Implicit FEM-FCT Algorithm for the Compressible Euler Equations

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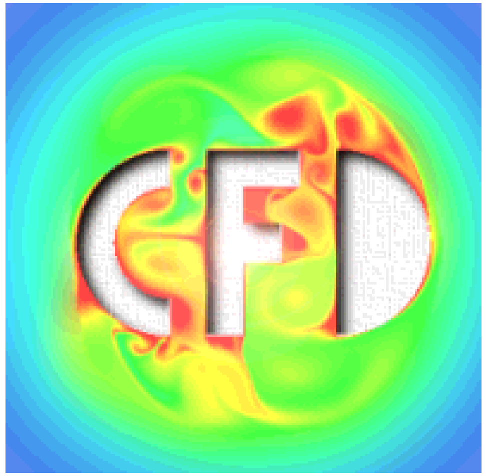


- State of the art: discretization techniques
- Discrete upwinding for scalar equations
- Iterative defect correction scheme
- Generalized FEM-FCT formulation
- Matrix assembly for the Euler equations
- Construction of artificial viscosities
- Solution strategies for coupled equations
- Implementation of boundary conditions

Finite element FCT procedure   Löhner *et al.* (1987)

step 1: High-order finite element scheme

$$M_C(u^H - u^n) = b$$

step 2: Low-order finite element scheme

$$M_L(u^L - u^n) = b + c_d[M_C - M_L]u^n$$

step 3: Antidiffusive Element Contributions

$$F_e = M_L^{-1}\Big|_e [\hat{M}_L - \hat{M}_C]((c_d - 1)\hat{u}^n + \hat{u}^H)$$

step 4: Zalesak's limiter

$$F_e^* = \alpha_e F_e, \quad 0 \le \alpha_e \le 1$$

step 5: Element-by-element correction

$$u_i^{n+1} = u_i^L + \sum_e F_{e,i}^*$$

- CFL restriction due to explicit time stepping
- arbitrarily chosen constant artificial diffusion
- formulation in terms of AEC rather than fluxes

Discrete positivity criteria

Jameson (1993)

$$\boxed{\textbf{LED:}\quad \frac{\mathrm{d}u_i}{\mathrm{d}t} = \sum_{j\neq i} c_{ij}(u_j - u_i), \quad c_{ij} \geq 0}$$

$$u_i = \max_j u_j \quad \Rightarrow \quad u_j - u_i \leq 0 \quad \Rightarrow \quad \frac{\mathrm{d}u_i}{\mathrm{d}t} \leq 0$$

$$u_i = \min_j u_j \quad \Rightarrow \quad u_j - u_i \geq 0 \quad \Rightarrow \quad \frac{\mathrm{d}u_i}{\mathrm{d}t} \geq 0$$

in 1D: equivalence between LED and Harten's (1983) TVD conditions

Lemma. A discrete scheme of the form

$$Au^{n+1} = Bu^n, \quad u^n \geq 0$$

is positivity-preserving if A is an *M-matrix* and all entries of B are non-negative.

Discrete diffusion operators $D = \{d_{ij}\}$ $\qquad d_{ij} = d_{ji}, \quad \sum_j d_{ij} = \sum_i d_{ij} = 0$

Flux decomposition of diffusive terms $\qquad (Du)_i = \sum_j d_{ij}u_j = \sum_{j\neq i} d_{ij}(u_j - u_i) = \sum_{j\neq i} f_{ij}$

Skew-symmetric (anti-)diffusive fluxes $\qquad \boxed{f_{ij} = d_{ij}(u_j - u_i), \quad f_{ji} = -f_{ij}}$

Discrete upwinding

Scalar transport equation $\quad \dfrac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u - \epsilon \nabla u) = 0 \qquad u = \sum_j u_j \varphi_j, \quad \mathbf{v}u \approx \sum_j \mathbf{v}_j u_j \varphi_j$

Lumped mass Galerkin scheme $\quad \boxed{M_L \dfrac{\mathrm{d}u}{\mathrm{d}t} = Ku} \qquad m_i \dfrac{\mathrm{d}u_i}{\mathrm{d}t} = \sum_{j \neq i} k_{ij}(u_j - u_i) + \delta_i u_i$

$$M_L = \text{diag}\{m_i\} \qquad m_{ij} = \int_\Omega \varphi_i \varphi_j \mathrm{d}\mathbf{x} \qquad k_{ij} = -\mathbf{v}_j \cdot \mathbf{c}_{ij} - \epsilon\, s_{ij} \qquad m_i = \sum_j m_{ij}$$

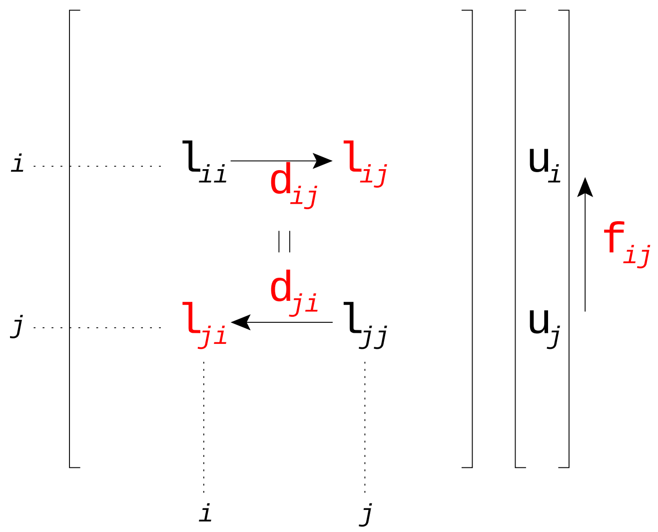
$$K = \{k_{ij}\} \qquad \mathbf{c}_{ij} = \int_\Omega \varphi_i \nabla \varphi_j \,\mathrm{d}\mathbf{x} \qquad s_{ij} = \int_\Omega \nabla \varphi_i \cdot \nabla \varphi_j \,\mathrm{d}\mathbf{x} \qquad \delta_i = \sum_j k_{ij} \approx \nabla \cdot \mathbf{v}$$

Adaptive artificial diffusion $\quad L = K + D$

$$d_{ij} = d_{ji} = \max\{0, -k_{ij}, -k_{ji}\}, \qquad d_{ii} = -\sum_{j \neq i} d_{ij}$$

