigrid in time
  3. Newton's method for linearization

Outlook/challenges

- Coarse grid solver
- Efficient damping strategy for Newton's method
- Improved preconditioner for convection-dominated flows
- Hardware-oriented implementation

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