

On the Design of Nonconforming High-Resolution Finite Element Schemes for Transport Problems

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Motivation

Algebraic Flux Correction, abbr. **AFC**, family of high-resolution schemes for convection-dominated transport and anisotropic diffusion problems.

- universal stabilization approach based on algebraic design criteria
- approved for conforming (multi-)linear finite element schemes
- found complicated to extend to higher-order finite elements

Objective: apply AFC schemes to nonconforming finite elements

- Do the algebraic design criteria apply to nonconforming elements?
- Is the accuracy of solutions comparable to P_1/Q_1 approximations?
- How to implement essential boundary conditions?
- Is there any benefit from using nonconforming elements?

Model problem

- Convection-diffusion equation

$$\int_{\Omega} w [\dot{u} + \nabla \cdot \mathbf{f}(u)] \, d\mathbf{x} = 0, \quad \mathbf{f}(u) = \mathbf{v}u - d\nabla u$$

- Fletcher's group representation

$$u(\mathbf{x}, t) \approx \sum_j \varphi_j(\mathbf{x}) u_j(t), \quad \mathbf{f}(u) \approx \sum_j \varphi_j(\mathbf{x}) \mathbf{f}(u_j)$$

- Semi-discrete high-order scheme

$$\sum_j m_{ij} \dot{u}_j = \sum_j k_{ij} u_j + \sum_j s_{ij} u_j$$

$$k_{ij} = -\mathbf{v}_j \cdot \int_{\Omega} \varphi_i \nabla \varphi_j \, d\mathbf{x}, \quad s_{ij} = -d \int_{\Omega} \nabla \varphi_i \cdot \nabla \varphi_j \, d\mathbf{x}$$

Model problem

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$$\int_{\Omega} w [\dot{u} + \nabla \cdot \mathbf{f}(u)] \, d\mathbf{x} = 0, \quad \mathbf{f}(u) = \mathbf{v}u - d\nabla u$$

- Fletcher's group representation

$$u(\mathbf{x}, t) \approx \sum_j \varphi_j(\mathbf{x}) u_j(t), \quad \mathbf{f}(u) \approx \sum_j \varphi_j(\mathbf{x}) \mathbf{f}(u_j)$$

- Semi-discrete high-order scheme

$$\sum_j m_{ij} \dot{u}_j = \sum_{j \neq i} k_{ij} (u_j - u_i) + \delta_i u_i$$


$$\mathbf{M}_C \dot{\mathbf{u}} = \mathbf{K} \mathbf{u}$$

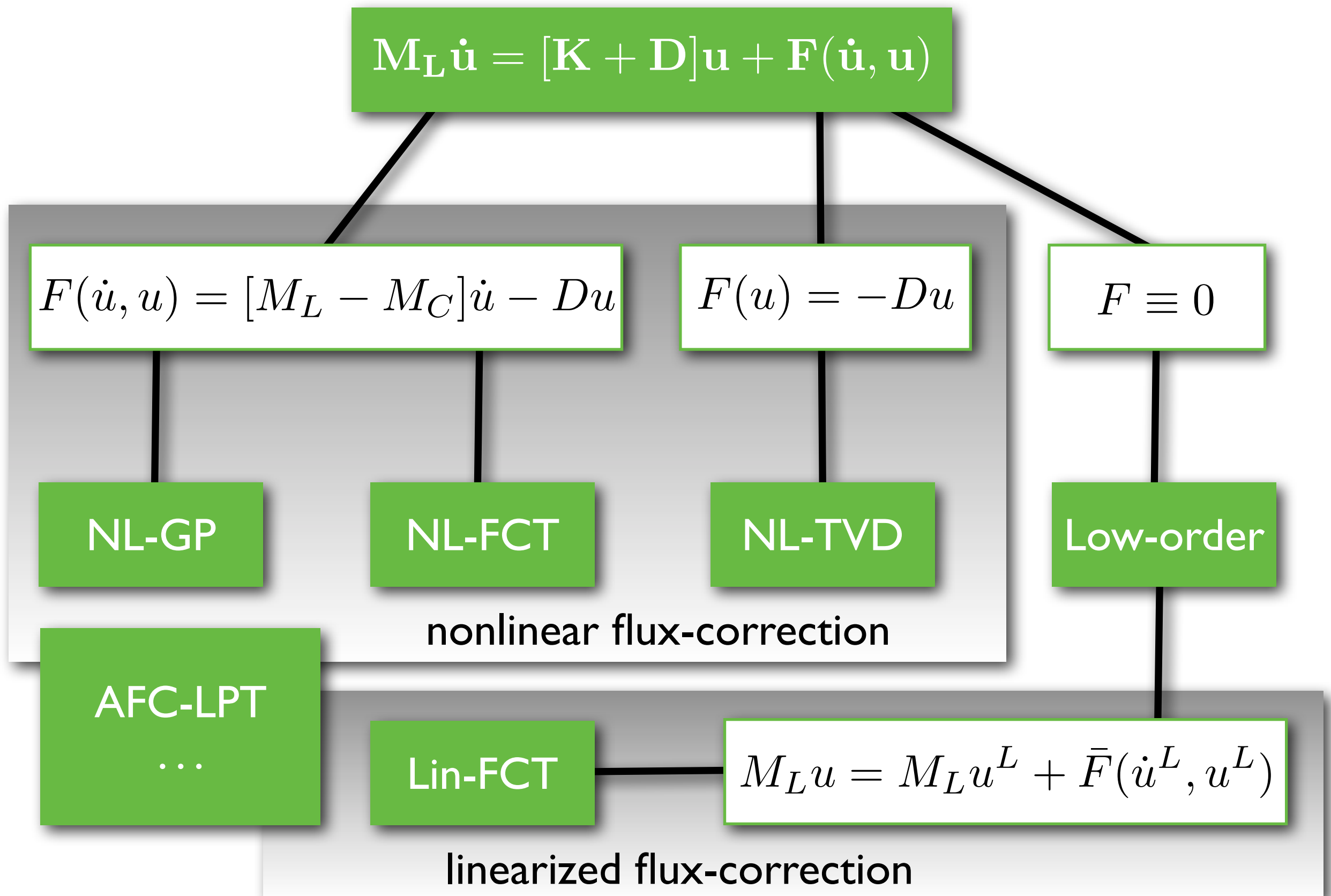
$$k_{ij} = -\mathbf{v}_j \cdot \int_{\Omega} \varphi_i \nabla \varphi_j \, d\mathbf{x}, \quad \delta_i = \sum_j k_{ij}$$

$$\mathbf{M}_L \dot{\mathbf{u}} = [\mathbf{K} + \mathbf{D}] \mathbf{u} + \mathbf{F}(\dot{\mathbf{u}}, \mathbf{u})$$

lumped mass matrix

antidiffusive correction

artificial diffusion operator



Review of algebraic design principles

- Jameson's Local Extremum Diminishing criterion

IF $m_i \dot{u}_i = \sum_{j \neq i} \sigma_{ij} (u_j - u_i)$

 positive  not negative

THEN local solution maxima/minima do not increase/decrease

- Semi-discrete high-resolution AFC scheme

$$m_i \dot{u}_i = \sum_{j \neq i} \underbrace{(k_{ij} + d_{ij})}_{\text{not negative by construction}} (u_j - u_i) + \delta_i u_i + \sum_{j \neq i} \underbrace{\alpha_{ij} f_{ij}}_{\text{controlled by flux limiter}}$$



$m_i = \sum_j m_{ij}$

need to check the positivity of mass matrix coefficients for each finite element by hand!

Finite element spaces

- Parametric quadrilateral finite elements

$$\mathcal{Q}(T) = \{q = \hat{q} \circ \Psi_T^{-1}, \hat{q} \in \hat{\mathcal{Q}}(\hat{T})\}$$

- Bilinear one-to-one mapping

$$\Psi_T : \hat{T} := [-1, 1] \times [-1, 1] \mapsto T \in \mathcal{T}_h$$

$Q_1(T)$	$\hat{Q}_1(\hat{T}) = \text{span}\langle 1, \hat{x}, \hat{y}, \hat{x}\hat{y} \rangle$
$Q_1^{\text{rot}}(T)$ Rannacher & Turek '92	$\hat{Q}_1^{\text{rot}}(\hat{T}) = \text{span}\langle 1, \hat{x}, \hat{y}, \hat{x}^2 - \hat{y}^2 \rangle$ $Q_{1,\text{par}}^{\text{rot}}(T) = \text{span}\langle 1, \xi, \eta, \xi^2 - \eta^2 \rangle$

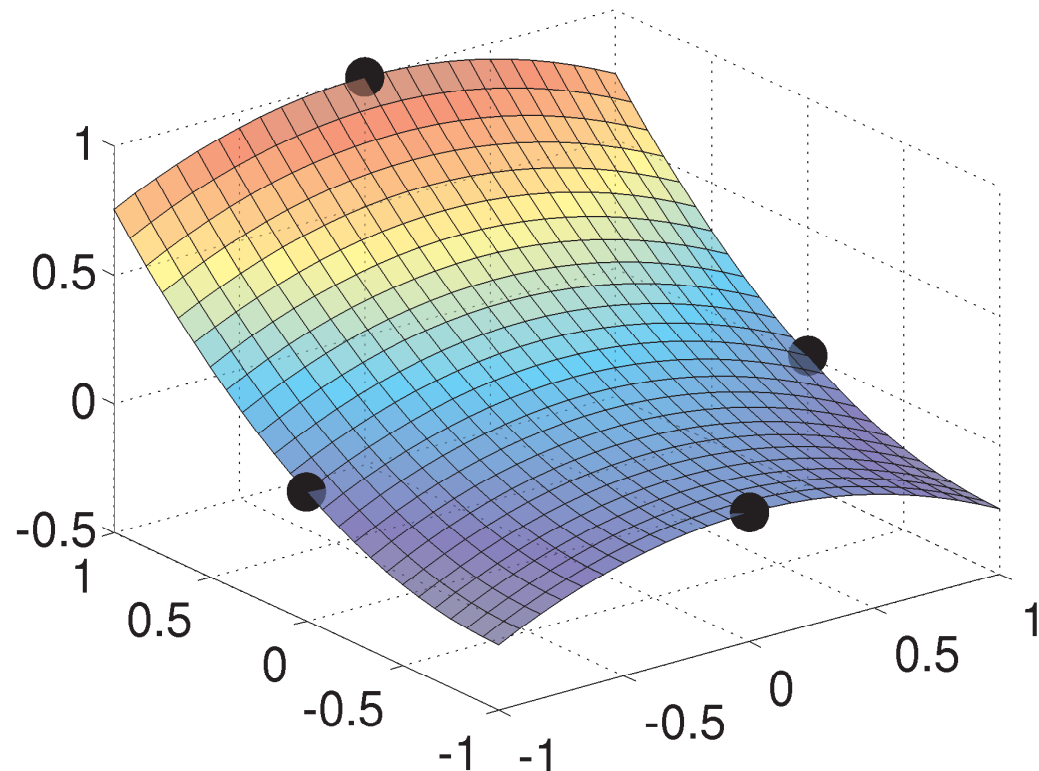
Nonconforming shape functions

$$\hat{\varphi}_1(\hat{\mathbf{x}}) = \frac{1}{4} - \frac{1}{2}\hat{y} - \beta(\hat{x}^2 - \hat{y}^2), \quad \hat{\varphi}_3(\hat{\mathbf{x}}) = \frac{1}{4} + \frac{1}{2}\hat{y} - \beta(\hat{x}^2 - \hat{y}^2)$$

$$\hat{\varphi}_2(\hat{\mathbf{x}}) = \frac{1}{4} + \frac{1}{2}\hat{x} + \beta(\hat{x}^2 - \hat{y}^2), \quad \hat{\varphi}_4(\hat{\mathbf{x}}) = \frac{1}{4} - \frac{1}{2}\hat{x} + \beta(\hat{x}^2 - \hat{y}^2)$$

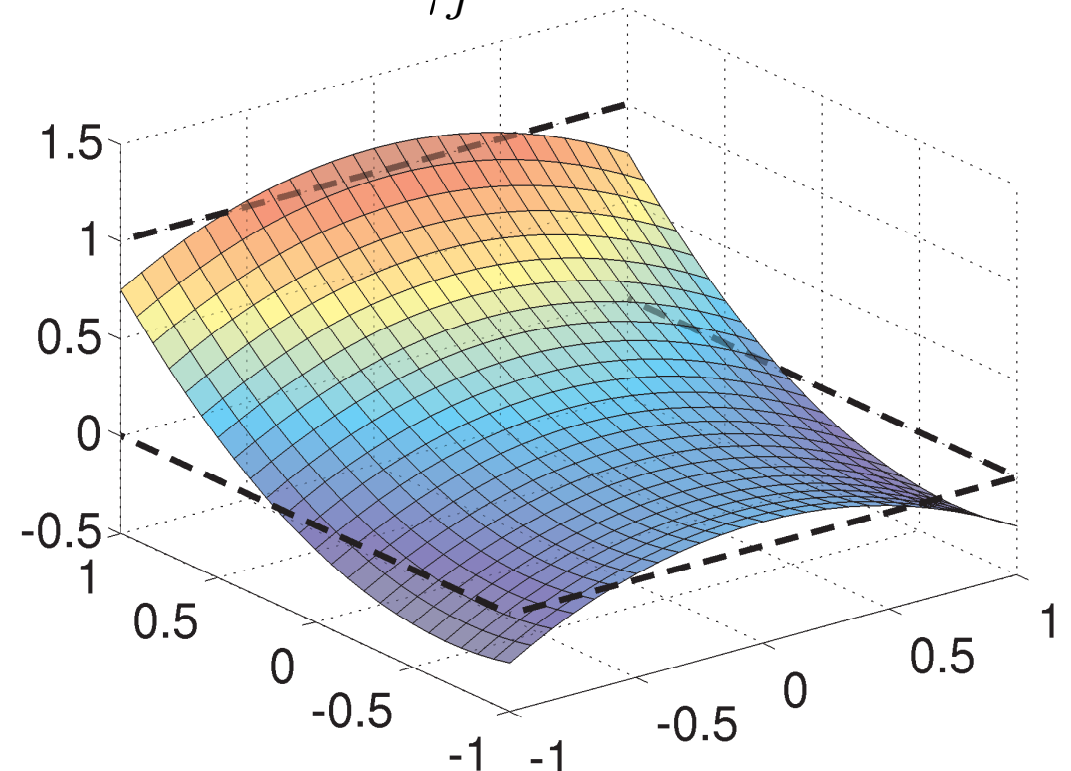
■ Midpoint based variant

$$\beta = \frac{1}{4} : \hat{\varphi}_i(\hat{\mathbf{m}}_j) = \delta_{ij}$$



■ Mean value based variant

$$\beta = \frac{3}{8} : |\hat{\gamma}_j|^{-1} \int_{\hat{\gamma}_j} \hat{\varphi}_i(\hat{\mathbf{x}}) d\gamma = \delta_{ij}$$



Matrix analysis - part I

- Consistent mass matrix on reference element

$$\hat{M} = \begin{pmatrix} \frac{16}{45}\beta^2 + \frac{7}{24} & -\frac{16}{45}\beta^2 + \frac{1}{8} & \frac{16}{45}\beta^2 - \frac{1}{24} & -\frac{16}{24}\beta^2 + \frac{1}{8} \\ -\frac{16}{24}\beta^2 + \frac{1}{8} & \frac{16}{45}\beta^2 + \frac{7}{24} & -\frac{16}{45}\beta^2 + \frac{1}{8} & \frac{16}{45}\beta^2 - \frac{1}{24} \\ \frac{16}{45}\beta^2 - \frac{1}{24} & -\frac{16}{24}\beta^2 + \frac{1}{8} & \frac{16}{45}\beta^2 + \frac{7}{24} & -\frac{16}{45}\beta^2 + \frac{1}{8} \\ -\frac{16}{45}\beta^2 + \frac{1}{8} & \frac{16}{45}\beta^2 - \frac{1}{24} & -\frac{16}{24}\beta^2 + \frac{1}{8} & \frac{16}{45}\beta^2 + \frac{7}{24} \end{pmatrix}$$

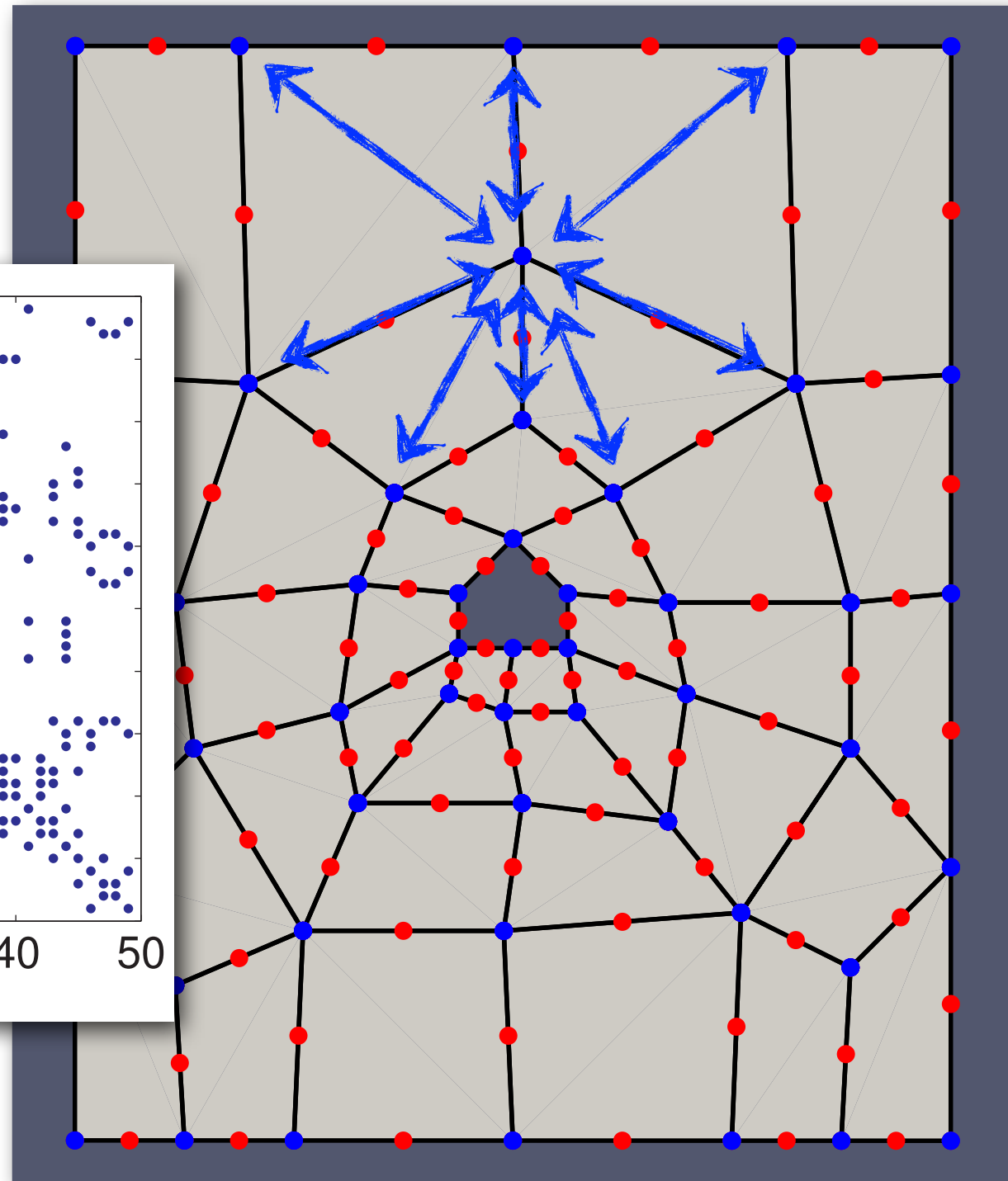
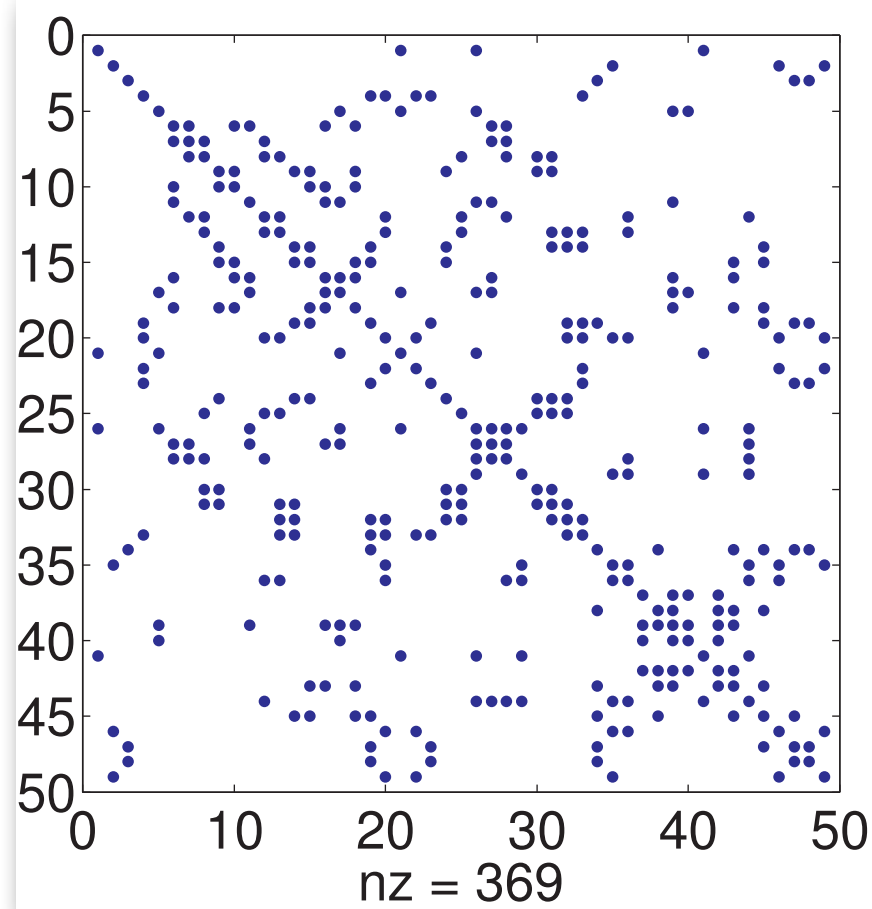
- Positivity criterion

$$0.3423 \approx \frac{1}{16}\sqrt{30} < |\beta| < \frac{3}{16}\sqrt{10} \approx 0.5929$$