Edge-based elimination of negative off-diagonal entries

$$L := K \quad \rightarrow \quad \begin{array}{ll} l_{ii} := l_{ii} - d_{ij}, & l_{ij} := l_{ij} + d_{ij} \\ l_{ji} := l_{ji} + d_{ij}, & l_{jj} := l_{jj} - d_{ij} \end{array}$$



Generalized FEM-FCT formulation

Nonlinear system for an implicit FEM-FCT algorithm (standard $\theta-$ scheme)

$$M_L \frac{u^{n+1} - u^n}{\Delta t} = \theta L(u^{n+1})u^{n+1} + (1-\theta)L(u^n)u^n + f(u^{n+1}, u^n)$$

Successive approximation $\boxed{A(u^{(m)})u^{(m+1)} = b^{(m+1)}}$ $\quad b^{(m+1)} = b^n + f(u^{(m)}, u^n)$

Low-order contribution $A(u^{(m)}) = M_L - \theta \Delta t L(u^{(m)}) \quad b^n = M_L u^n + (1-\theta)\Delta t L(u^n)u^n$

Raw antidiffusion (difference between high- and low-order scheme)

$$\begin{aligned} f^{(m)} &= \left[(M_C - M_L) - (1-\theta)\Delta t(L(u^n) - K(u^n))\right] u^n \\ &- \left[(M_C - M_L) + \theta \Delta t \Big(L(u^{(m)}) - K(u^{(m)})\Big)\right] u^{(m)} \end{aligned}$$

Limited antidiffusion $f_i = \sum_{j \neq i} \alpha_{ij}^{(m)} f_{ij}^{(m)}, \quad f_{ji}^{(m)} = -f_{ij}^{(m)}, \quad \alpha_{ij}^{(m)} = \alpha_{ji}^{(m)}$

Antidiffusive fluxes $f_{ij}^{(m)} = \left[m_{ij} - (1-\theta)\Delta t d_{ij}^n\right](u_j^n - u_i^n) - \left[m_{ij} + \theta \Delta t d_{ij}^{(m)}\right](u_j^{(m)} - u_i^{(m)})$

Basic FEM-FCT formulation

Successive approximation $\boxed{A(u^{(m)})u^{(m+1)} = b^{(m+1)}}$ $\quad u^{(0)} = u^n, \qquad m = 0, 1, 2, \dots$

By construction, A is an M-matrix which is easy to 'invert' and satisfies the positivity criterion.

Strategy: *Design the correction factors α_{ij} so that there exists a matrix $B \geq 0$ and a positivity-preserving auxiliary solution $\tilde{u}$ such that $b^{(m+1)} = B\tilde{u}$.*

Positivity transfer cycle $\quad u^n \geq 0 \quad \Rightarrow \quad \tilde{u} \geq 0 \quad \Rightarrow \quad u^{(m+1)} \geq 0$

Modified right-hand side $\boxed{b_i^{(m+1)} = m_i \tilde{u}_i^n + \sum_{j \neq i} \alpha_{ij}^{(m)} f_{ij}^{(m)}}$ where $\quad \tilde{u}^n = M_L^{-1} b^n$

Correction factors $\quad \alpha_{ij}^{(m)} = \alpha_{ij}(\tilde{u}^n, f_{ij}^{(m)}), \quad 0 \leq \alpha_{ij} \leq 1$ Zalesak's limiter

The solution $\tilde{u}^n$ to the explicit subproblem

$$M_L \tilde{u}^n = M_L u^n + (1-\theta)\Delta t L(u^n) u^n, \quad 0 < \theta \leq 1$$

proves to be positivity-preserving for $\quad \Delta t \leq \frac{1}{1-\theta} \min_i \{-m_i/l_{ii} | l_{ii} < 0\}$

Remark: The admissible percentage of the antidiffusive flux depends on the magnitude of the time step Δt.

Iterative FEM-FCT formulation

Strategy: *Build accepted antidiffusion into $\tilde{u}$ and limit only the rejected portion.*

Variable auxiliary solution $M_L\tilde{u}^{(m)} = b^{(m)}, \qquad b^{(0)} = b^n$

Rejected antidiffusive fluxes $\Delta f_{ij}^{(m)} = f_{ij}^{(m)} - g_{ij}^{(m)}, \qquad \Delta f_{ij}^{(0)} = f_{ij}^{(0)}$

Correction factors $\alpha_{ij}^{(m)} = \alpha_{ij}(\tilde{u}^{(m)}, \Delta f_{ij}^{(m)}), \qquad 0 \leq \alpha_{ij}^{(m)} \leq 1$ Zalesak's limiter

Accepted antidiffusive fluxes $g_{ij}^{(m+1)} = g_{ij}^{(m)} + \alpha_{ij}^{(m)} \Delta f_{ij}^{(m)}, \qquad g_{ij}^{(0)} = 0$

Modified right-hand side

$$\boxed{b_i^{(m+1)} = b_i^{(m)} + \sum_{j \neq i} \alpha_{ij}^{(m)} \Delta f_{ij}^{(m)}}$$

Consistency: $\alpha_{ij}^{(m)} \equiv 1 \quad \Rightarrow \quad b_i^{(m+1)} = b_i^n + \sum_{j \neq i} \left(g_{ij}^{(m)} + \Delta f_{ij}^{(m)} \right) = b_i^n + \sum_{j \neq i} f_{ij}^{(m)}$

Remark: The iterative flux limiter makes it possible to 'recycle' the rejected antidiffusion.

Multidimensional Zalesak's limiter

(0.) Prelimiting: set $f_{ij} := 0$ if $f_{ij}(\tilde{u}_i - \tilde{u}_j) \leq 0$

1. Sum of positive / negative antidiffusive fluxes $P_i^{\pm} = \frac{1}{m_i} \sum_{j \neq i} \begin{smallmatrix}\max\\ \min\end{smallmatrix} \{0, f_{ij}\}$

2. Upper / lower bounds

$$Q_i^{\pm} = \tilde{u}_i^{\begin{smallmatrix}\max\\ \min\end{smallmatrix}} - \tilde{u}_i, \qquad \tilde{u} \approx u(t^{n+1-\theta})$$

