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Efficient FEM-multigrid solver for granular material

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Generalized Navier-Stokes equation

Conservation of mass

$$\nabla \cdot u = 0 \qquad \text{in } \Omega$$

Conservation of momentum

$$\rho \frac{Du}{Dt} = -\nabla p + \nabla \cdot \sigma + f \qquad \text{in } \Omega$$

along with some initial and boundary conditions

Flow Rheology

$$\sigma = 2\eta(\gamma_{II}, p)\dot{\gamma};$$

$$\text{where} \quad \dot{\gamma} = \frac{1}{2}(\nabla u + \nabla u^T),$$

$$\text{and} \quad 2\gamma_{II} = \dot{\gamma} : \dot{\gamma} = |\dot{\gamma}|^2 = tr[(\dot{\gamma})^2]$$

Nonlinear and coupled equations

Miscellaneous models

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Fluid models

$$\eta(\gamma_{II}, p) = \eta_0, \qquad \text{for Newtonian fluid}$$

$$= \eta_0(|\dot{\gamma}|^2 + \epsilon)^{\frac{m}{2}-1}, \qquad \text{for Power law fluid}$$

Granular materials

$$\eta(\gamma_{II}, p) = \sqrt{2} p \sin\phi \frac{1}{|\dot{\gamma}|}, \qquad \text{Schaeffer model}$$

$$= \sqrt{2} p (\sin\phi + b\cos\phi |\dot{\gamma}|^n) \frac{1}{|\dot{\gamma}|}, \qquad \text{Schaeffer-Tardos model}$$

$$= \frac{\sqrt{2}}{2} \left(\frac{\alpha p}{|\dot{\gamma}|} + \frac{\beta d p}{\delta \sqrt{\frac{p}{\rho}} + |\dot{\gamma}| d} \right), \qquad \text{Poliquen model}$$

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Bilinear Form

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Find $(u, p) \in X \times M$ such that

$$\int_\Omega 2\eta(\gamma_{II}, p) D(u) : D(v) dx + \int_\Omega (u.\nabla u) v dx$$

$$+ \int_\Omega p \text{ div } v dx = \int_\Omega f v dx, \qquad \forall v \in X;$$

$$\text{with } \int_\Omega q \text{ div } u dx = 0, \qquad \forall q \in M$$

Compact form

Find $(u, p) \in X \times M$ such that

$$\underbrace{\langle L(u,p)u, v\rangle + \langle N(u)u, v\rangle}_{Au} + \langle Bp, v\rangle = \underbrace{\int_\Omega f v dx}_{g}, \qquad \forall v \in X;$$

$$\langle q, B^T u\rangle = 0, \qquad \forall q \in M$$

Nonlinear discrete system

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FEM discretization

We take $X_h \subset X$ and $M_h \subset M$

Find $(\tilde{u}, \tilde{p}) \in X_h \times M_h$ such that

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix}$$

Nonlinear saddle point problem!

Newton solver

Motivation

- Strongly coupled problem
- Solution $x^{n+1} = (\tilde{u}, p)$, Residual equation $R(x^n)$

  $$x^{n+1} = x^n + w^n \left[\frac{\partial R(x^n)}{\partial x}\right]^{-1} R(x^n)$$

- Automatic damping control w^n for each nonlinear step
- Quadratic convergence when iterative solutions are close to the actual one

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Jacobian with respect to the Diffusive term

$$\int_\Omega 2\eta(\gamma_{II}(u^l),p^l)D(u):D(v)dx$$
$$+\int_\Omega 2\partial_1\eta(\gamma_{II}(u^l),p^l)[D(u^l):D(u)][D(u^l):D(v)]dx$$
$$+\int_\Omega 2\partial_2\eta(\gamma_{II}(u^l),p^l)[D(u^l):D(v)]pdx$$
$$=\int_\Omega fvdx-\int_\Omega 2\eta(\gamma_{II}(u^l),p^l)D(u^l):D(v)dx,\forall v\in V$$

Jacobian with respect to the Convective term

$$\int_\Omega (u^l.\nabla u)vdx \quad +\int_\Omega (u.\nabla u^l)vdx \qquad \forall v\in X,$$

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Diffusive term

$$\int_\Omega 2\eta(\gamma_{II}(u^l), p^l) D(u) : D(v) dx \quad {\color{red}L}$$
$$+ \int_\Omega 2\partial_1\eta(\gamma_{II}(u^l), p^l)[D(u^l) : D(u)][D(u^l) : D(v)] dx \quad {\color{red}L^*}$$
$$+ \int_\Omega 2\partial_2\eta(\gamma_{II}(u^l), p^l)[D(u^l) : D(v)] p dx \quad {\color{red}B^*}$$
$$= \int_\Omega f v dx - \int_\Omega 2\eta(\gamma_{II}(u^l), p^l) D(u^l) : D(v) dx, \forall v \in V$$

Convective term

$$\int_\Omega (u^l.\nabla u) v dx \; {\color{red}N} \; + \int_\Omega (u.\nabla u^l) v dx \quad {\color{red}N^*} \quad \forall v \in X,$$

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Final discrete problem

Compute $\tilde{u}$ and $\tilde{p}$ by solving

$$\begin{pmatrix} A & \tilde{B} \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} Res_{\tilde{u}} \\ Res_{\tilde{p}} \end{pmatrix}$$

where $A\tilde{u} = [(L + \delta_d L^*)(\tilde{u}^l, \tilde{p}^l) + (N + \delta_c N^*)(\tilde{u}^l)]\tilde{u},$

$$\tilde{B}\tilde{p} = [B + \delta_{\tilde{p}} B^*(\tilde{u}^l, \tilde{p}^l)]\tilde{p}$$

Unusual saddle point problem!