- midpoint based variant ($\beta = \frac{1}{4}$) has negative matrix coefficients
- mean value based variant ($\beta = \frac{3}{8}$) has positive matrix coefficients
- lower bound is invariant to shape of quadrilateral element

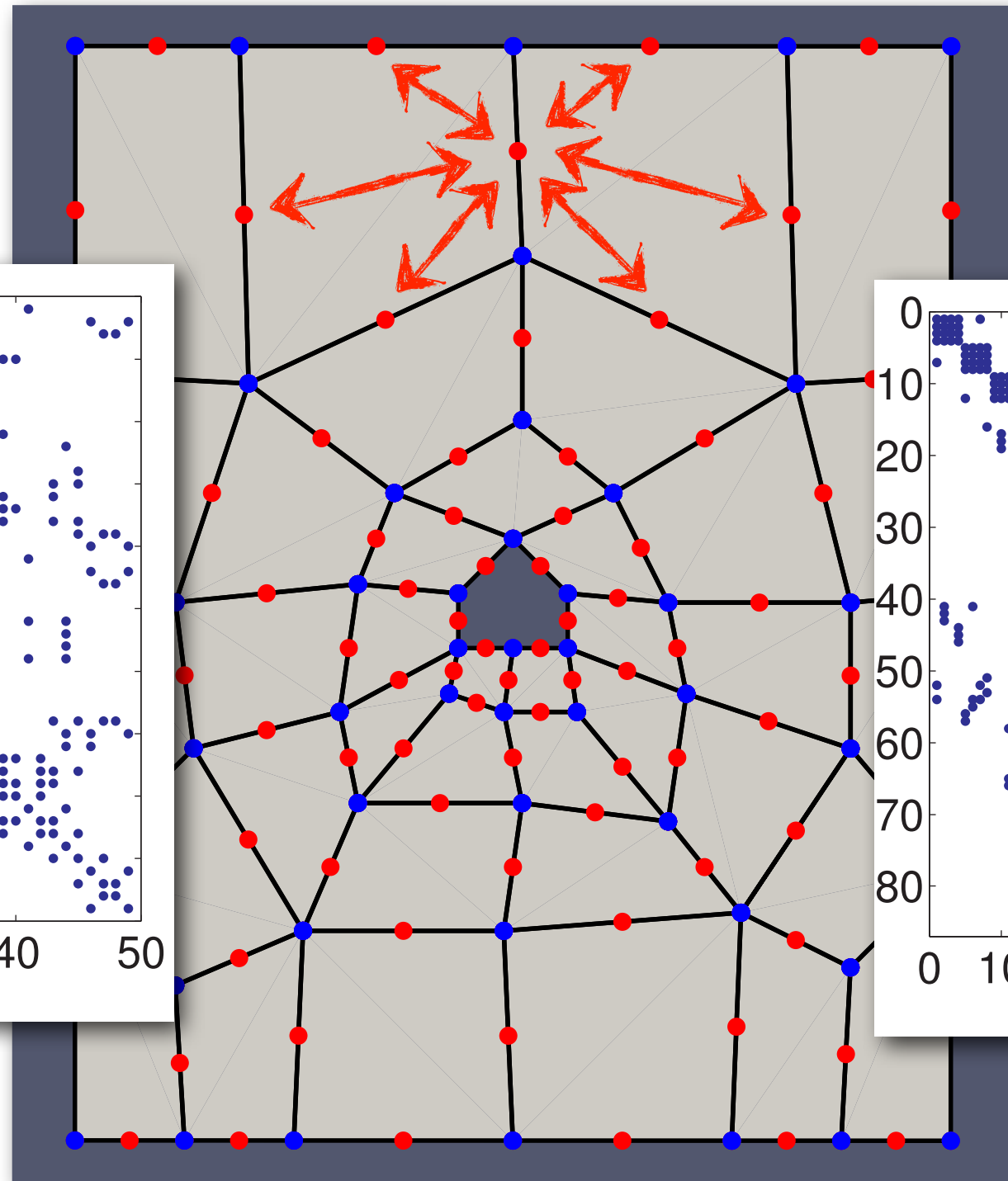
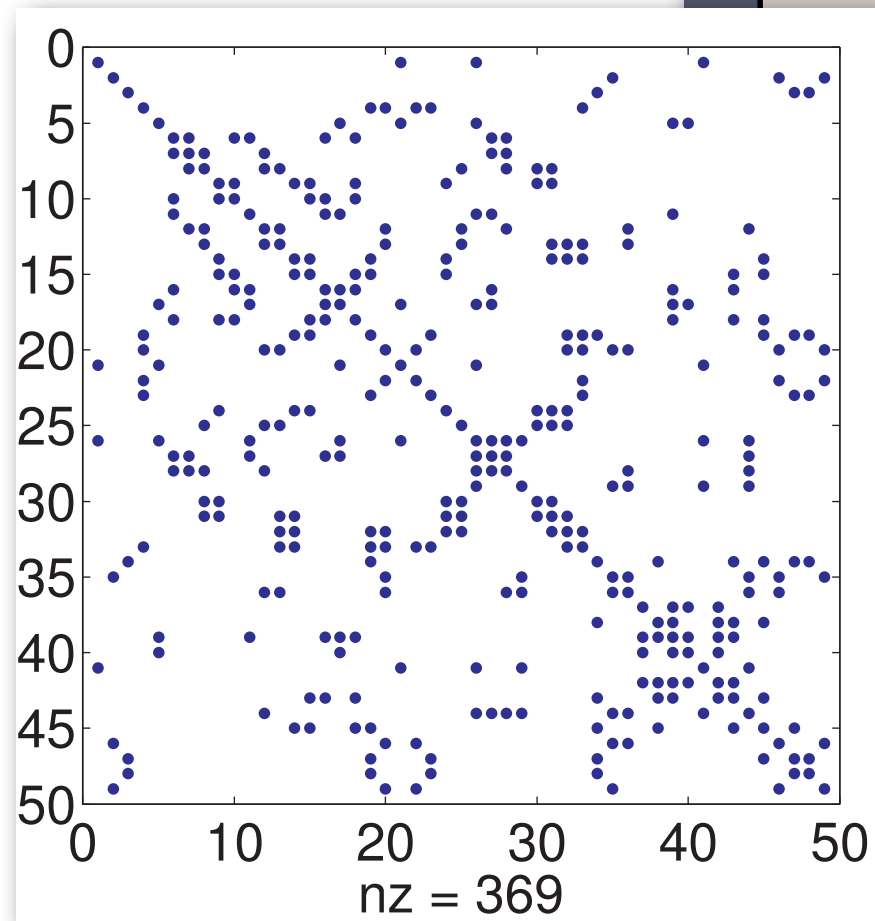
Matrix analysis - part 2

Q_1 FE

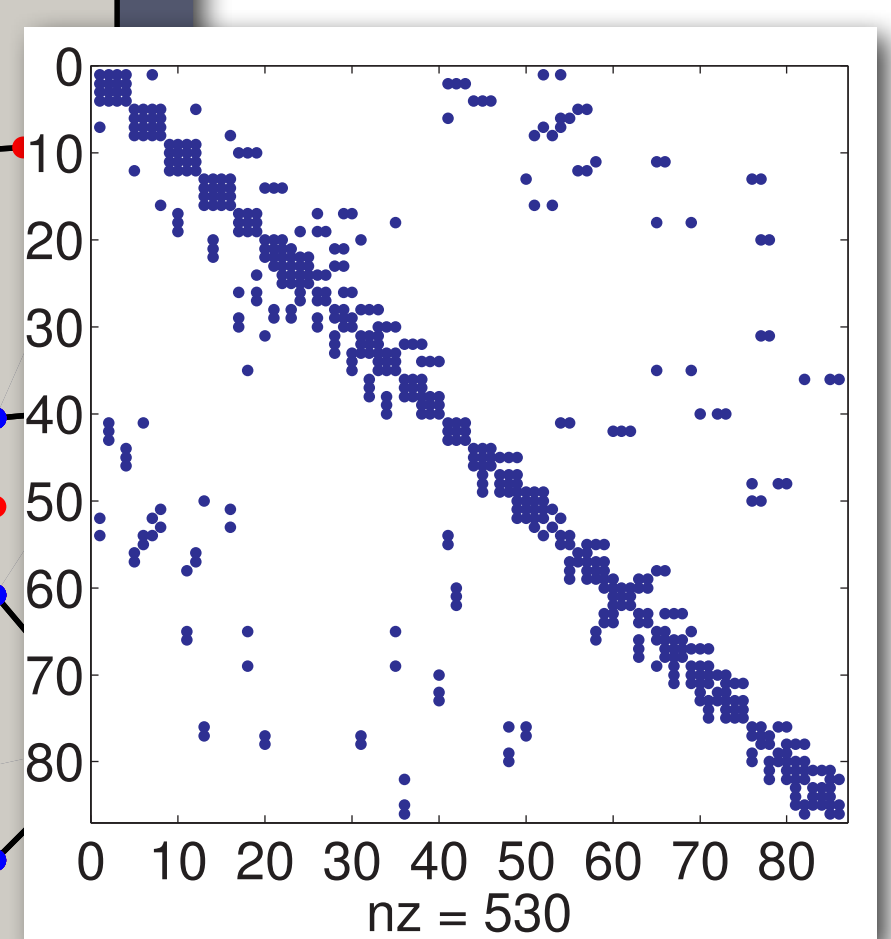


Matrix analysis - part 2

Q_I FE



Q_I^{rot} FE

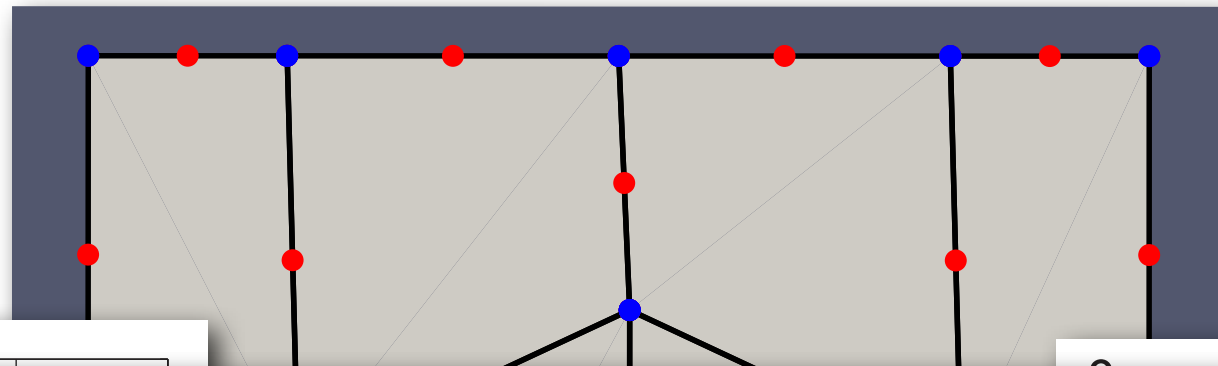


Matrix analysis - part 2

Storage format:

■ CRS, CCS

Q_I FE

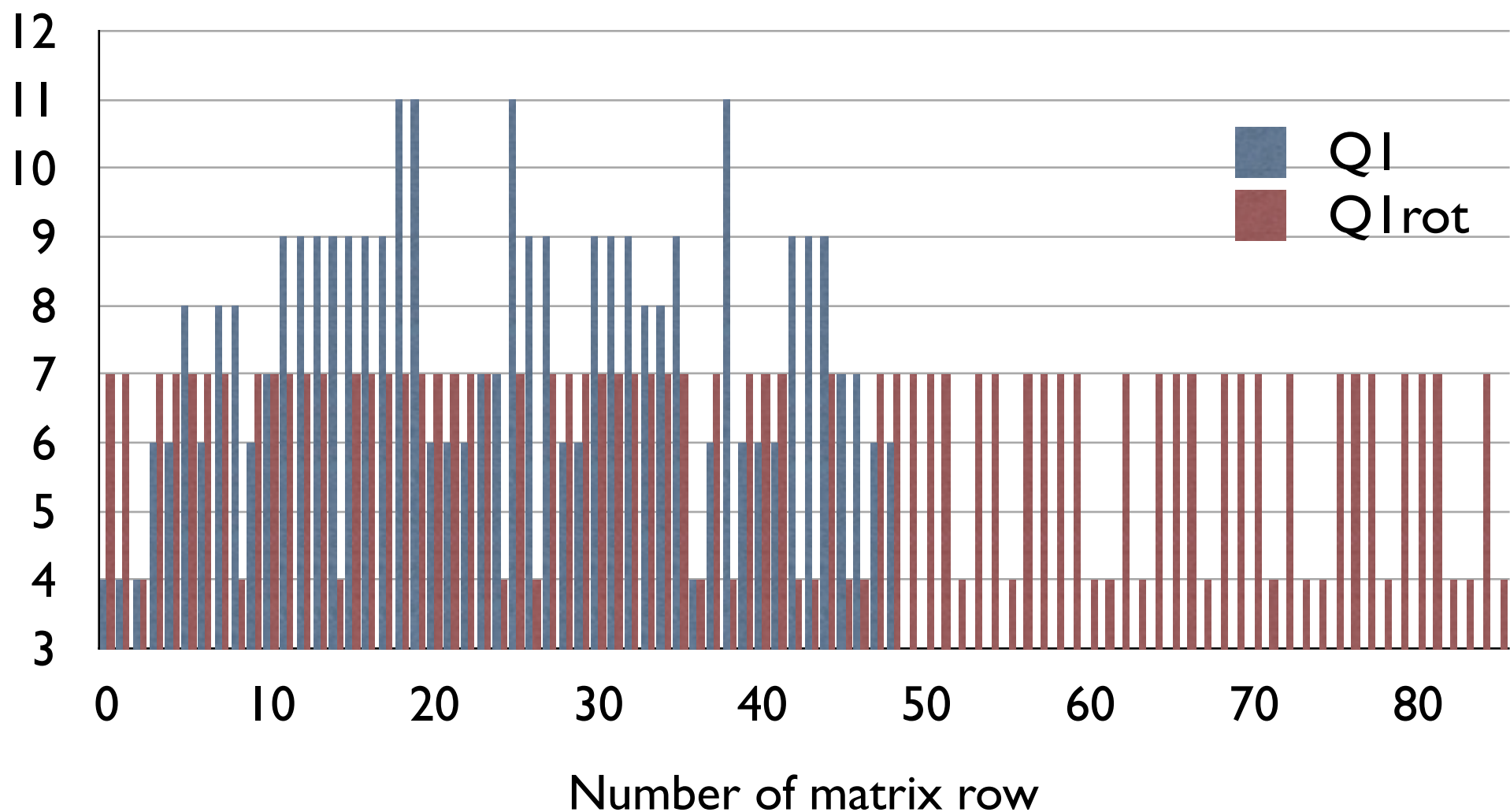


Storage format:

■ ELLPACK

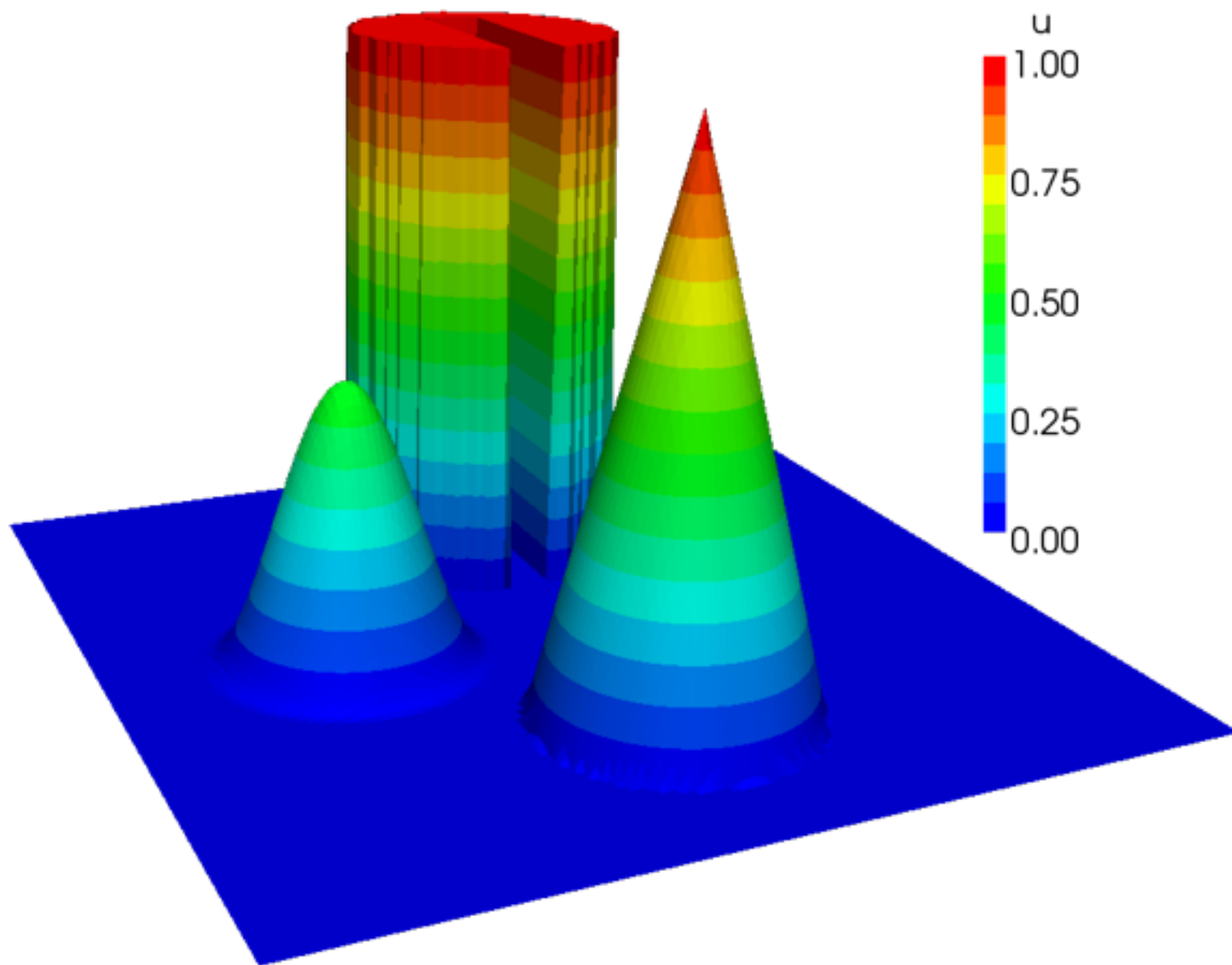
Q_I^{rot} FE

Number of non-zero entries per matrix row



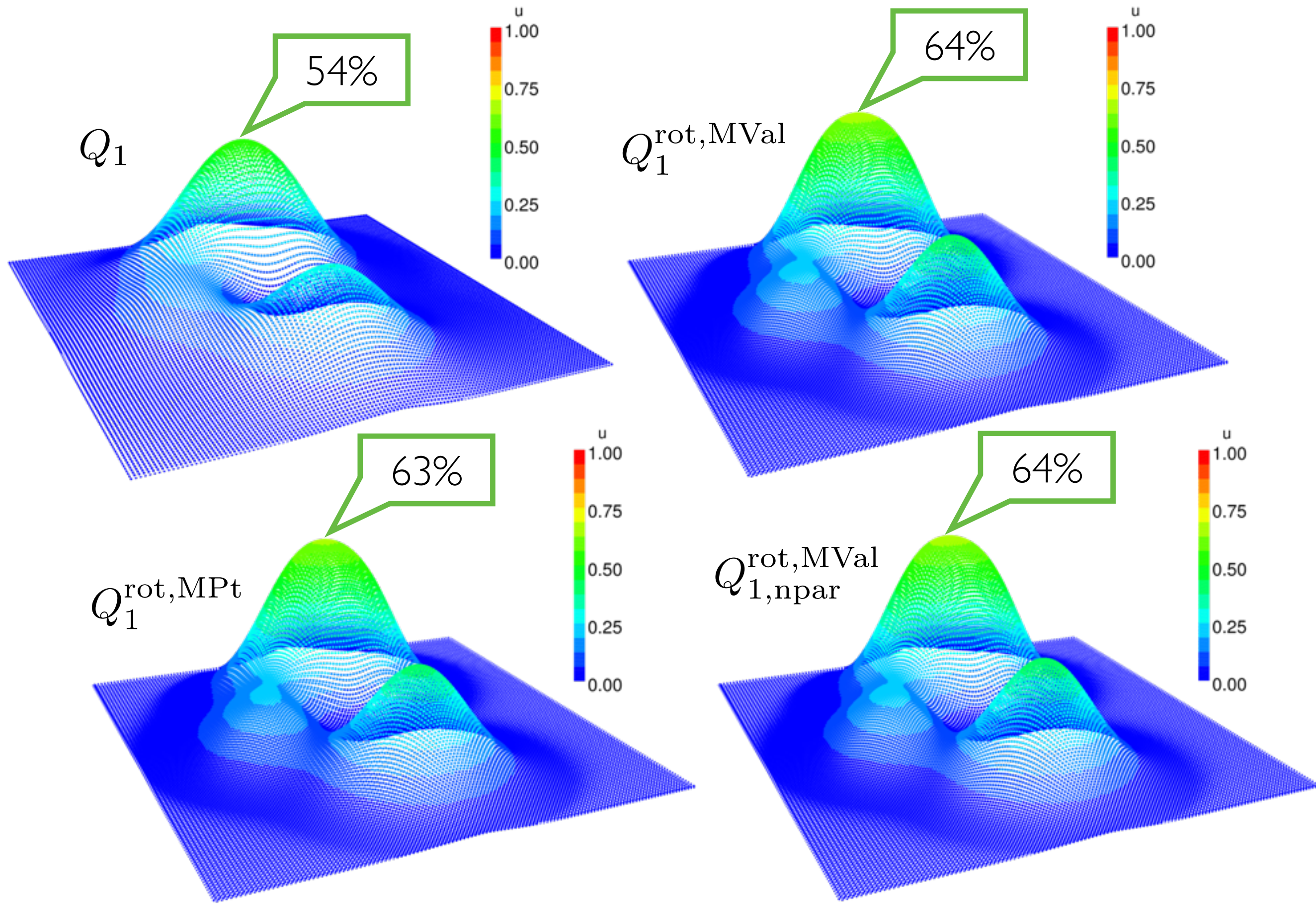
Solid body rotation

$$\begin{aligned} \dot{u} + \nabla \cdot (\mathbf{v}u) &= 0 \quad \text{in } (0, 1)^2 \\ u &= 0 \quad \text{on } \Gamma_{\text{inflow}} \end{aligned}$$