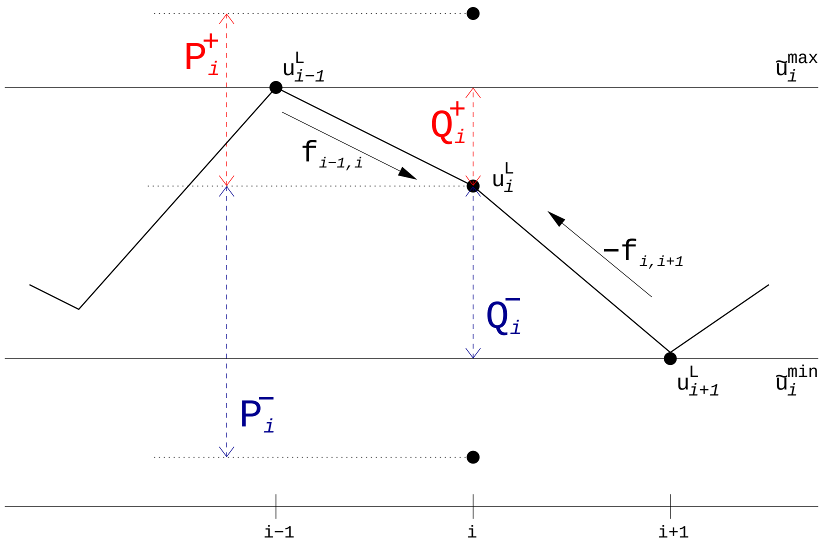
3. Nodal correction factors

$$R_i^{\pm} = \begin{cases} \min\{1, Q_i^{\pm}/P_i^{\pm}\}, & \text{if } P^{\pm} \neq 0 \\ 0, & \text{if } P_i^{\pm} = 0 \end{cases}$$

(3.') Postlimiting: set $R_i^{\pm} := 1$ at in- / outlet where $u_i^{\begin{smallmatrix}\max\\ \min\end{smallmatrix}}$ may be estimated wrongly

4. Final correction factors

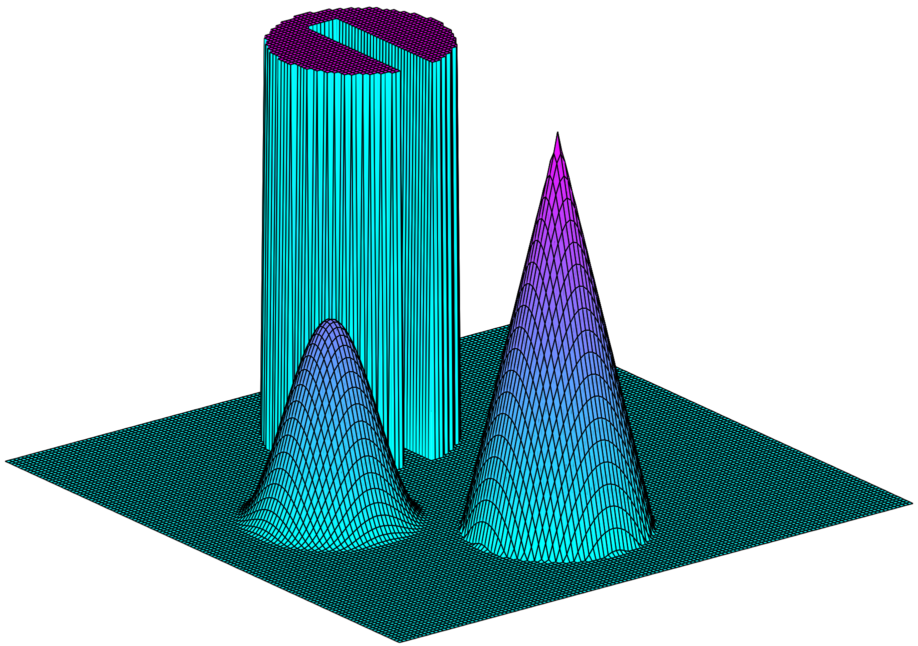
$$\alpha_{ij} = \begin{cases} \min\{R_i^+, R_j^-\}, & \text{if } f_{ij} \geq 0 \\ \min\{R_j^+, R_i^-\}, & \text{if } f_{ij} < 0 \end{cases}$$



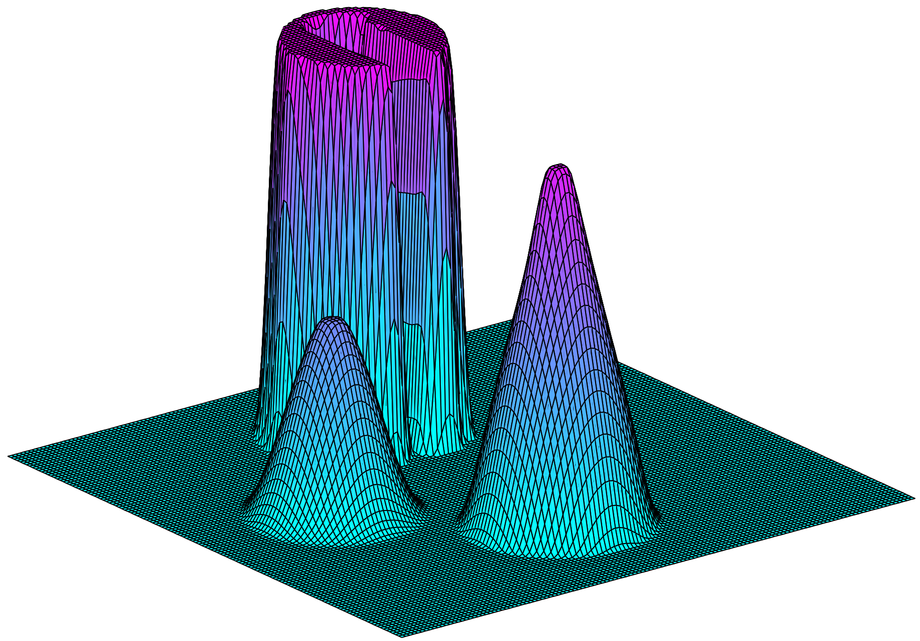
Remark: Zalesak's limiter enforces the positivity constraint by tuning the correction factors.