Linear solver

Multigrid techniques

- Direct Gauss elimination as coarse-grid solver
- General VANKA smoother with block-diagonal preconditioner
- F-cycle multigrid
- Intergrid transfer and coarse grid correction based on the underlying mesh hierarchy and the finite elements

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Benchmark problem

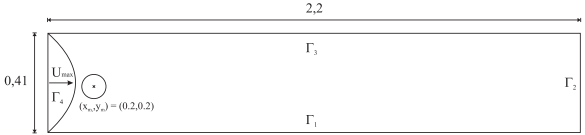

Figure : Geometry for the 'flow around cylinder' configuration

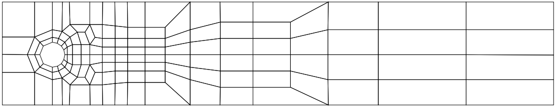

Figure : Computational mesh for the 'flow around cylinder' configuration

Featflow

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Why Featflow

- Basic flow solver for incompressible fluids
- Supports higher order (space/time) FEM
- Use of unstructured meshes
- Dynamic adaptive grid formulation
- FEM based tools

FEM characteristics

- Stable FE spaces for velocity/pressure and velocity/stress interpolation; e.g. $Q2/P1$
- Special treatments of the convective term - EO FEM, TVD/FCT

Inner solver

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Solver

- Nonlinearity handled by Newton method
- Monolithic Multigrid techniques for the auxiliary linearized problem
- Vanishing shear rate taken care by the regularization parameter
- Appropriate module to ensure unique and positive pressure

Advantages

- Inf-sup stable (LBB condition) for velocity and pressure
- Higher order is good for accuracy
- Discontinuous pressure is good for the solver

Results

Shear-thinning fluid

$$\eta(\gamma_{II}, p) = \eta_0(|\dot{\gamma}|^2 + \epsilon)^{\frac{m}{2}-1}, \qquad \eta_0 = 10^{-3}, m = 1.5, \epsilon = 0.1$$

MG	*Drag*		*Lift*		*Solver Statistics*	
Level	*Ref*[1]	Feat2	*Ref*[1]	Feat2	*Ref*[1]	Feat2
2	3.19922	3.08944	-0.01238	-0.01212	9/3	8/3
3	3.26246	3.22926	-0.01334	-0.01313	4/2	4/2
4	3.27553	3.26638	-0.01335	-0.01334	3/2	3/2
5	3.27781	3.27536	-0.01332	-0.01330	2/2	3/2

Efficient nonlinear and linear solver

Ref[1]: Damanik et al. Monolithic Newton-multigrid solution techniques for incompressible nonlinear flow models, International Journal for Numerical Methods in Fluids, Volume 71, Issues 2, pages 208-222.

Shear-thickening fluid

$$\eta(\gamma_{II}, p) = \eta_0(|\dot{\gamma}|^2 + \epsilon)^{\frac{m}{2}-1}, \qquad \eta_0 = 10^{-3}, m = 3.0, \epsilon = 0.1$$

MG	*Drag*		*Lift*		*Solver Statistics*	
Level	*Ref*[1]	Feat2	*Ref*[1]	Feat2	*Ref*[1]	Feat2
2	13.66925	13.97544	0.34424	0.34748	7/2	6/3
3	13.78575	13.86550	0.35232	0.35351	3/2	3/2
4	13.81625	13.83644	0.35148	0.35193	3/2	3/2
5	13.82490	13.82989	0.35248	0.35265	3/1	3/1

Ref[1]: Damanik et al. Monolithic Newton-multigrid solution techniques for incompressible nonlinear flow models, International Journal for Numerical Methods in Fluids, Volume 71, Issues 2, pages 208-222.

Test case

$$\eta(\gamma_{II}, p) = \eta_0 \exp(p), \qquad \eta_0 = 10^{-3}$$

Fluid	Level	Drag	Lift	*Solver Statistics*
Test case	2	5.648734E+00	1.118633E-02	5/3
	3	5.667088E+00	1.226157E-02	3/2
	4	5.672566E+00	1.227602E-02	3/1.7
	5	5.673991E+00	1.227831E-02	2/1.5

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Conclusion

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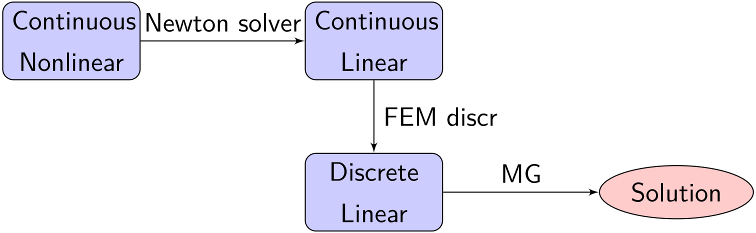
Summary

- Newton solver for nonlinearity
- Multigrid techniques for the linearized problem
- Numerical examples
    1. non-Newtonian fluids including shear thinning and shear thickening fluid
    2. Test case with pressure dependant viscosity



Outlook

Current work

- Benchmarking of Poliquen model
- Test with split-bottom geometry
- To incorporate Tardos model

$$\rho \frac{Du}{Dt} = -\nabla p + \nabla \cdot \left[\frac{q(p,\rho)}{||D - \frac{1}{n}\nabla \cdot uI||} \left(D - \frac{1}{n} \nabla \cdot uI \right) \right] + \rho g; \qquad n = 2, 3$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$$

$$\nabla \cdot u = \frac{\partial q(p,\rho)}{\partial p} ||D - \frac{1}{n} \nabla \cdot uI||$$

Questions

- How to fix the pressure range?
- 2D version of split-bottom geometry?

THANK YOU!