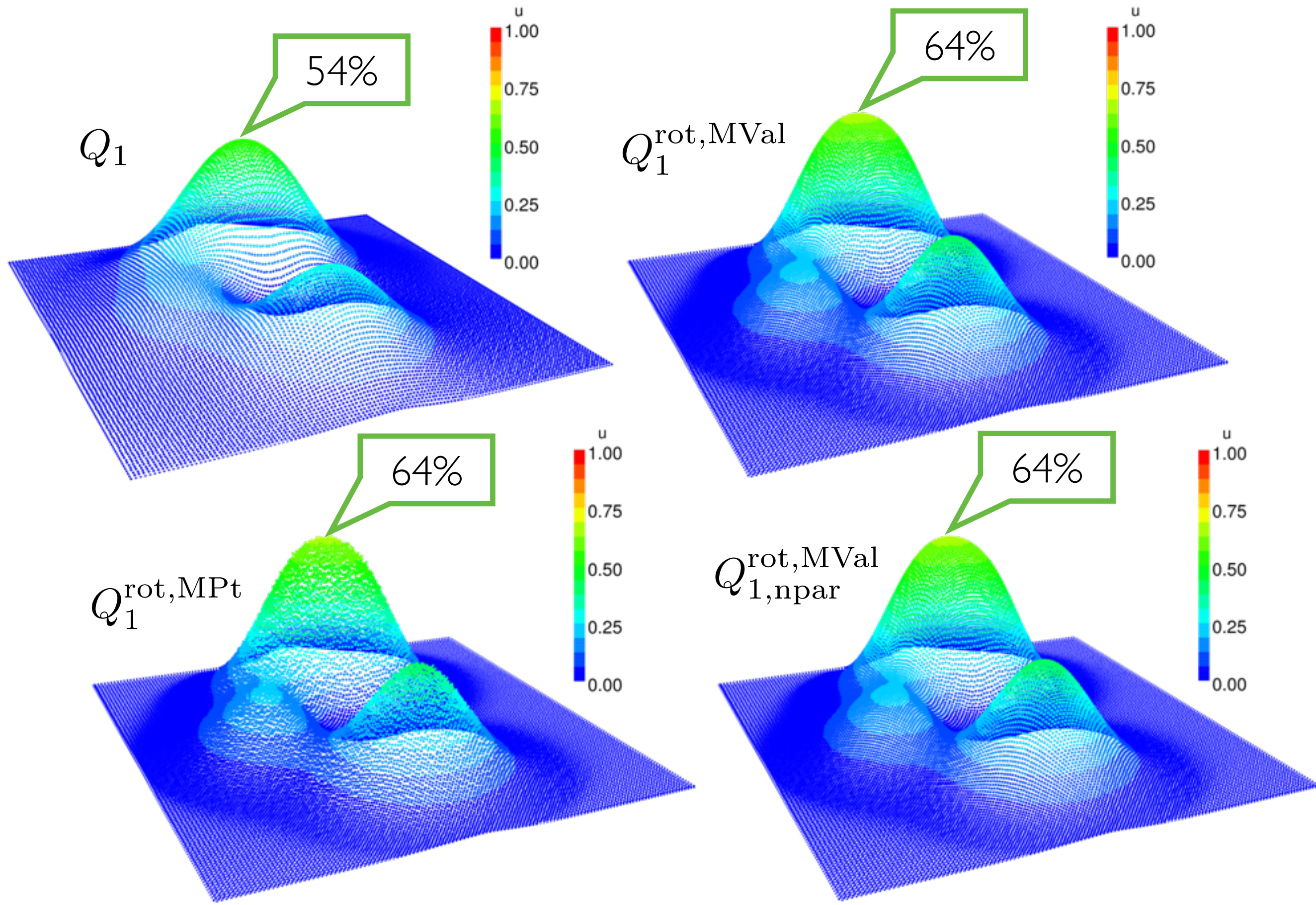


- Velocity field
 $\mathbf{v}(x, y) = (0.5 - y, x - 0.5)$
- Grid size
 $h = 1/2^l, l = 5, 6, \dots$
- Stochastic grid disturbance
 $\delta \in \{0\%, 1\%, 5\%\}$
- Time step in Crank-Nicolson $\Delta t = 1.28 \cdot h$
- Initial = exact solution at
 $t = 2\pi k, k \in \mathbb{N}$

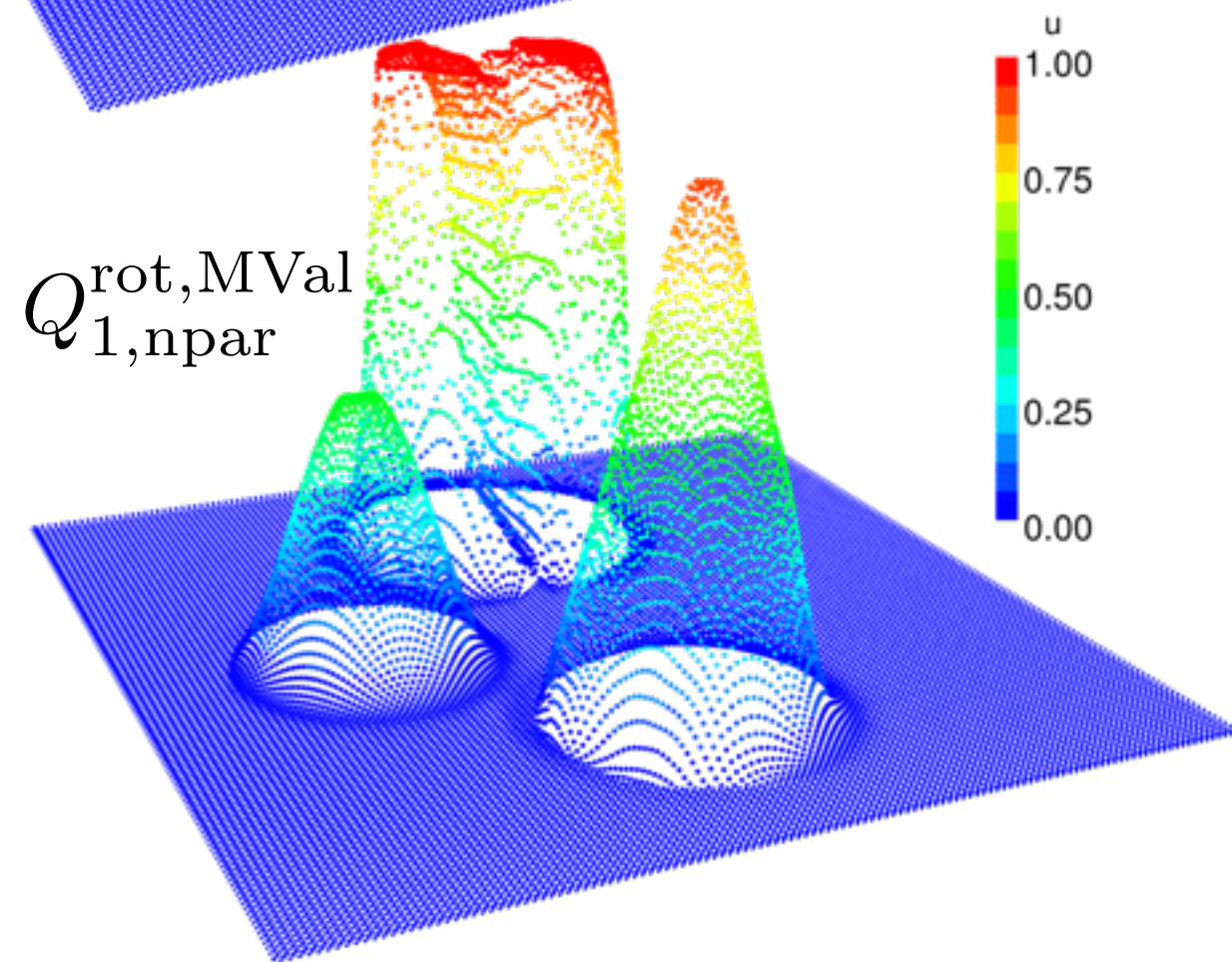
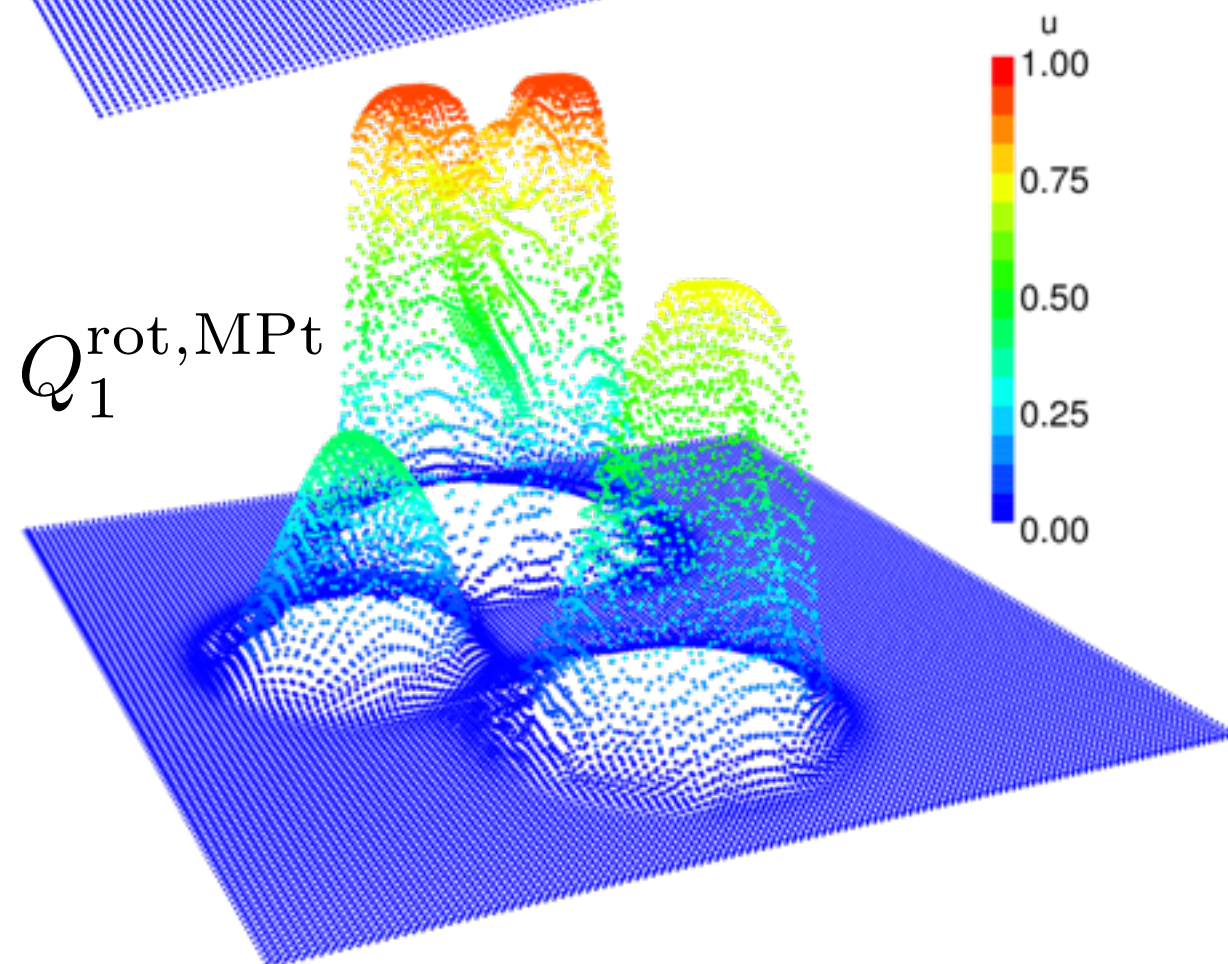
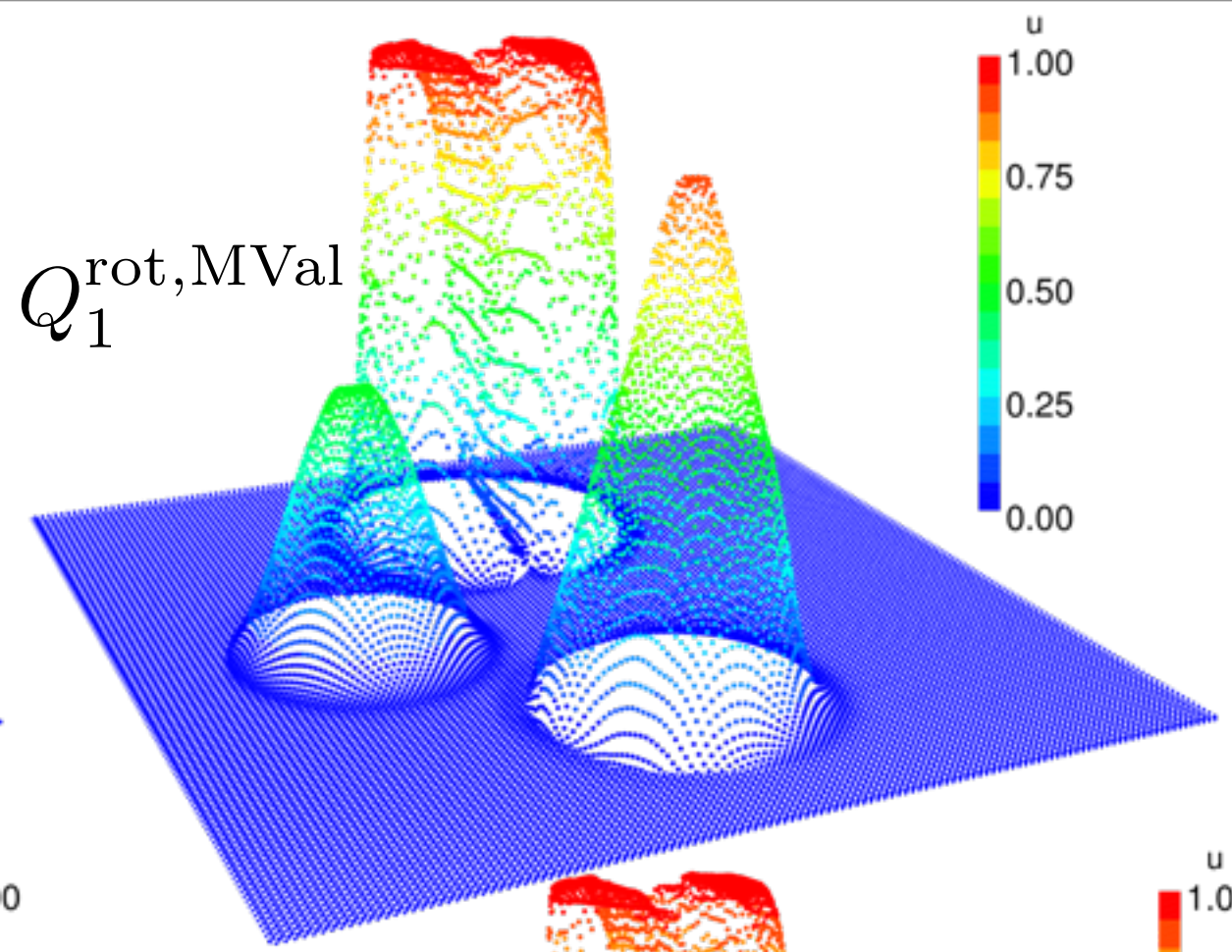
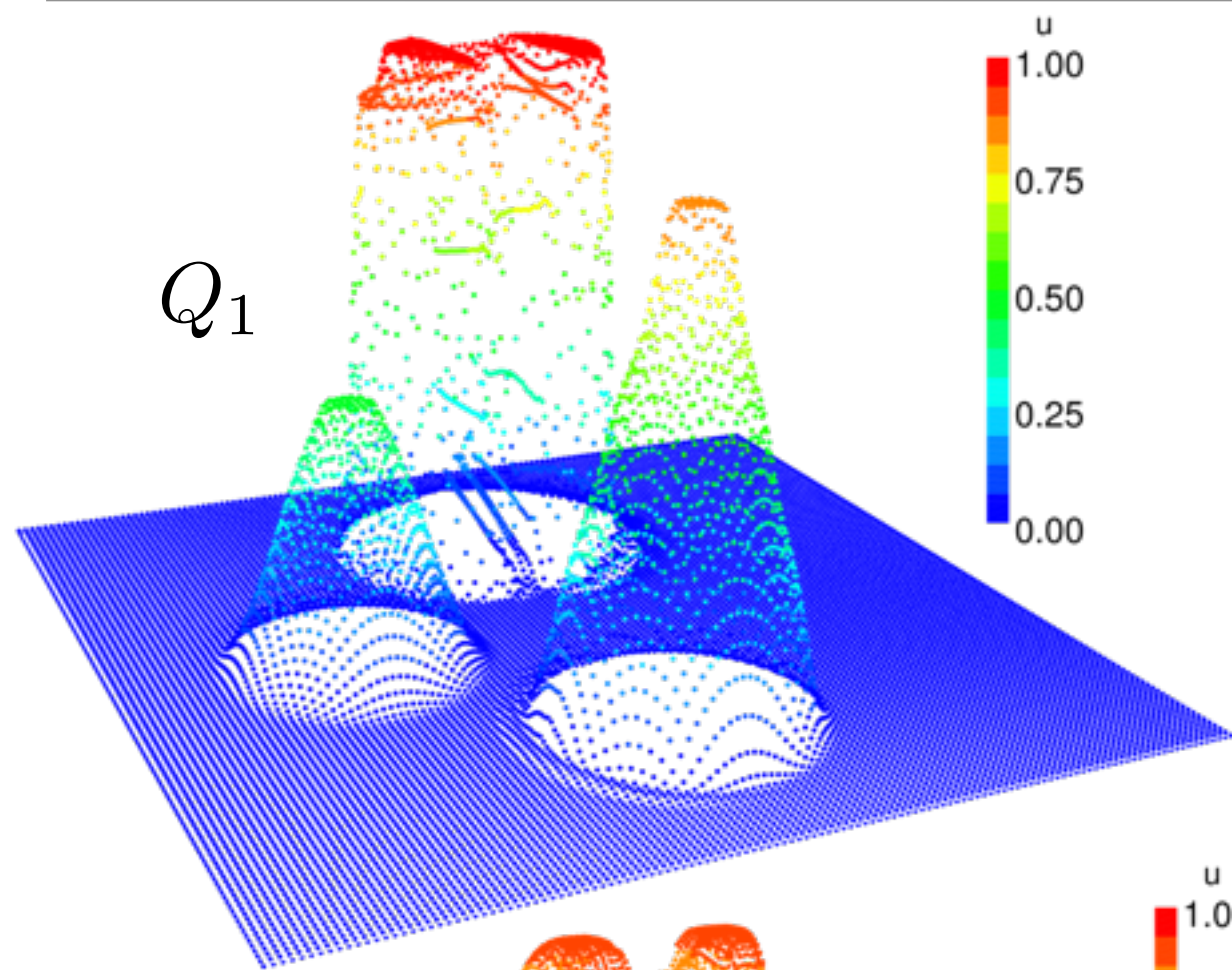
SBR: low-order solutions (0% mesh perturbation)