Solid body rotation

exact solution / initial data

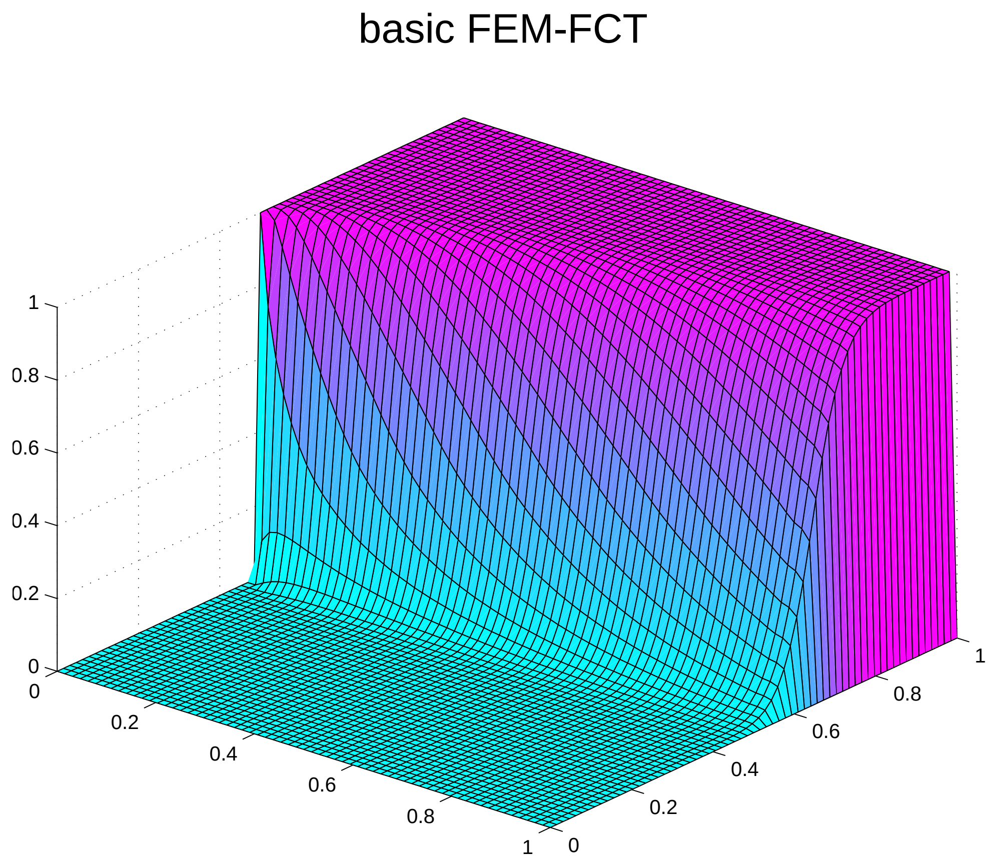


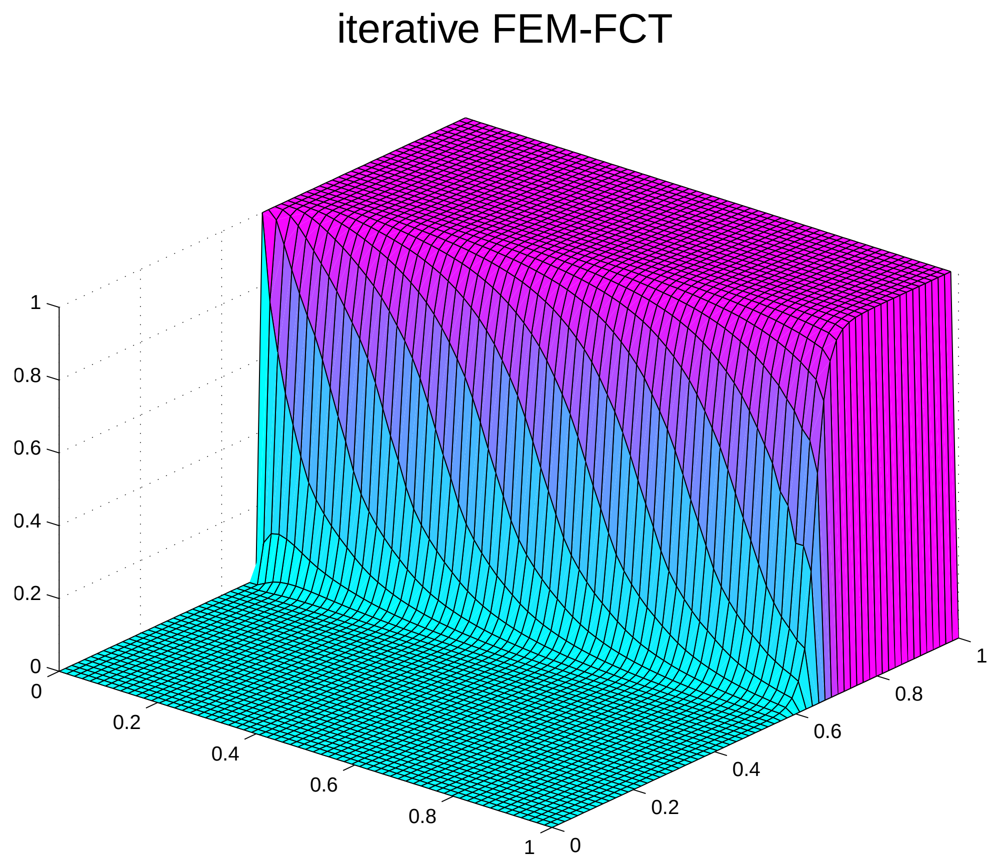
iterative FEM-FCT



Crank-Nicolson time-stepping, $\Delta t = 10^{-3}$, 128×128 Q_1 elements

Steady-state convection-diffusion





Backward Euler time-stepping, $\mathbf{v} = (\cos 10^\circ, \sin 10^\circ),\ \Delta t = 10^{-1},\ \epsilon = 10^{-3},\ 64 \times 64\ Q_1$ elements

Compressible Euler equations

Divergence form $\boxed{\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = 0}$ where $\nabla \cdot \mathbf{F} = \sum_{d=1}^{3} \frac{\partial F^d}{\partial x_d}$

Conservative variables and fluxes

$$U = (\rho, \rho\mathbf{v}, \rho E)^T \qquad \mathbf{F} = \begin{bmatrix} \rho\mathbf{v} \\ \rho\mathbf{v} \otimes \mathbf{v} + p\mathcal{I} \\ \rho H\mathbf{v} \end{bmatrix} \qquad H = E + \frac{p}{\rho}$$

$$\mathbf{F} = (F^1, F^2, F^3) \qquad \gamma = c_p/c_v$$

Equation of state $p = (\gamma - 1)\rho\left(E - |\mathbf{v}|^2/2\right)$ for a polytropic gas