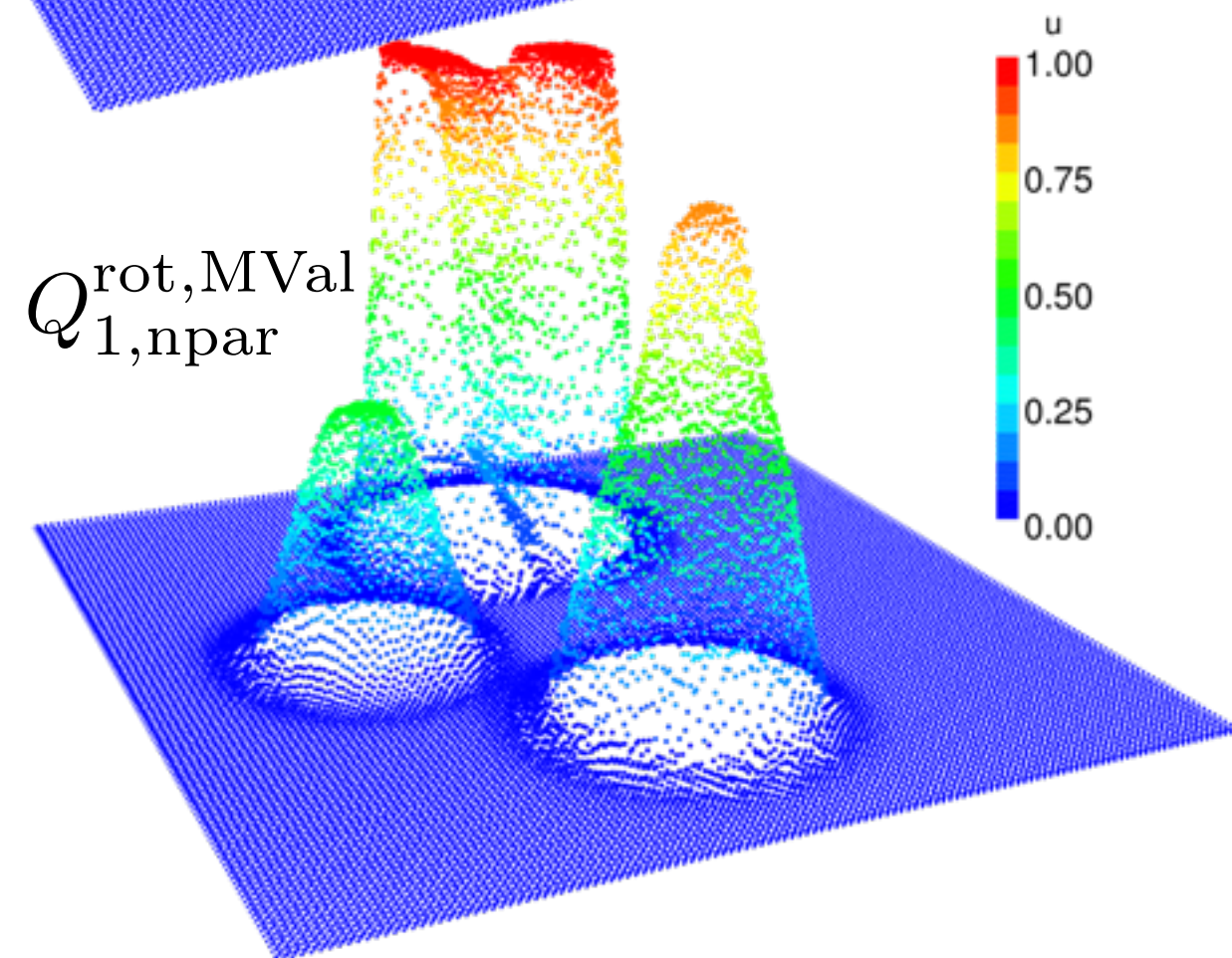
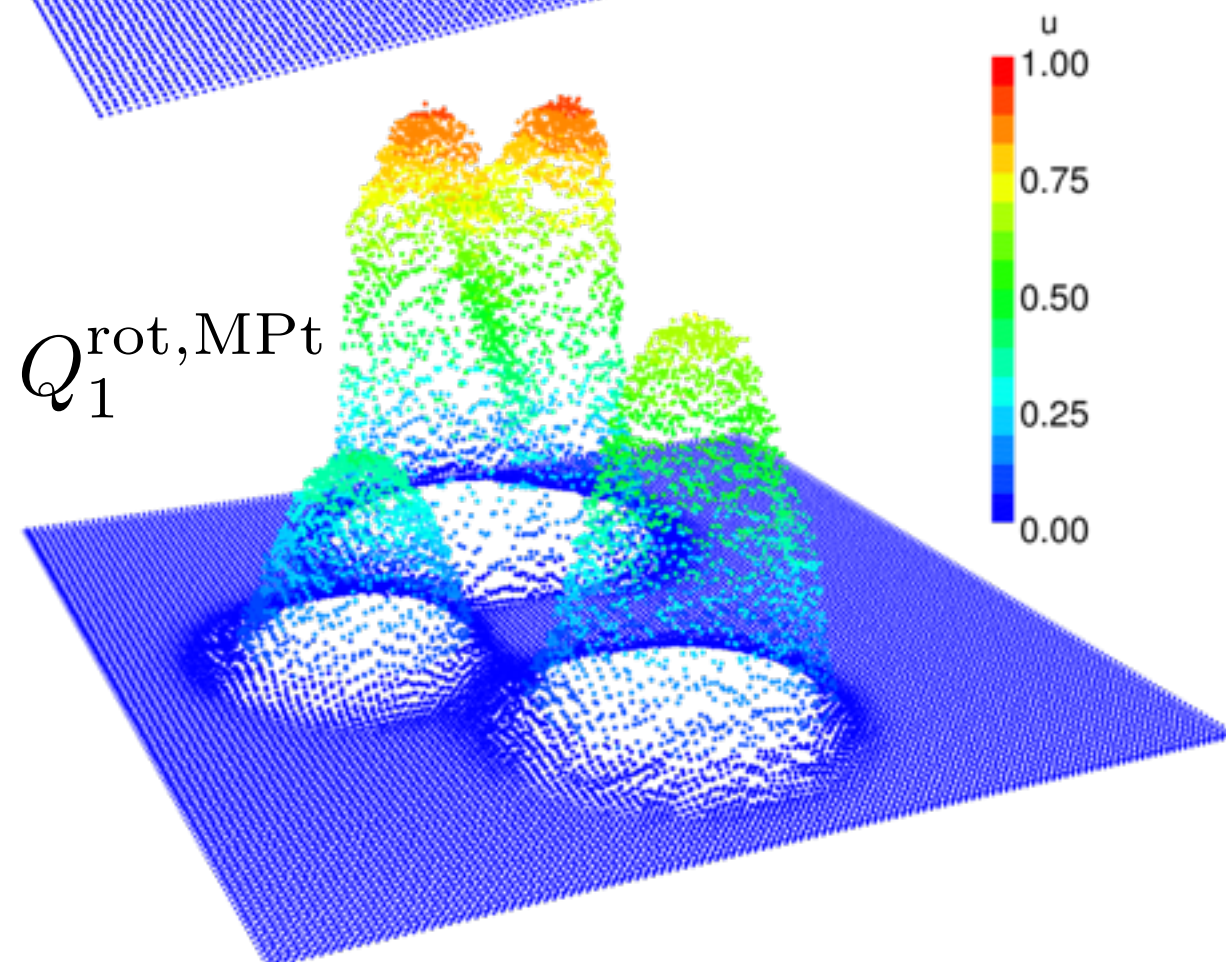
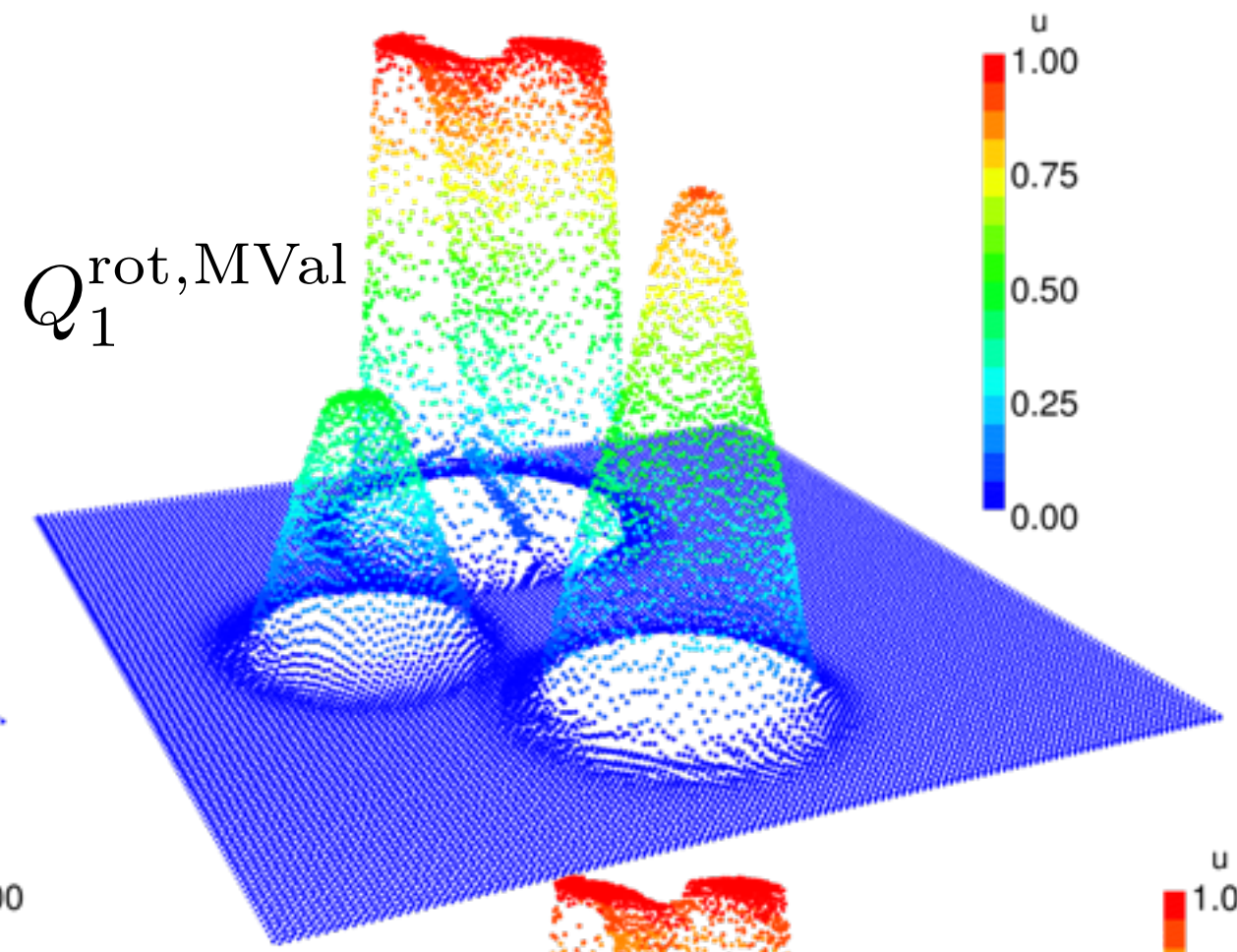
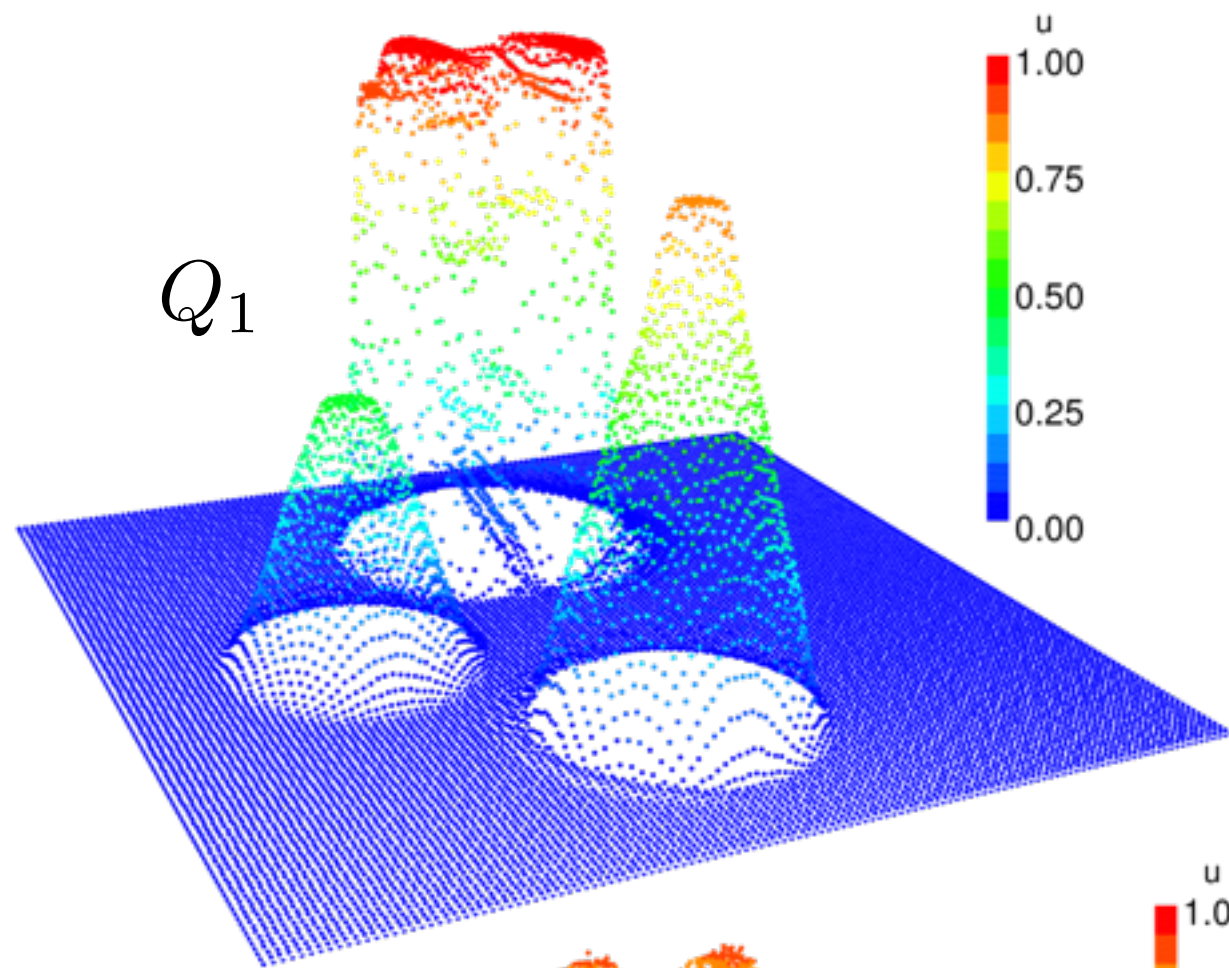
SBR: *low-order solutions (5% mesh perturbation)*



SBR: *NL-FCT* solutions (0% mesh perturbation)



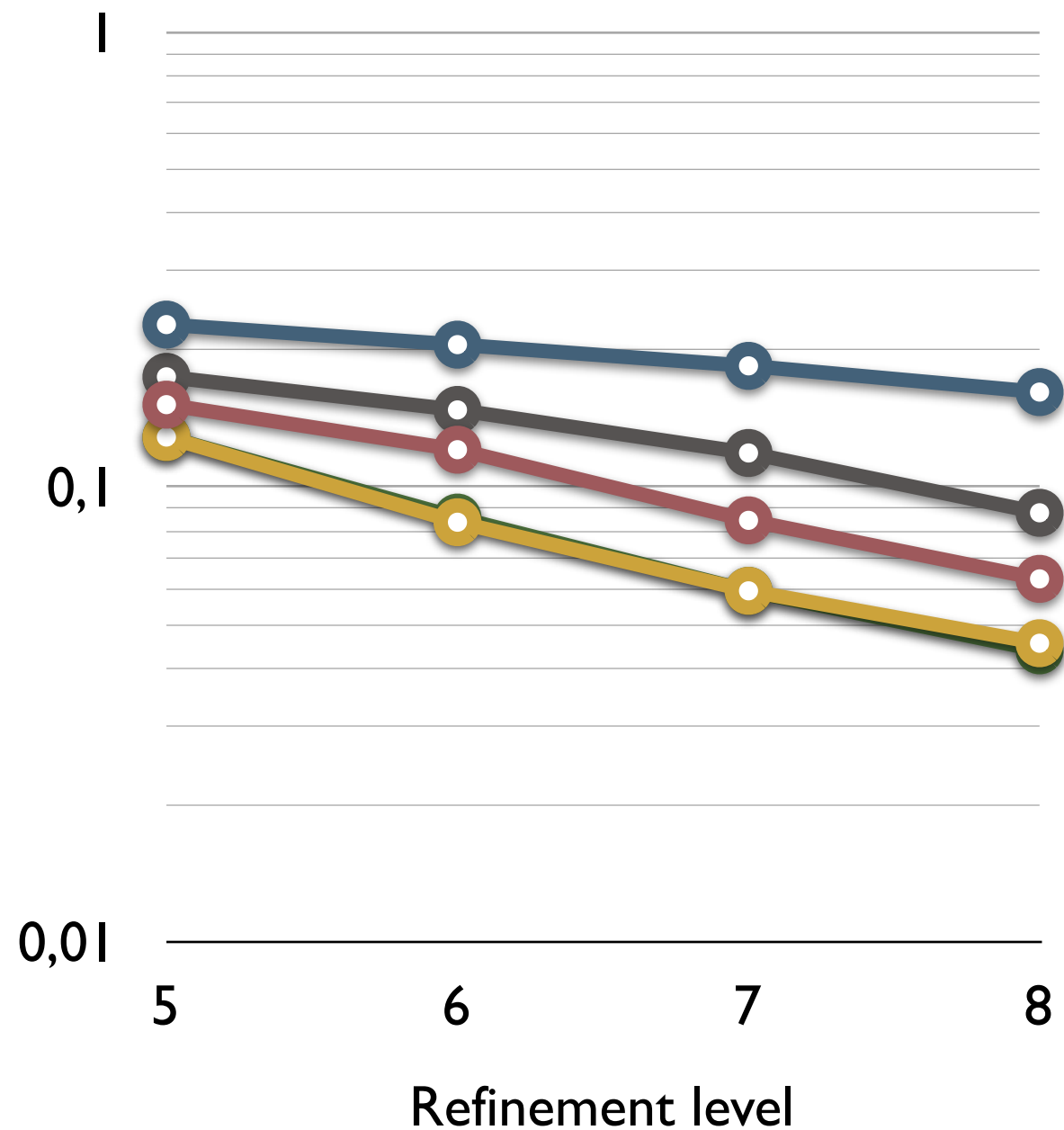
SBR: *NL-FCT* solutions (5% mesh perturbation)



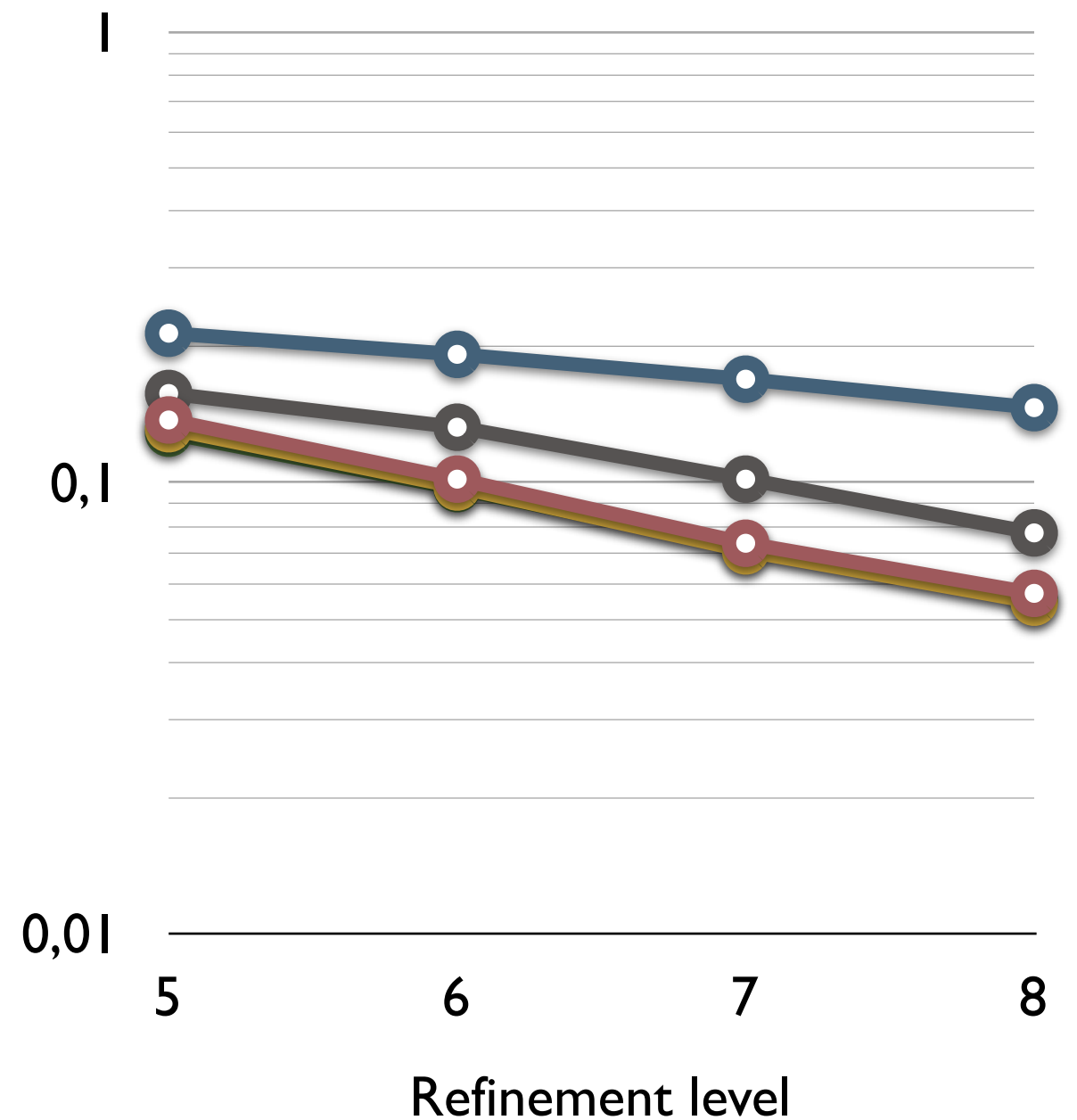
SBR: L_2 -error (5% mesh perturbation)

Low-order Lin-FCT NL-FCT NL-GP NL-TVD

conforming FE

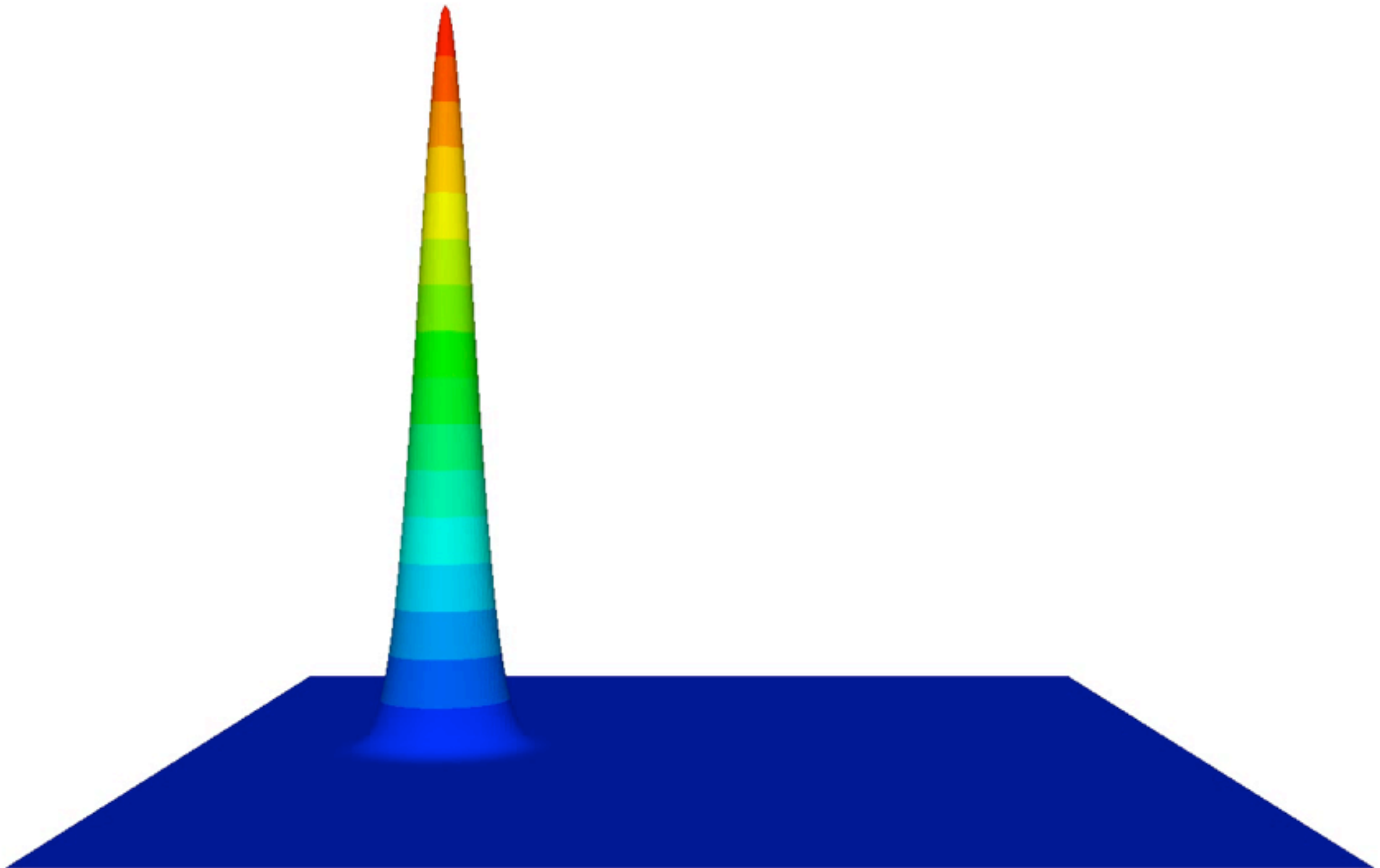


nonconforming FE



Rotation of a Gaussian hill

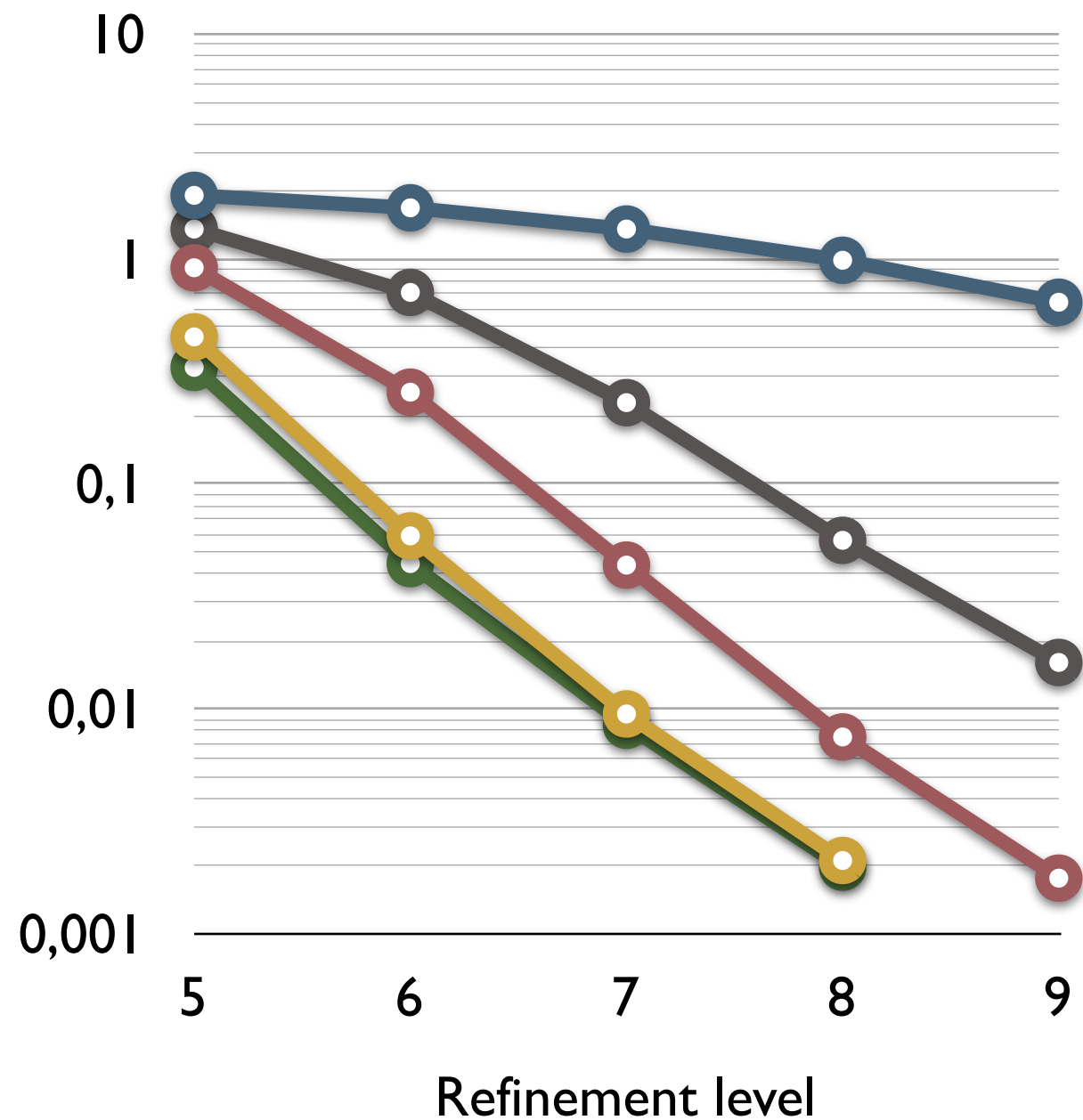
$$\dot{u} + \nabla \cdot (\mathbf{v}u - d\nabla u) = 0 \quad \text{in } (-1, 1)^2, \quad \mathbf{v}(x, y) = (-y, x), \quad d = 0.001$$



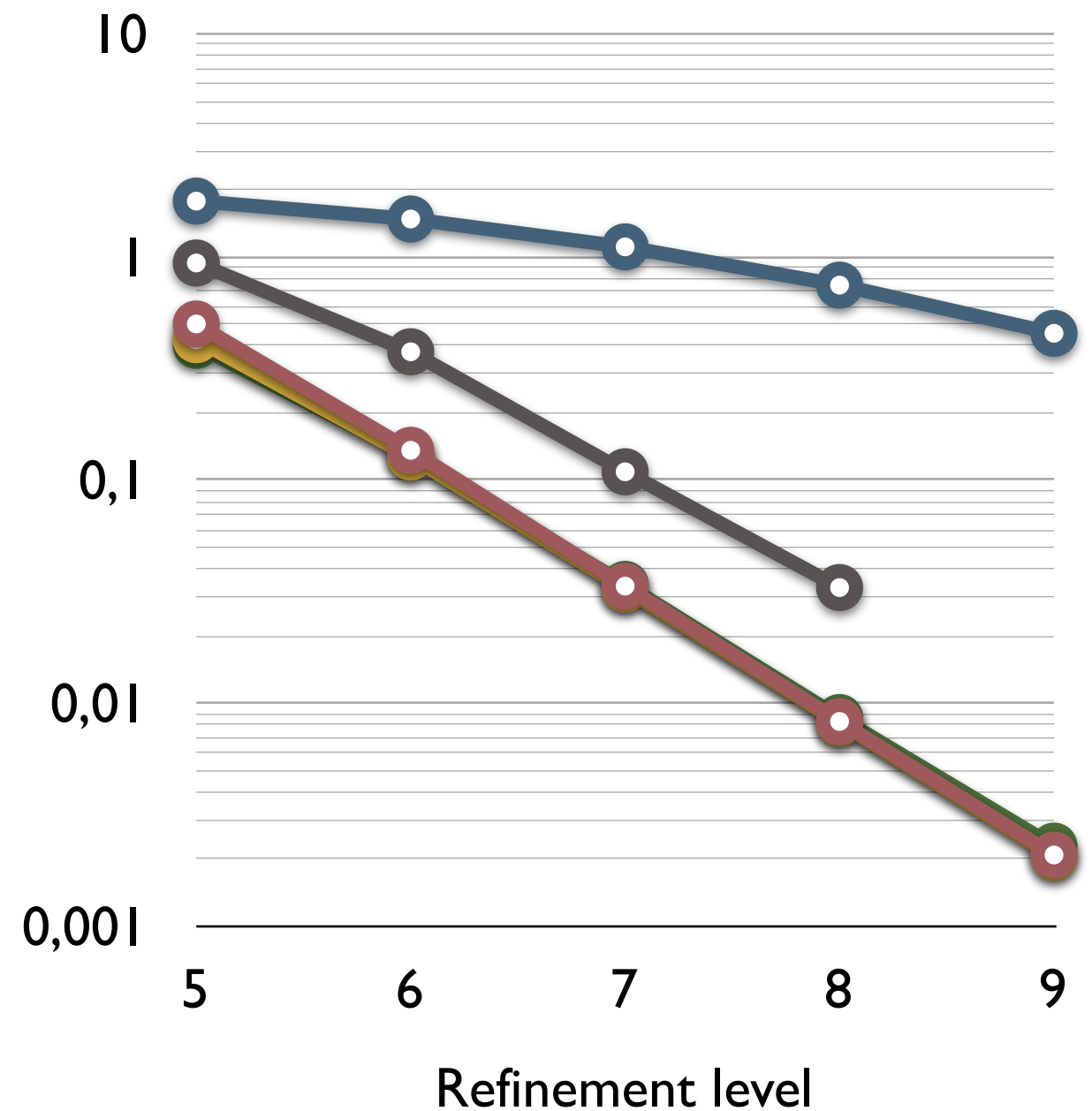
RGH: L_2 -error (5% mesh perturbation)

Low-order Lin-FCT NL-FCT NL-GP NL-TVD

conforming FE



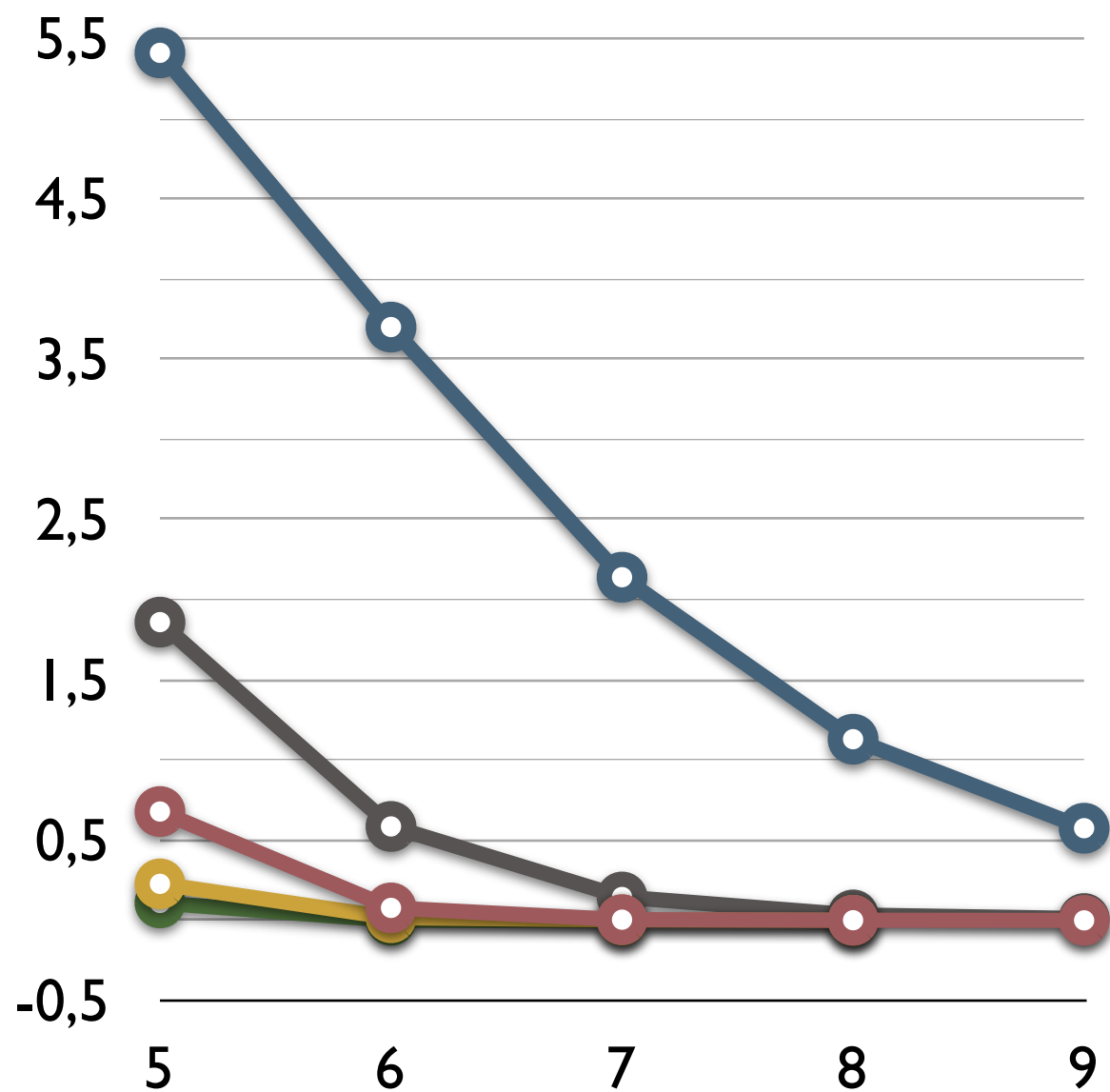
nonconforming FE



RGH: *dispersion-error* (5% mesh perturbation)

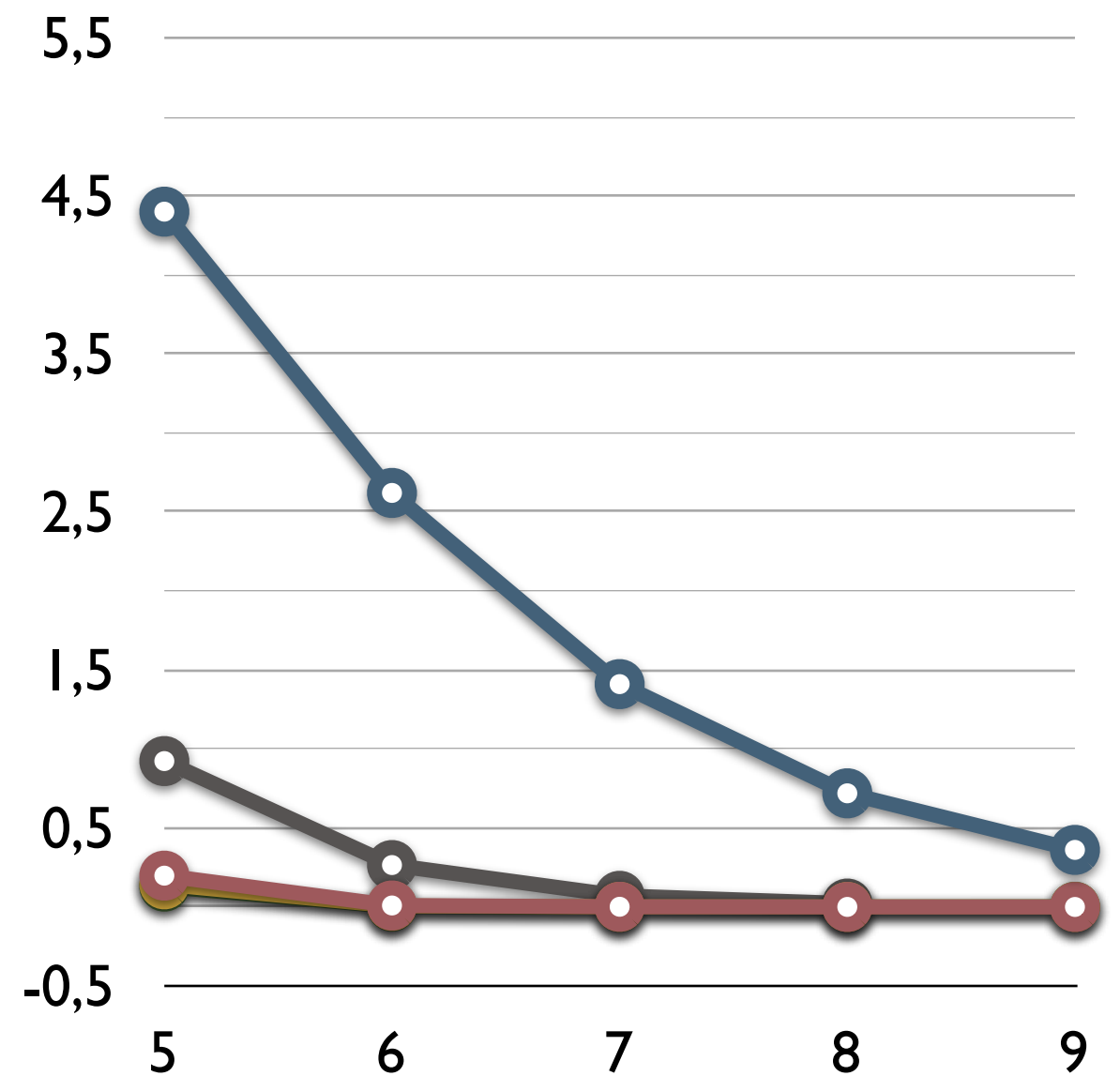
Low-order Lin-FCT NL-FCT NL-GP NL-TVD

conforming FE



Refinement level

nonconforming FE



Refinement level

Taxonomy of finite elements

conforming FE		nonconforming FE
good	accuracy	good
small	numerical diffusion	small(er)
smaller	#DOFs, #edges	larger
irregular	sparsity pattern	regular