Quasi-linear form $\boxed{\frac{\partial U}{\partial t} + \mathbf{A} \cdot \nabla U = 0}$ where $\mathbf{A} \cdot \nabla U = \sum_{d=1}^{3} A^d \frac{\partial U}{\partial x_d}$

Jacobian matrices $\mathbf{A} = (A^1, A^2, A^3)$ $\quad F^d = A^d U, \quad A^d = \frac{\partial F^d}{\partial U}, \quad d = 1, 2, 3$

Galerkin discretization

Group FEM formulation $\boxed{M_C \frac{\mathrm{d}U}{\mathrm{d}t} = KU}$ $(KU)_i = -\sum_j \mathbf{c}_{ij} \cdot \mathbf{F}_j = -\sum_{j \neq i} \mathbf{c}_{ij} \cdot (\mathbf{F}_j - \mathbf{F}_i)$

due to the fact that for the sum of basis functions $\sum_j \varphi_j \equiv 1 \quad \Rightarrow \quad \mathbf{c}_{ii} = -\sum_{j \neq i} \mathbf{c}_{ij}$

Roe averaging $\mathbf{F}_j - \mathbf{F}_i = \hat{\mathbf{A}}_{ij}(U_j - U_i)$ where $\hat{\mathbf{A}}_{ij} = \mathbf{A}(\hat{\rho}_{ij}, \hat{\mathbf{v}}_{ij}, \hat{H}_{ij})$

$$\hat{\rho}_{ij} = \sqrt{\rho_i \rho_j}, \qquad \hat{\mathbf{v}}_{ij} = \frac{\sqrt{\rho_i}\mathbf{v}_i + \sqrt{\rho_j}\mathbf{v}_j}{\sqrt{\rho_i} + \sqrt{\rho_j}}, \qquad \hat{H}_{ij} = \frac{\sqrt{\rho_i}H_i + \sqrt{\rho_j}H_j}{\sqrt{\rho_i} + \sqrt{\rho_j}}$$

Quasi-linear formulation

$$(KU)_i = -\sum_{j \neq i} \mathbf{c}_{ij} \cdot \hat{\mathbf{A}}_{ij}(U_j - U_i) = \sum_{j \neq i} (\mathsf{A}_{ij} + \mathsf{B}_{ij})(U_j - U_i)$$

Cumulative Roe matrices

$$\mathsf{A}_{ij} = \mathbf{a}_{ij} \cdot \hat{\mathbf{A}}_{ij}, \qquad \mathbf{a}_{ij} = -\frac{\mathbf{c}_{ij} - \mathbf{c}_{ji}}{2}$$

$$\mathsf{B}_{ij} = \mathbf{b}_{ij} \cdot \hat{\mathbf{A}}_{ij}, \qquad \mathbf{b}_{ij} = -\frac{\mathbf{c}_{ij} + \mathbf{c}_{ji}}{2}$$

Contribution of edge $\vec{ij}$

$$(\mathsf{A}_{ij} + \mathsf{B}_{ij})(U_j - U_i) \quad \longrightarrow \quad (KU)_i$$

$$(\mathsf{A}_{ij} - \mathsf{B}_{ij})(U_j - U_i) \quad \longrightarrow \quad (KU)_j$$

Galerkin matrix assembly

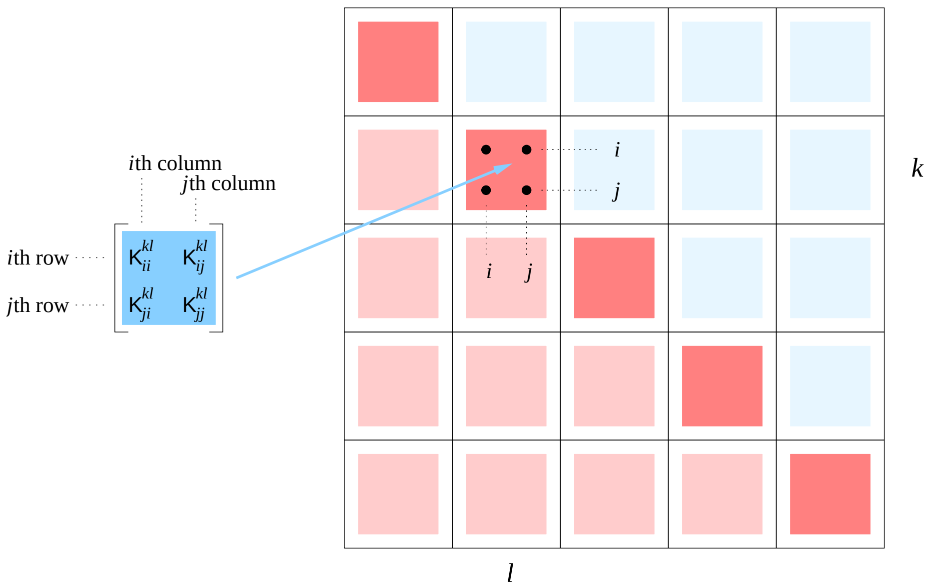
Edge contribution to the operator K

$$\mathsf{K}_{ii} = -\mathsf{A}_{ij} - \mathsf{B}_{ij} \qquad \mathsf{K}_{ij} = \mathsf{A}_{ij} + \mathsf{B}_{ij}$$
$$\mathsf{K}_{ji} = -\mathsf{A}_{ij} + \mathsf{B}_{ij} \qquad \mathsf{K}_{jj} = \mathsf{A}_{ij} - \mathsf{B}_{ij}$$

Edge contribution to the operator L

$$\mathsf{L}_{ii} = -\mathsf{A}_{ij} - \mathsf{D}_{ij} \qquad \mathsf{L}_{ij} = \mathsf{A}_{ij} + \mathsf{D}_{ij}$$
$$\mathsf{L}_{ji} = -\mathsf{A}_{ij} + \mathsf{D}_{ij} \qquad \mathsf{L}_{jj} = \mathsf{A}_{ij} - \mathsf{D}_{ij}$$

Structure of the global matrix



Raw antidiffusive flux for the edge $\vec{ij}$

$$\mathsf{F}_{ij} = -\left(\mathsf{M}_{ij}\frac{\mathrm{d}}{\mathrm{d}t} + \mathsf{D}_{ij} - \mathsf{B}_{ij}\right)(U_j - U_i), \quad \mathsf{F}_{ji} = -\mathsf{F}_{ij} \qquad \text{(semi-discrete)}$$

where $\mathsf{M}_{ij} = m_{ij}I$ is a block of M_C and D_{ij} is the tensorial artificial diffusion.