Boundary conditions

Convection-diffusion equation

$$\begin{aligned}\nabla \cdot (\mathbf{v}u - d\nabla u) &= f && \text{in } \Omega \\ u &= u_D && \text{on } \Gamma_D \\ (d\nabla u) \cdot \mathbf{n} &= g && \text{on } \Gamma_N\end{aligned}$$

hyperbolic limit $d \rightarrow 0$

$$\begin{aligned}\nabla \cdot (\mathbf{v}u) &= f && \text{in } \Omega \\ (\mathbf{v}u) \cdot \mathbf{n} &= h && \text{on } \Gamma_{\text{in}}\end{aligned}$$

- Y. Basilevs, T. Hughes, *Weak imposition of Dirichlet boundary conditions in fluid mechanics*, Computers & Fluids 32 (1) 2007, 12-26
 - $\gamma = 1$ consistent, adjoint-consistent
 - $\gamma = -1$ consistent, adjoint-inconsistent
- E. Burman, *A penalty free non-symmetric Nitsche type method for the weak imposition of boundary conditions*, eprint arXiv:1106.5612v2 (Nov 2011)
 - $\gamma = -1, \beta \equiv 0$

$$\beta_b = \frac{Cd}{h_b}$$

Weak imposition of boundary conditions

$$\begin{aligned}
 & \int_{\Omega} -\nabla w_h \cdot (\mathbf{v}u_h - d\nabla u_h) d\mathbf{x} + \int_{\Gamma} w_h(\mathbf{v}u_h) \cdot \mathbf{n} ds \\
 & - \int_{\Gamma_D} w_h(d\nabla u_h) \cdot \mathbf{n} ds - \int_{\Gamma_D} (\gamma d\nabla w_h) \cdot \mathbf{n} u_h ds \\
 & - \int_{\Gamma_D \cap \Gamma_{\text{in}}} (\mathbf{v}w_h) \cdot \mathbf{n} u_h ds + \sum_{b=1}^{N_{\text{eb}}} \int_{\Gamma_D \cap \Gamma_b} \beta_b w_h u_h ds \\
 & = \int_{\Omega} w_h f d\mathbf{x} + \int_{\Gamma_N} w_h g ds - \int_{\Gamma_D} \gamma (d\nabla w_h) \cdot \mathbf{n} u_D ds \\
 & - \int_{\Gamma_D \cap \Gamma_{\text{in}}} (\mathbf{v}w_h) \cdot \mathbf{n} u_D ds + \sum_{b=1}^{N_{\text{eb}}} \int_{\Gamma_D \cap \Gamma_b} \beta_b w_h u_D ds
 \end{aligned}$$

$$\gamma = \pm 1, \quad \beta_b = \frac{Cd}{h_b}$$

Weak imposition of boundary conditions

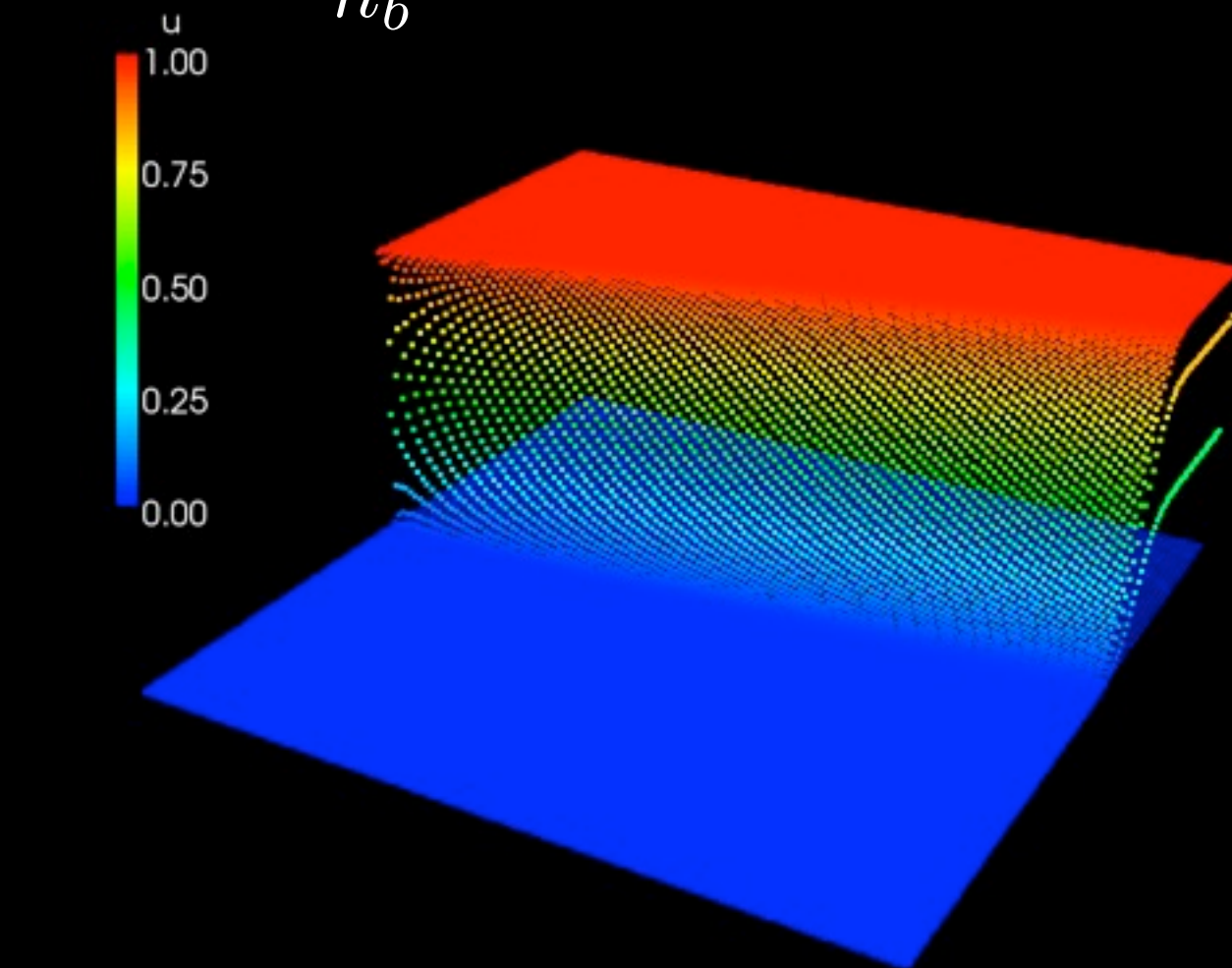
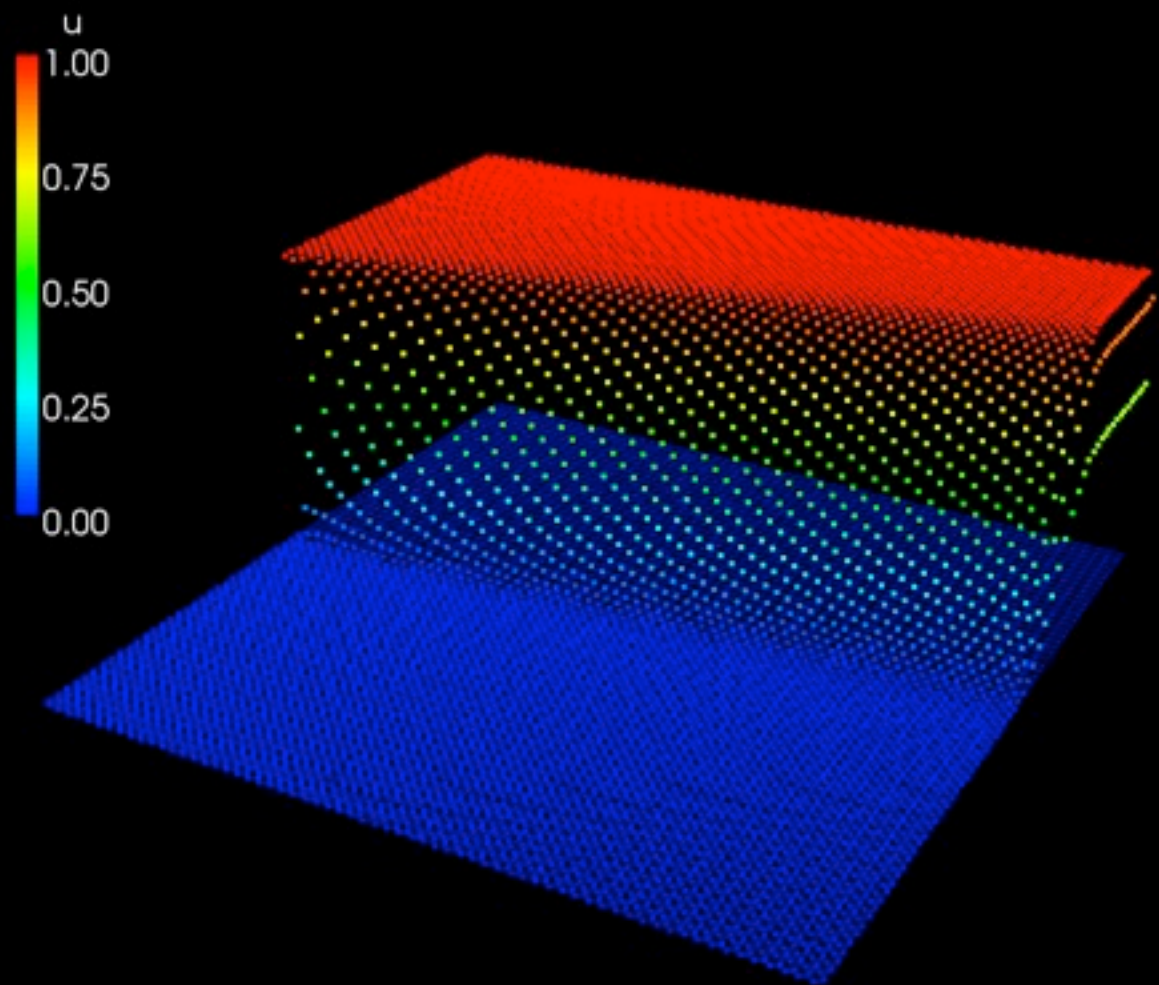
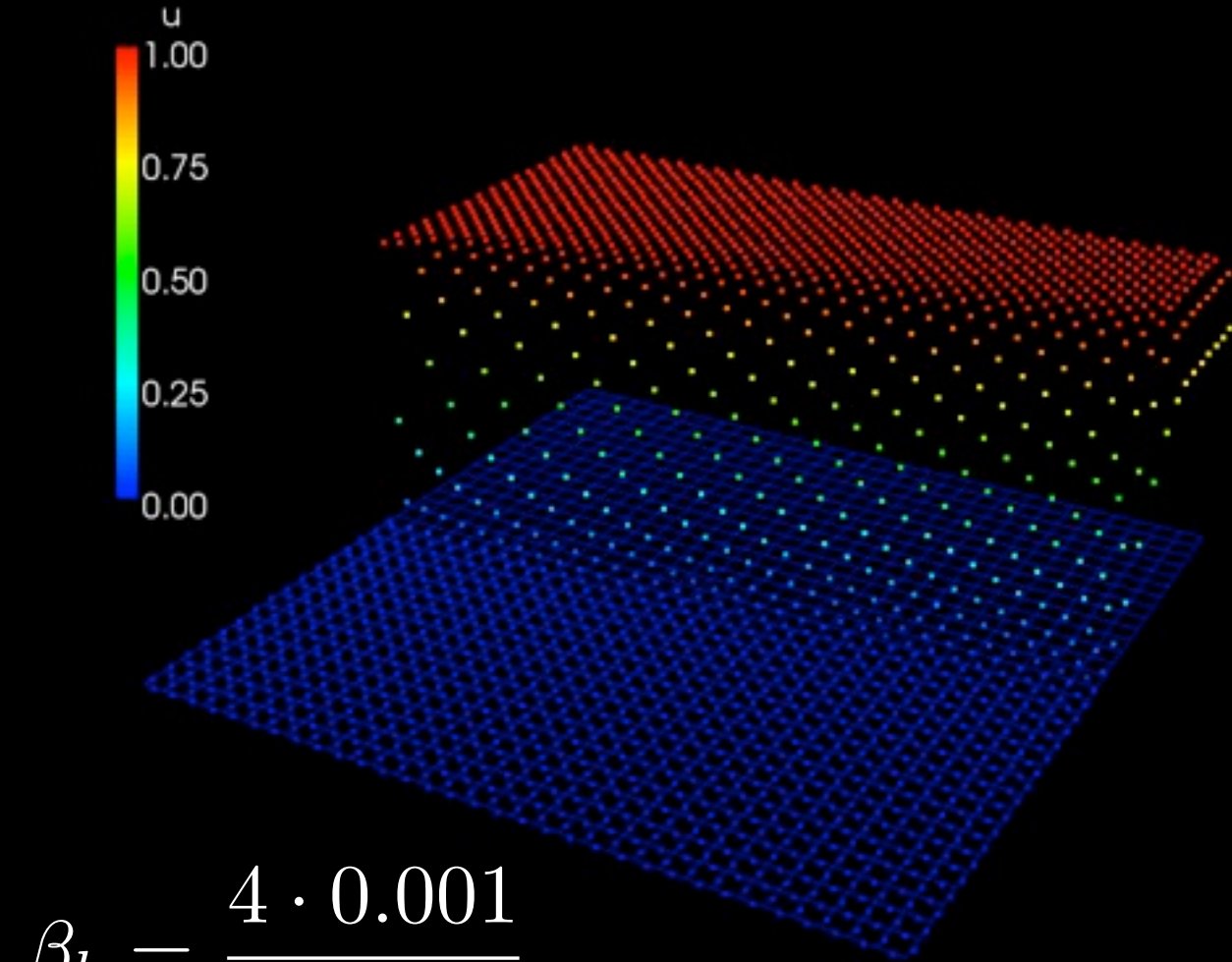
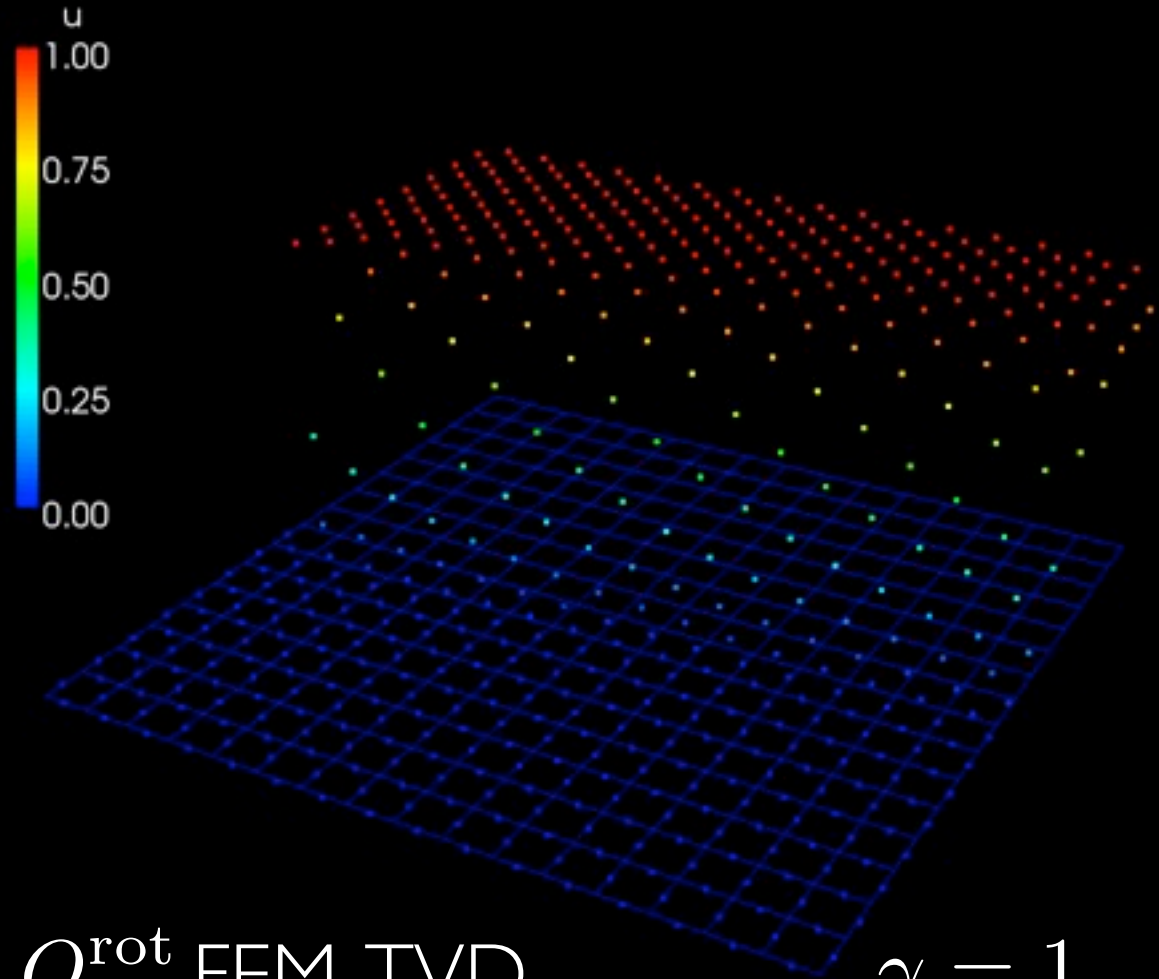
$$\int_{\Omega} -\nabla w_h \cdot (\mathbf{v} u_h - d \nabla u_h) d\mathbf{x} + \int_{\Gamma} w_h (\mathbf{v} u_h) \cdot \mathbf{n} ds$$

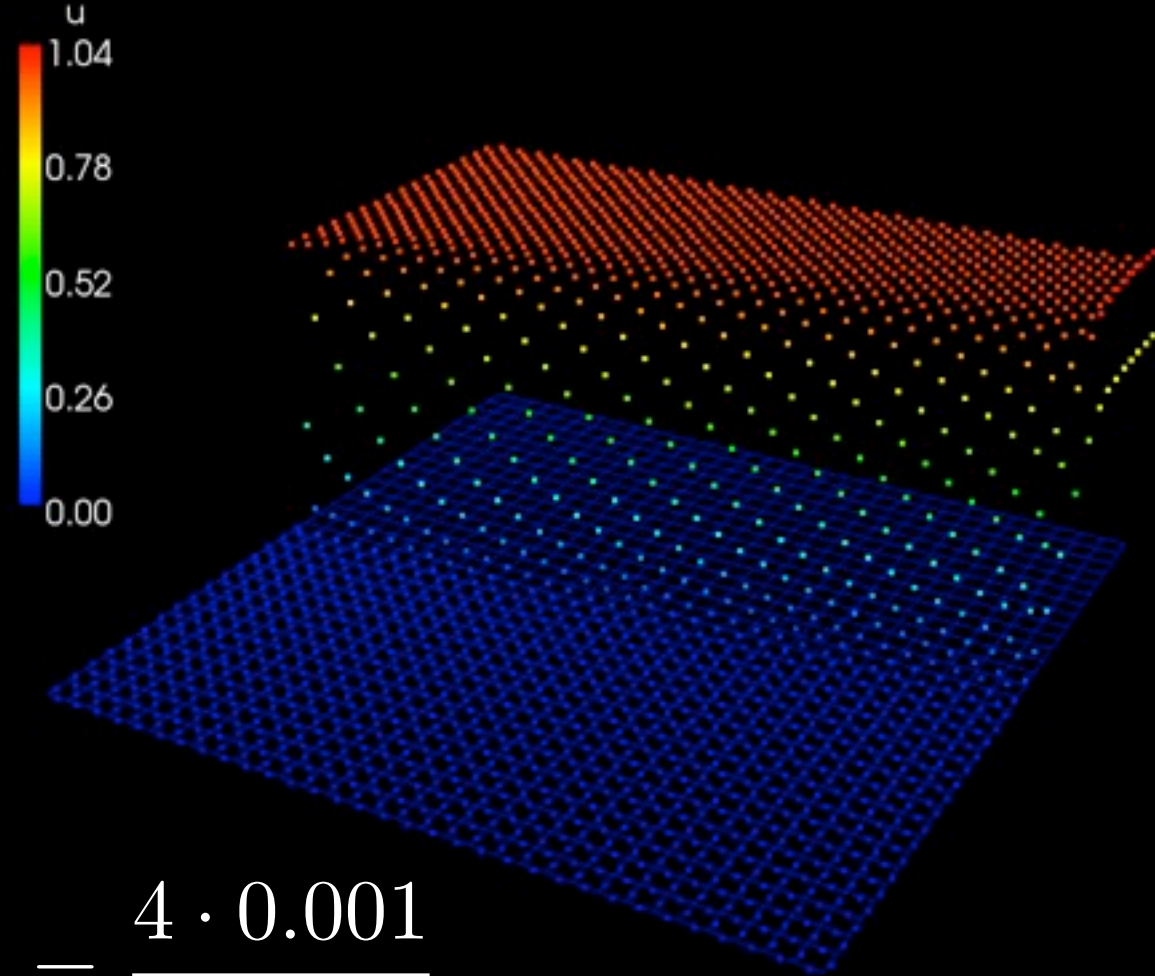
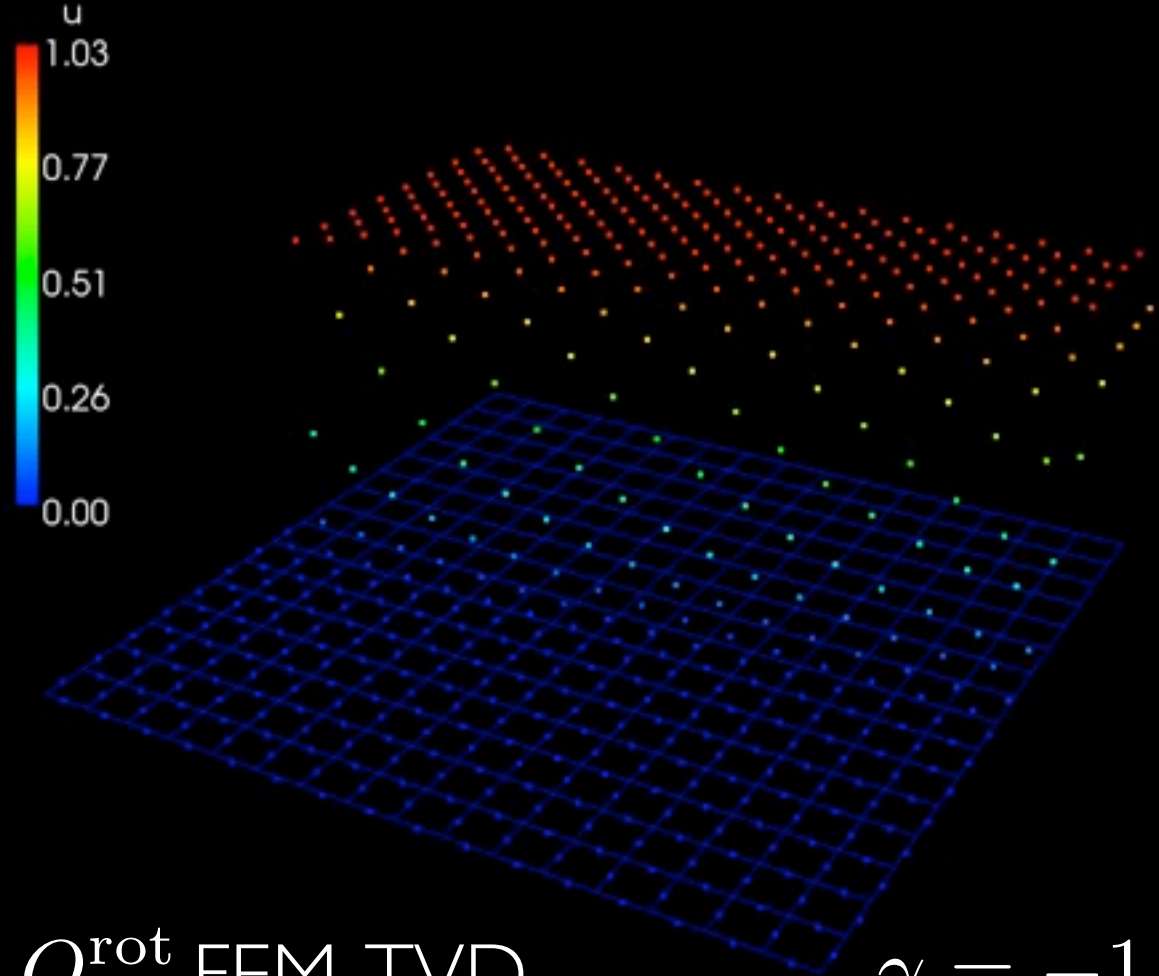


$$- \int_{\Gamma_D \cap \Gamma_{\text{in}}} (\mathbf{v} w_h) \cdot \mathbf{n} u_h ds \quad \longrightarrow \quad \int_{\Gamma_{\text{out}}} w_h \mathbf{v} \cdot \mathbf{n} u_h ds$$

$$= \int_{\Omega} w_h f d\mathbf{x}$$

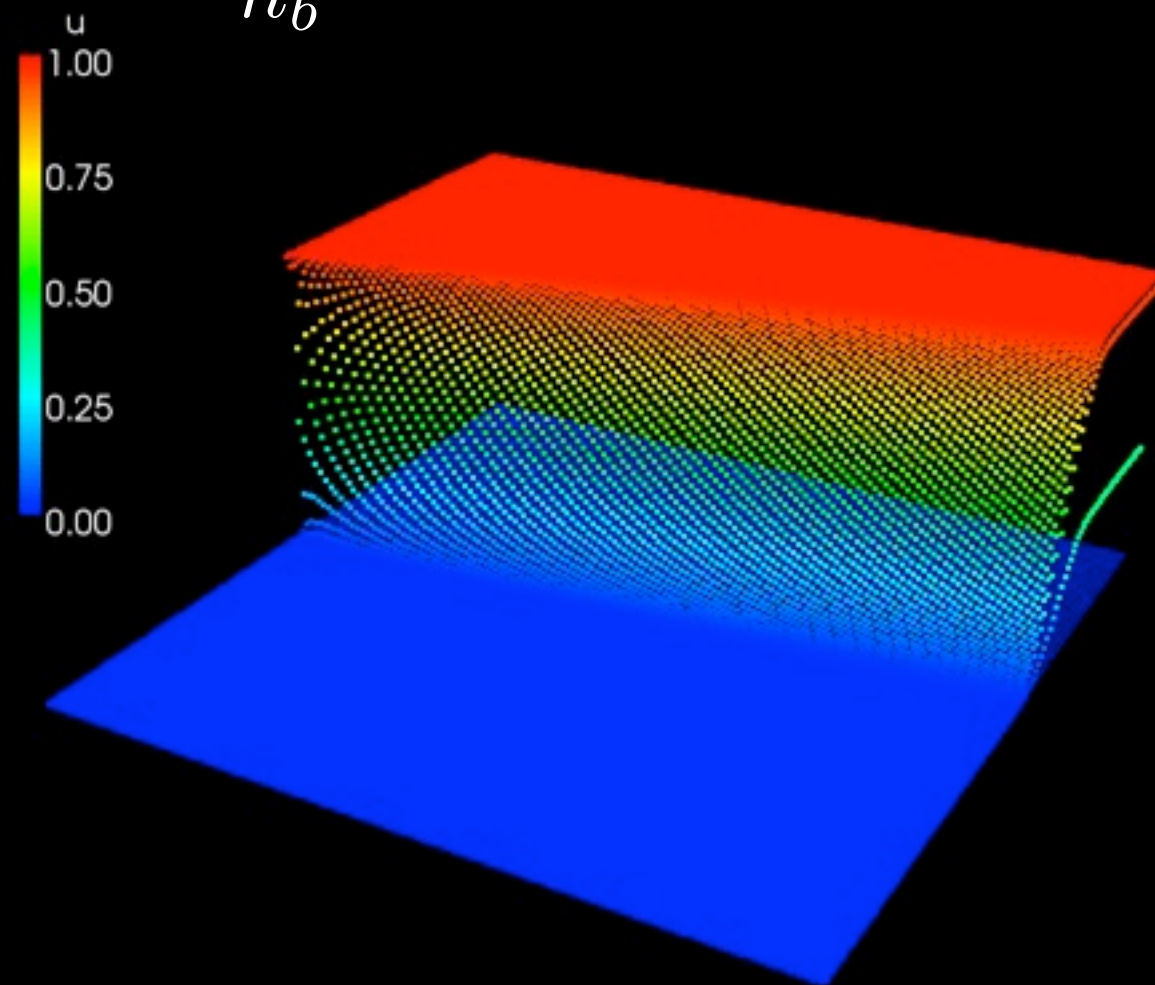
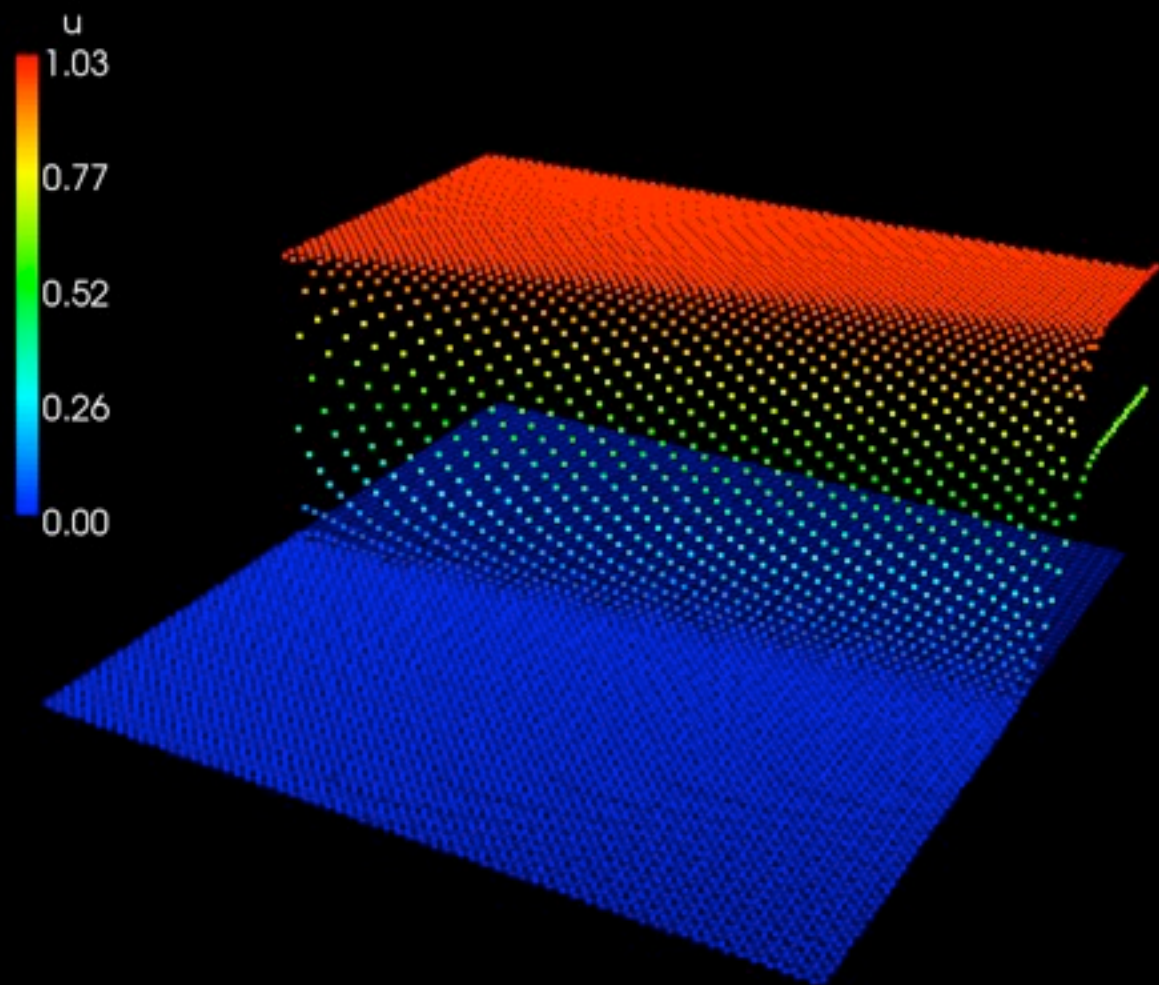
$$- \int_{\Gamma_D \cap \Gamma_{\text{in}}} (\mathbf{v} w_h) \cdot \mathbf{n} u_D ds$$

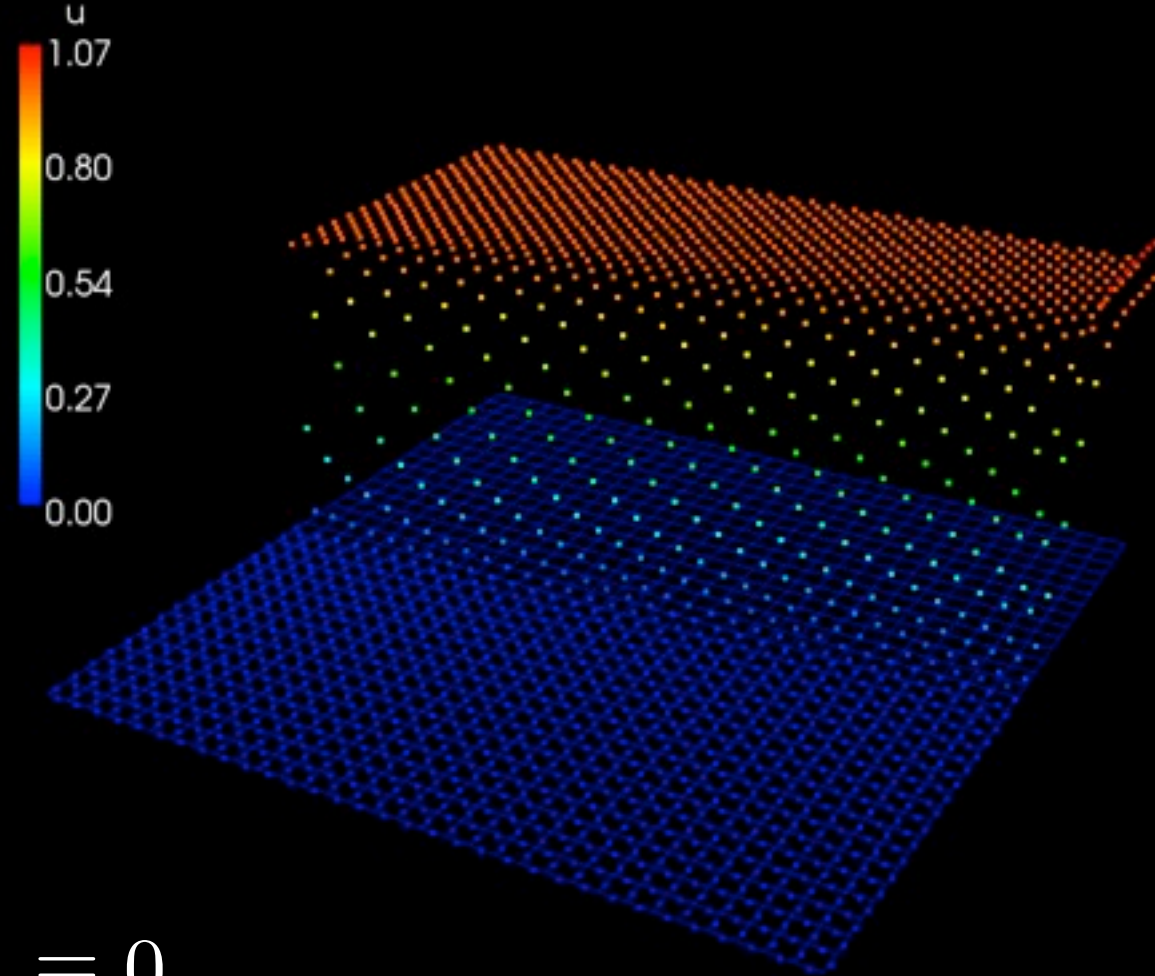
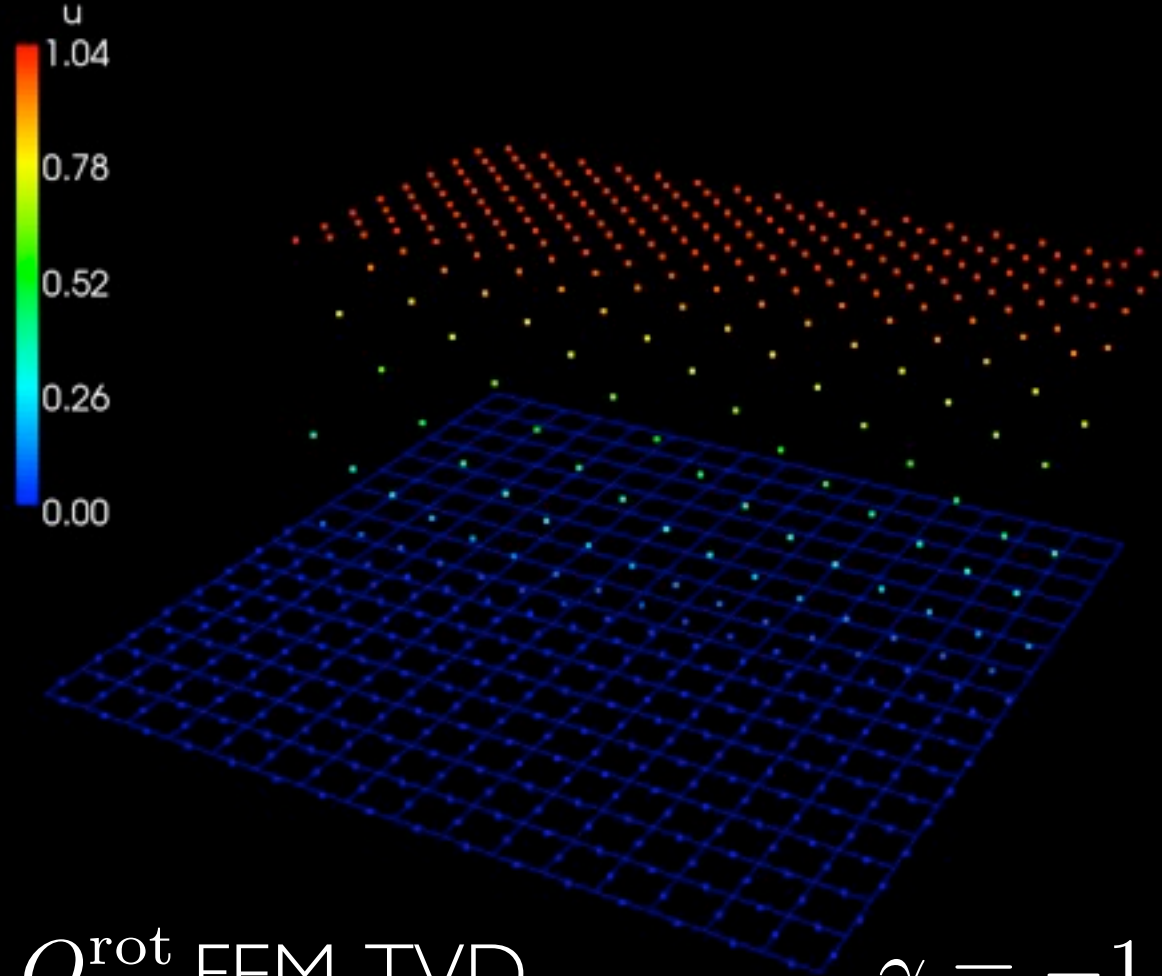