Remark: Depending on the solution strategy, only ‘a few’ blocks of the global matrix need to be assembled and stored.

Artificial viscosity

Generalization of the LED-principle for systems of hyperbolic conservation laws

Off-diagonal blocks of the global matrix should be positive definite

Characteristic decomposition $\boxed{\mathsf{A}_{ij} = R_{ij}\Lambda_{ij}R_{ij}^{-1}}$ $\qquad |\mathbf{a}_{ij}| = \sqrt{\mathbf{a}_{ij}\cdot\mathbf{a}_{ij}}$

where $\Lambda_{ij} = |\mathbf{a}_{ij}|\mathrm{diag}\,\{\lambda_1,\ldots,\lambda_5\}$ and R_{ij} is the matrix of right eigenvectors.

Eigenvalues of the cumulative Roe matrix A_{ij}

$$\lambda_1 = \hat{v}_{ij} - \hat{c}_{ij}, \quad \lambda_2 = \lambda_3 = \lambda_4 = \hat{v}_{ij}, \quad \lambda_5 = \hat{v}_{ij} + \hat{c}_{ij}$$

Characteristic velocities $\quad \hat{v}_{ij} = \dfrac{\mathbf{a}_{ij}\cdot\hat{\mathbf{v}}_{ij}}{|\mathbf{a}_{ij}|}, \qquad \hat{c}_{ij} = \sqrt{(\gamma-1)\left(\hat{H}_{ij} - \dfrac{|\hat{\mathbf{v}}_{ij}|^2}{2}\right)}$

System upwinding (expensive)

$$\mathsf{D}_{ij} = |\mathsf{A}_{ij}| = R_{ij}|\Lambda_{ij}|R_{ij}^{-1}$$

Generalization of Roe's approximate Riemann solver *(1981)*

Scalar dissipation (efficient)

$$\mathsf{D}_{ij} = d_{ij}I \quad \text{where} \quad d_{ij} = |\mathbf{a}_{ij}|\max_i |\lambda_i|$$

optimal for FCT since excessive artificial diffusion is removed by the flux limiter

Iterative defect correction

Successive approximation $A(U^{(m)})U^{(m+1)} = B^{(m+1)}$ where

$$B^{(m+1)} = B^{n} \quad + F$$

or

$$B^{(m+1)} = B^{(m)} + \Delta F, \qquad B^{(0)} = B^{n}$$

How to solve this system?

$$\begin{bmatrix} A_{11}^{(m)} & A_{12}^{(m)} & A_{13}^{(m)} & A_{14}^{(m)} & A_{15}^{(m)} \\ A_{21}^{(m)} & A_{22}^{(m)} & A_{23}^{(m)} & A_{24}^{(m)} & A_{25}^{(m)} \\ A_{31}^{(m)} & A_{32}^{(m)} & A_{33}^{(m)} & A_{34}^{(m)} & A_{35}^{(m)} \\ A_{41}^{(m)} & A_{42}^{(m)} & A_{43}^{(m)} & A_{44}^{(m)} & A_{45}^{(m)} \\ A_{51}^{(m)} & A_{52}^{(m)} & A_{53}^{(m)} & A_{54}^{(m)} & A_{55}^{(m)} \end{bmatrix} \begin{bmatrix} u_1^{(m)} \\ u_2^{(m)} \\ u_3^{(m)} \\ u_4^{(m)} \\ u_5^{(m)} \end{bmatrix} = \begin{bmatrix} b_1^{(m+1)} \\ b_2^{(m+1)} \\ b_3^{(m+1)} \\ b_4^{(m+1)} \\ b_5^{(m+1)} \end{bmatrix}$$

(Preconditioned) defect correction

$$U^{(m+1)} = U^{(m)} + [A^{(m)}]^{-1} R^{(m)}$$

$$R^{(m)} = B^{(m+1)} - A(U^{(m)})U^{(m)}$$

Practical implementation

$$A^{(m)} \Delta U^{(m)} = R^{(m)}$$

$$U^{(m+1)} = U^{(m)} + \Delta U^{(m)}, \quad U^{(0)} = U^{n}$$

Subdivide into supdiagonal, superdiagonal and maindiagonal matrix $A^{(m)} = W^{(m)} + J^{(m)} + E^{(m)}$

Decoupling of the Euler equations

Block-diagonal approximation

$$J_{kk}^{(m)} = M_L - \theta \Delta t L_{kk}^{(m)},\ \forall k, \qquad J_{kl}^{(m)} = 0,\ \forall l \neq k$$

$$\begin{bmatrix} \boxed{A_{11}^{(m)}} & 0 & 0 & 0 & 0 \\ 0 & \boxed{A_{22}^{(m)}} & 0 & 0 & 0 \\ 0 & 0 & \boxed{A_{33}^{(m)}} & 0 & 0 \\ 0 & 0 & 0 & \boxed{A_{44}^{(m)}} & 0 \\ 0 & 0 & 0 & 0 & \boxed{A_{55}^{(m)}} \end{bmatrix} \begin{bmatrix} \Delta u_1^{(m)} \\ \Delta u_2^{(m)} \\ \Delta u_3^{(m)} \\ \Delta u_4^{(m)} \\ \Delta u_5^{(m)} \end{bmatrix} = \begin{bmatrix} r_1^{(m)} \\ r_2^{(m)} \\ r_3^{(m)} \\ r_4^{(m)} \\ r_5^{(m)} \end{bmatrix}$$

Sequence of scalar subproblems

$$J_{kk}^{(m)} \Delta u_k^{(m)} = r_k^{(m)}, \qquad k = 1, \ldots, 5$$

$$u_k^{(m+1)} = u_k^{(m)} + \Delta u_k^{(m)}, \quad u_k^{(0)} = u_k^n$$

- $\oplus$ only 5 blocks need to be assembled and stored
- $\oplus$ equations can be solved separately or in parallel (BiCGSTAB, geometric multi-grid)
- $\ominus$ poor / no convergence for large time steps

Gekoppelter Löseransatz

Block-Gauss-Seidel *Vorkonditionierer für einen globalen BiCGSTAB-Löser*

$$\begin{aligned} W_{kl}^{(m)} &= M_L - \theta\Delta t L_{kl}^{(m)}, \quad \forall l < k, \qquad W_{kl}^{(m)} = 0, \quad \forall l \geq k, \\ E_{kl}^{(m)} &= M_L - \theta\Delta t L_{kl}^{(m)}, \quad \forall l > k, \qquad E_{kl}^{(m)} = 0, \quad \forall l \leq k. \end{aligned}$$

Vorkonditionierter BiCGSTAB-Löser

$$J^{(m)}\Delta U^{(m+1)} = R^{(m)} - W^{(m)}\Delta U^{(m+1)} - E^{(m)}\Delta U^{(m)}$$

Jedes skalare Teilproblem wird wie zuvor (BiCGSTAB, Mehrgitter) gelöst.