Q_1^{rot} FEM-TVD

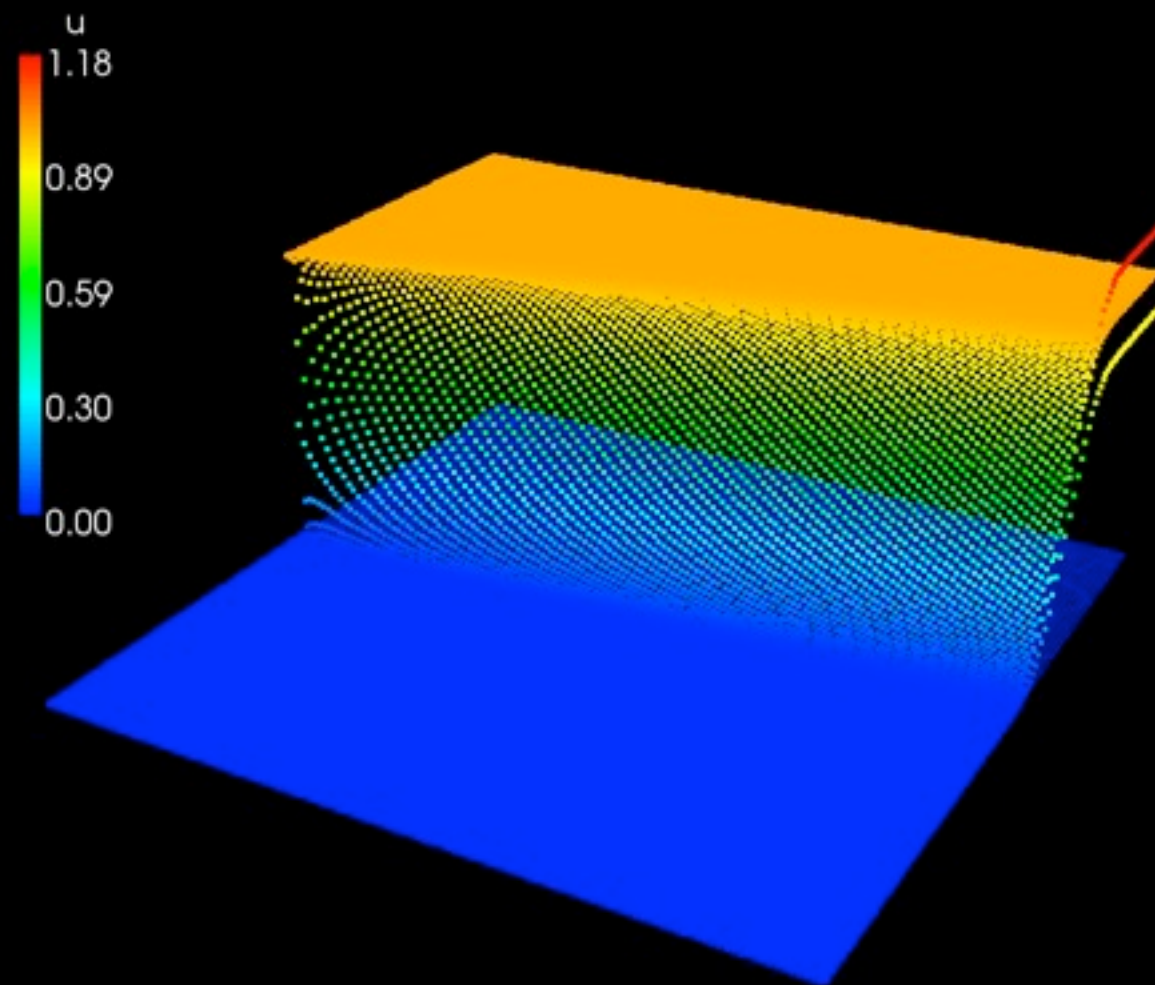
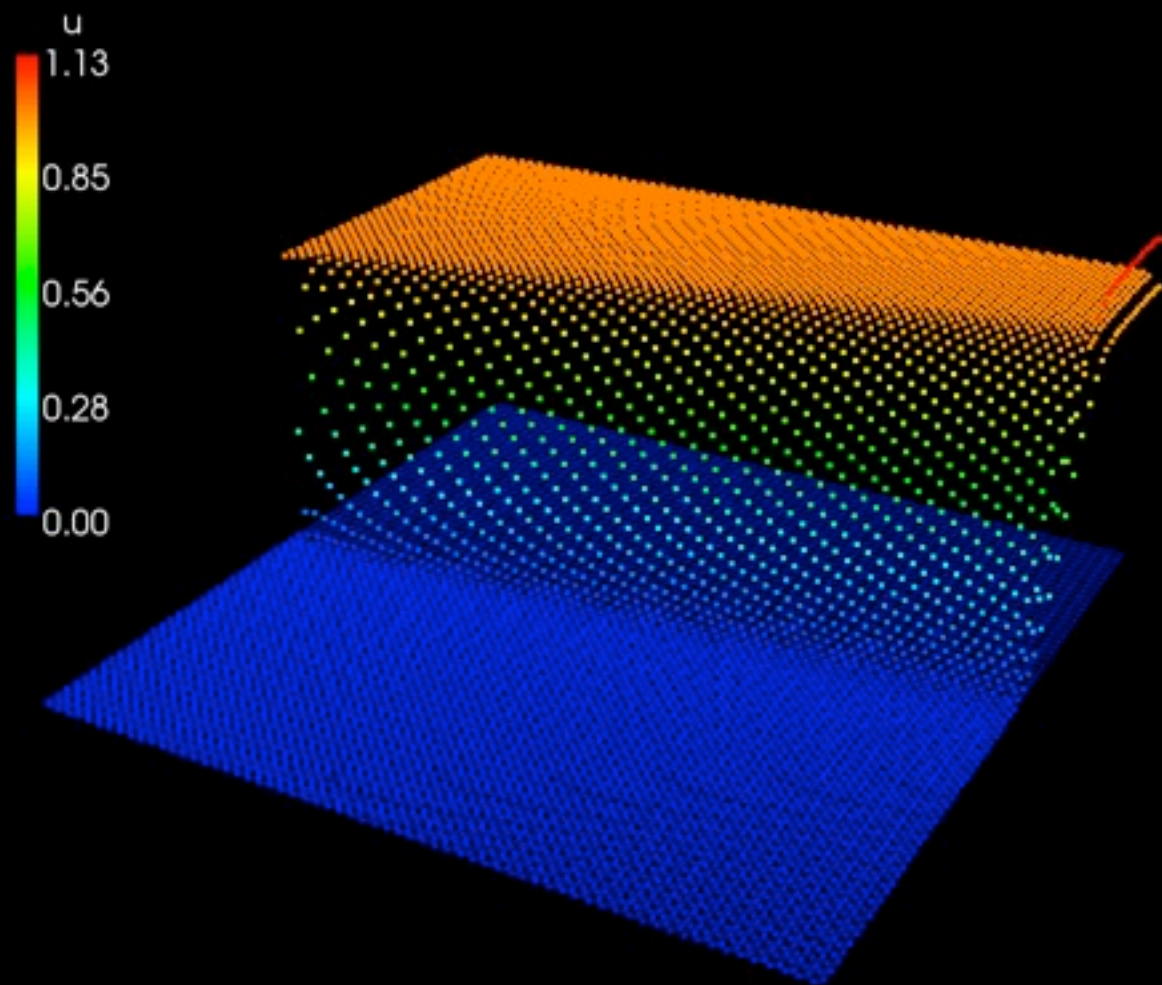
$$\gamma = -1, \quad \beta_b = \frac{4 \cdot 0.001}{h_b}$$





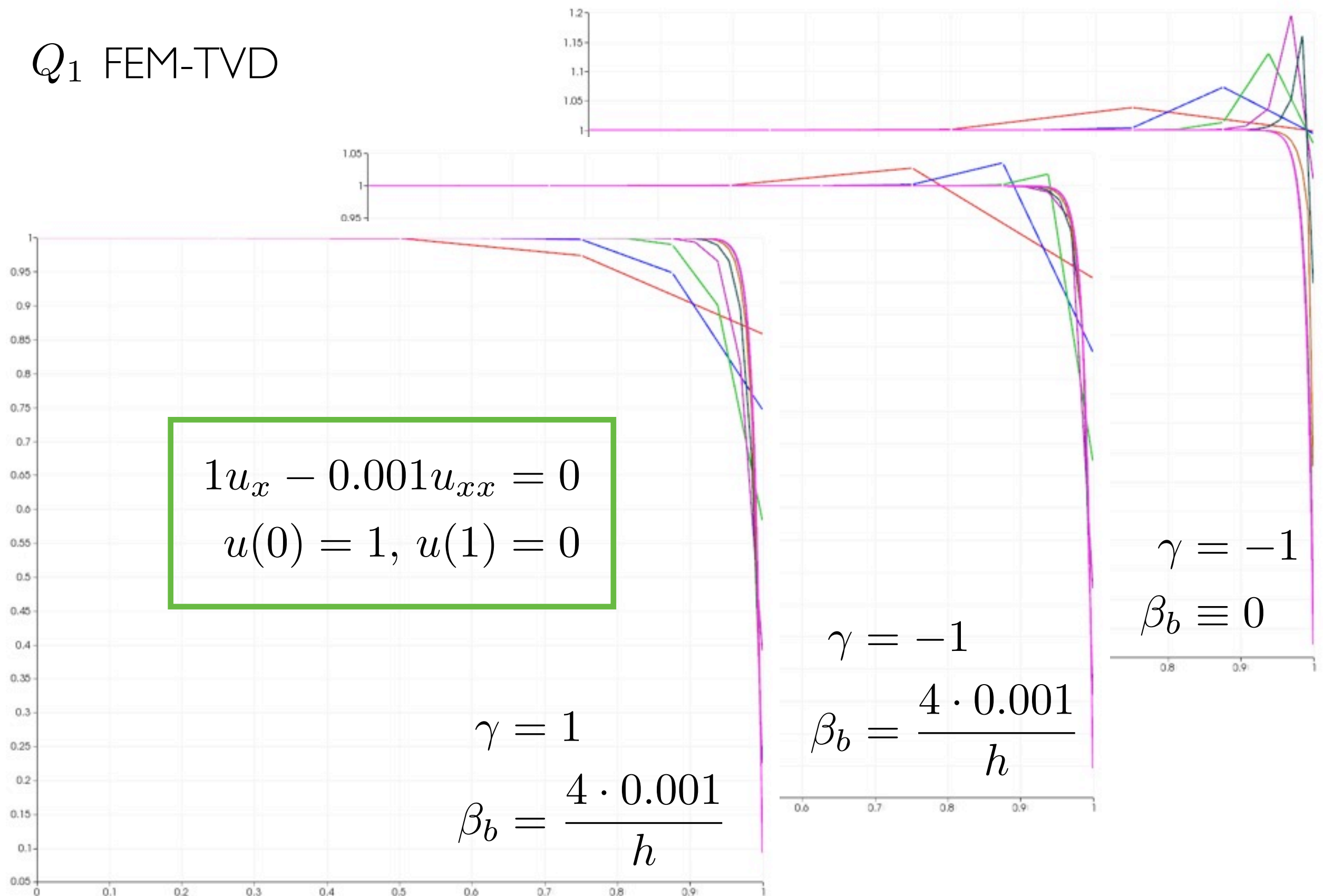
Q_1^{rot} FEM-TVD

$$\gamma = -1, \quad \beta_b \equiv 0$$



1D convection-diffusion equation

Q_1 FEM-TVD



Summary

- Do the algebraic design criteria apply to nonconforming elements?
Yes, for the integral mean value based variant of nonconforming rotated bilinear finite elements.
- Is the accuracy of solutions comparable to P_1/Q_1 approximations?
Yes, accuracy and numerical dissipation are comparable.
- How to implement essential boundary conditions?
Apply consistent, adjoint-consistent Nitsche-type method.
- Is there any benefit from using nonconforming elements?
Regular sparsity structure beneficial for parallel HPC.

Outlook

- Extend AFC schemes to other nonconforming finite elements
 - NC-Quad (Z. Cai, J. Douglas, Jr., X. Ye)
 - Composite Crouzeix-Raviart (F. Schieweck)
- Generalize consistent, adjoint-consistent approach towards the implementation of Dirichlet boundary conditions to periodic ones
- Improve (parallel) efficiency by exploring the regular sparsity structure
 - use variant of ELLPACK matrix storage format
 - reduce communication costs due to weaker coupling

Acknowledgment

AFC schemes

- D. Kuzmin (University of Erlangen-Nuremberg)

Featflow2

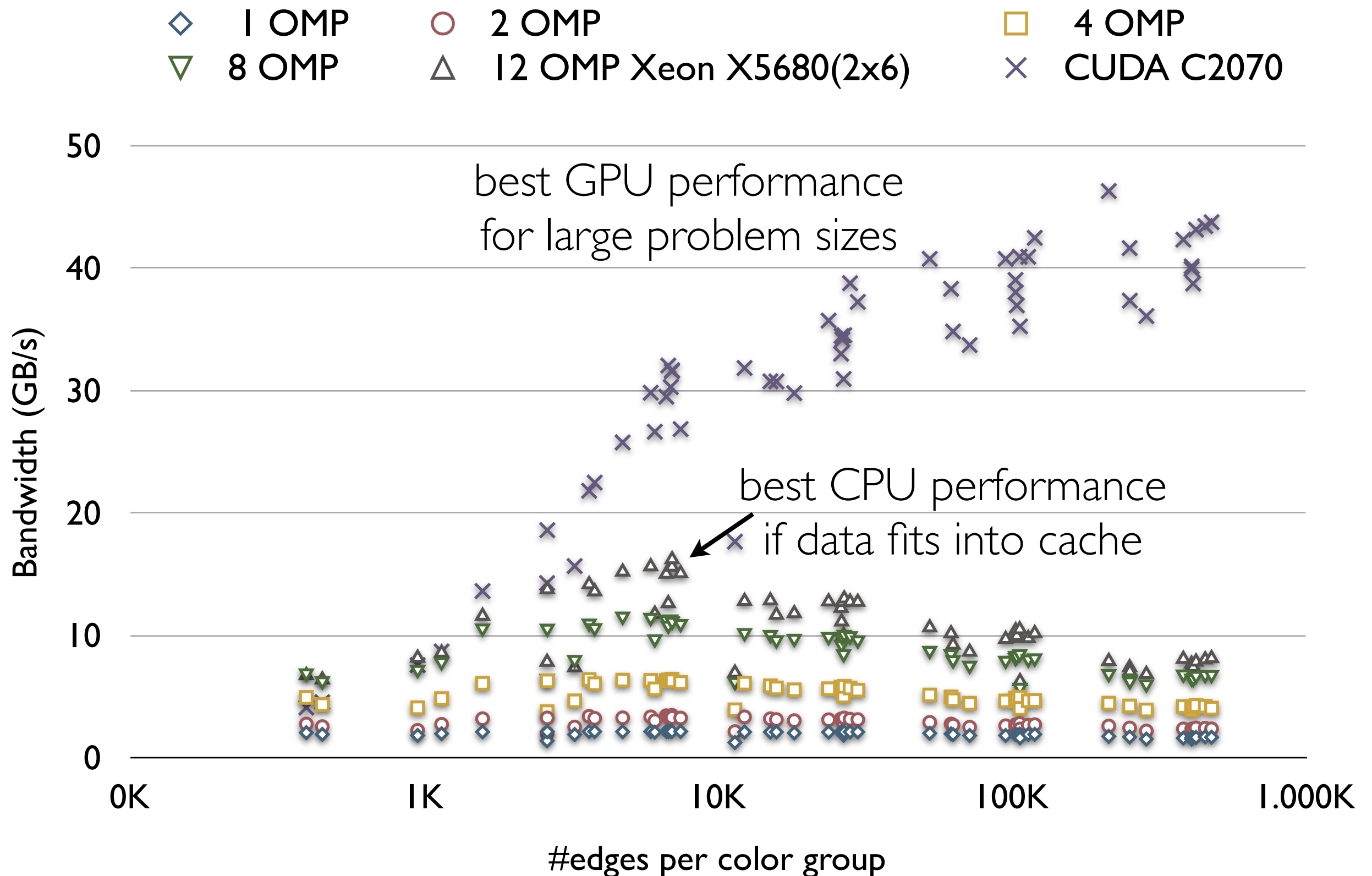
- M. Köster, P. Zajac (TU Dortmund)

Source code freely available at:

<http://www.featflow.de/en/software/featflow2.html>

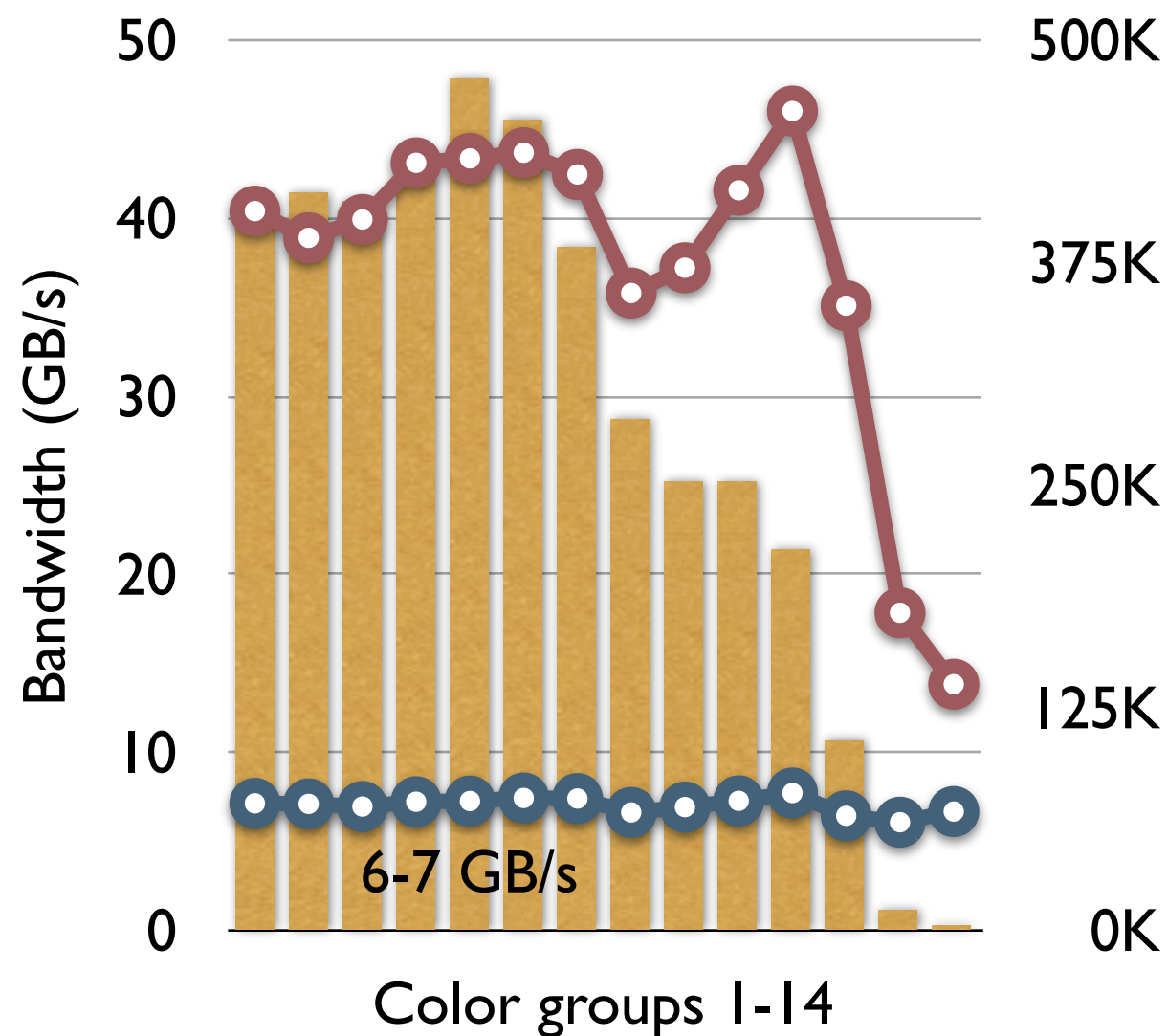
Edge-based solvers for the compressible
Euler equations on multicores and GPUs

Example: edge-based flux-assembly with Q_1 FE

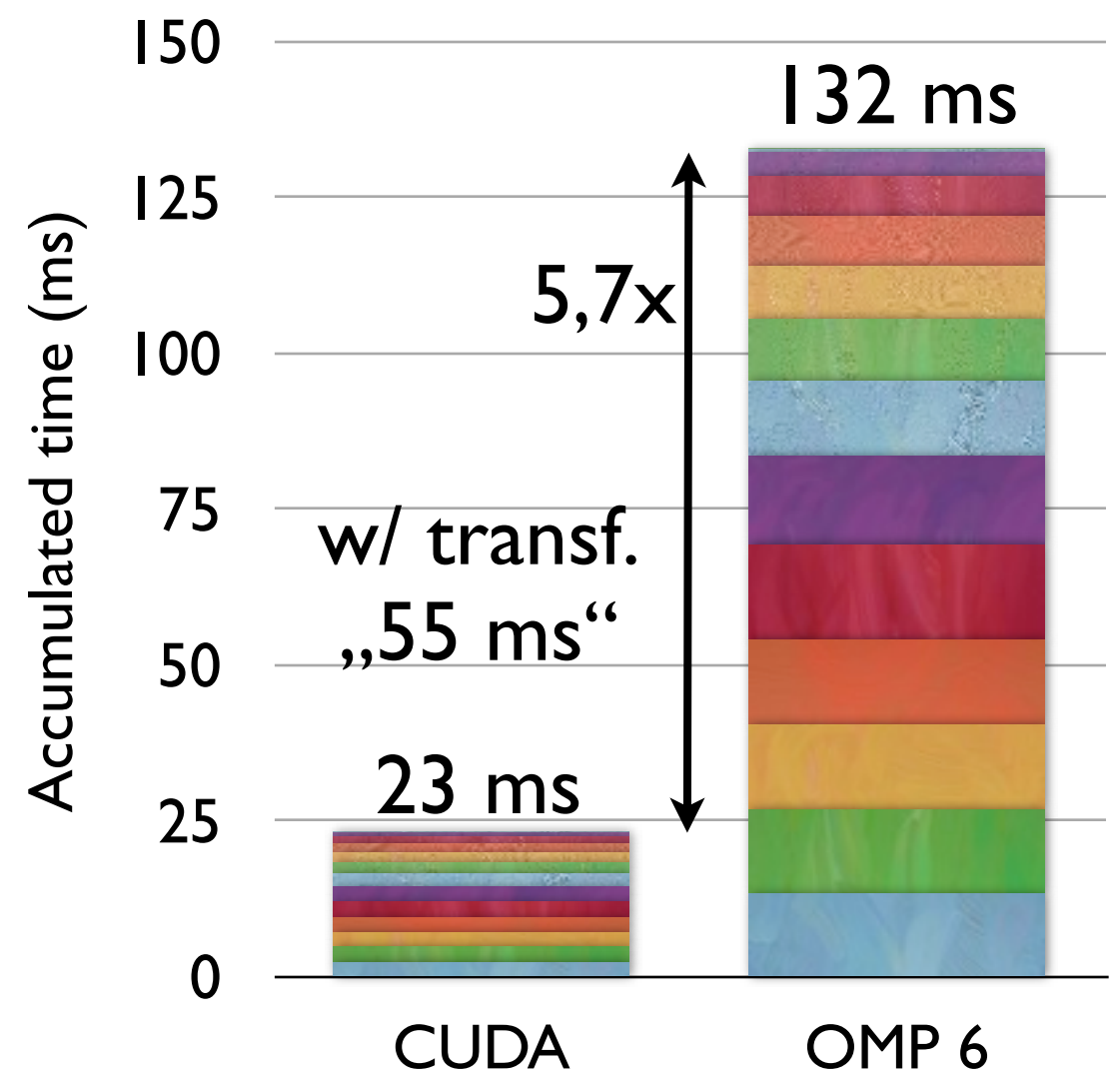


Example: edge-based flux assembly with Q_1 FE

- OMP 6 on Core i7 EP
- CUDA on C2070
- #edges per color group



$$F_i = \sum_{k=1}^{N_{\text{colors}}} \sum_{ij \in \text{CG}_k} \sigma_{ij} (U_j - U_i)$$



Example: edge-based flux assembly with Q_1^{rot} FE