- $\oplus$ Gutes Konvergenzverhalten auch für große Zeitschritte
- $\oplus$ Modularer Code lässt sich einfach anpassen
- $\ominus$ alle Blöcke der Matrix müssen gespeichert werden

Synchronisierung von Korrekturfaktoren $\alpha_{ij} = \mathcal{S}(\alpha_{ij}^1, \dots, \alpha_{ij}^5)$ Löhner (1987)

1. Wähle eine spezielle Indikatorvariable.
2. Verwende $\min\{\alpha_\rho, \alpha_E\}$ (für instationäre Strömungen geeignet).
3. Verwende $\min\{\alpha_\rho, \alpha_p\}$ (für stationäre Strömungen geeignet).

Linearization of nonlinear source/sink terms

Nonlinear system for an implicit FEM-FCT algorithm with nonlinear source/sink terms

$$M_L \frac{u^{n+1} - u^n}{\Delta t} = \theta L(u^{n+1})u^{n+1} + (1-\theta)L(u^n)u^n + f^*(u^{n+1}, u^n) + s(u)$$

Negative-slope linearization of source terms Patankar (1980)

$$s(u) = s_C + s_V u, \qquad \text{where} \quad s_C \geq 0 \quad \text{and} \quad s_V \leq 0$$

Let u^* be the best approximation to u then take $s_C = s^+$ and $s_V = -s^-/u^*$

in the useful identity $s(u) = s^+ - s^- = s^+ + \left(\frac{-s^-}{u}\right) u$

Discretization of u in time $u = \theta u^{n+1} + (1-\theta)u^n$

Set $u = u^{n+1}$ and define $S^+ = \text{diag}\{s^+/u^n\}, \quad S^- = \text{diag}\{s^-/u^*\}$

Nonlinear system for an implicit FEM-FCT algorithm with linearized source/sink terms

$$M_L \frac{u^{n+1} - u^n}{\Delta t} = [\theta L(u^{n+1}) + S^-]u^{n+1} + [(1-\theta)L(u^n) + S^+]u^n + f^*(u^{n+1}, u^n)$$

Implementation of characteristic boundary conditions

Physical vs. numerical boundary conditions

	subsonic inflow			supersonic inflow			subsonic outflow			supersonic outflow		
	1D	2D	3D	1D	2D	3D	1D	2D	3D	1D	2D	3D
N_p	2	3	4	3	4	5	1	1	1	0	0	0
N_n	1	1	1	0	0	0	2	3	4	3	4	5

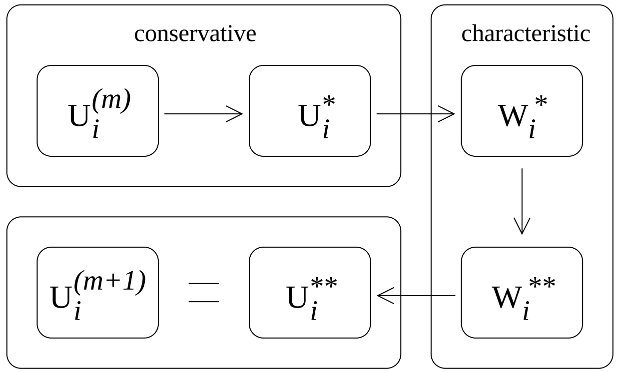
Algebraic manipulations for $\mathbf{x}_i \in \Gamma$

1. Prediction of $U_i = [u_{1,i}, \dots, u_{5,i}]^T$

$$a_{ij}^{kk} := 0 \qquad \boxed{u_{k,i}^* = u_{k,i}^{(m)} + r_{k,i}^{(m)} / a_{ii}^{kk}} \qquad r_{k,i}^{(m)} := 0$$

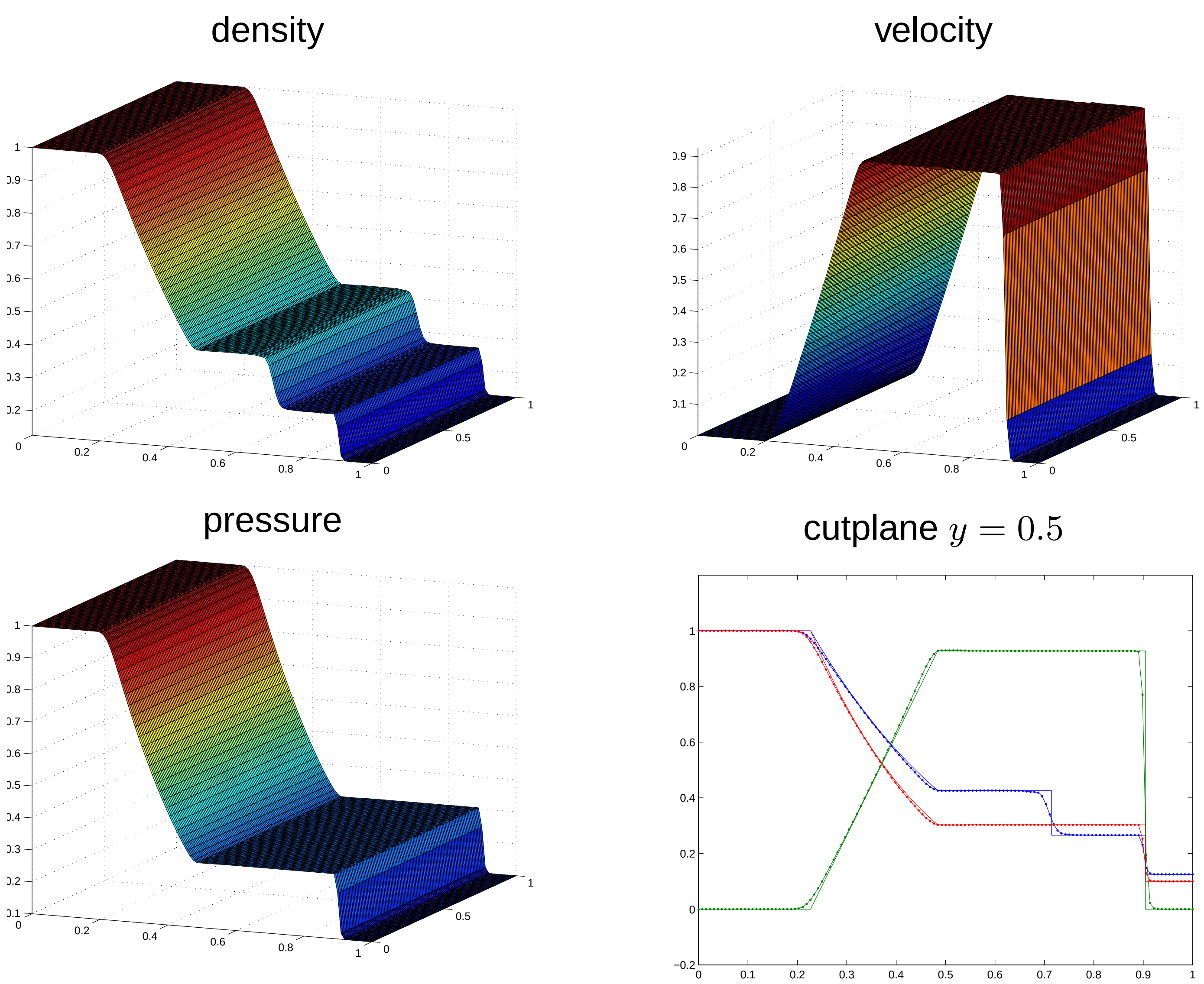
2. Correction of $W_i = [w_{1,i}, \dots, w_{5,i}]^T$

Variable transformations



- transform U_i^* into W_i and apply PBC for the incoming Riemann invariants
- convert the resulting vector W_i^* back to the conservative variables $U_i^{(m+1)}$

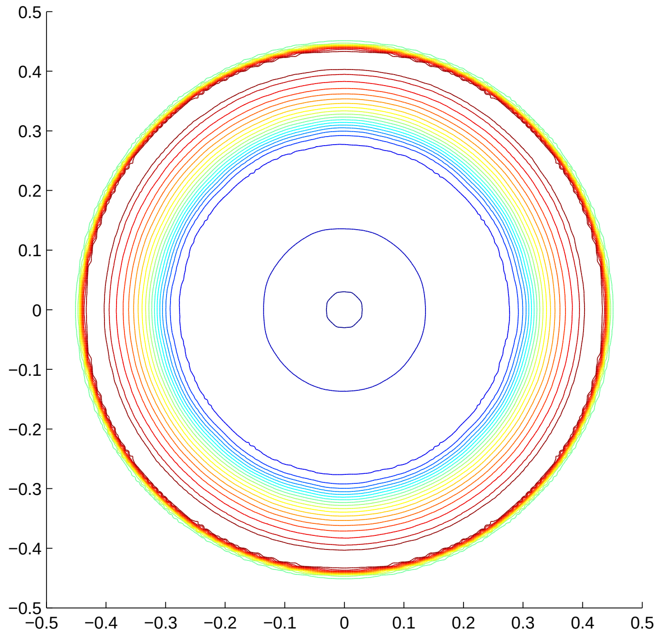
Shock tube problem



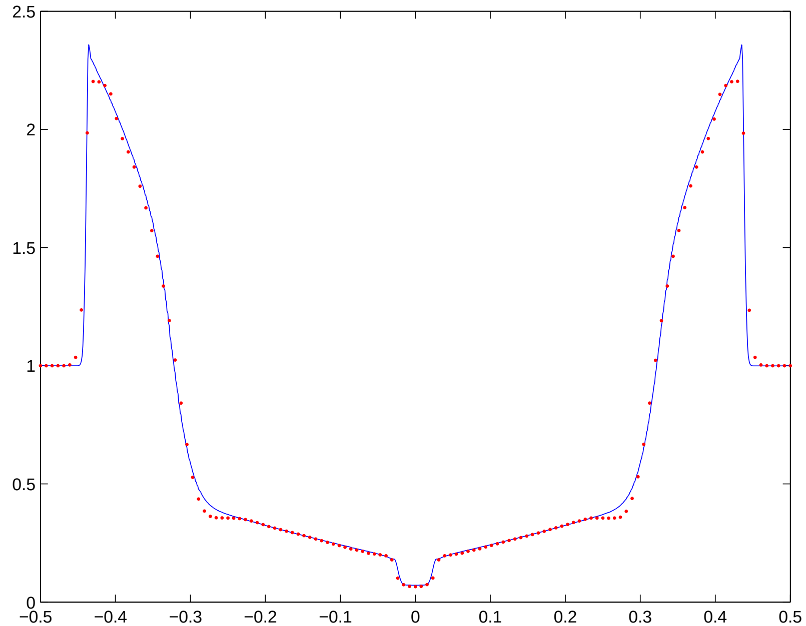
Crank-Nicolson time-stepping, $\Delta t = 10^{-3}$, 128×128 Q_1 elements

Radially symmetric Riemann problem

density

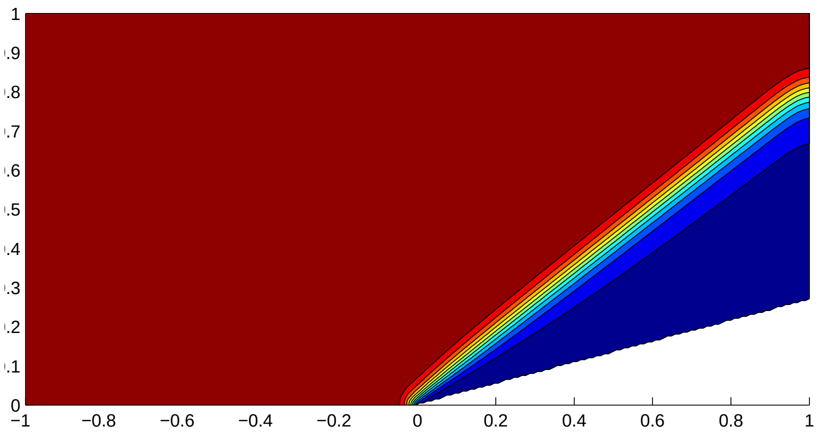


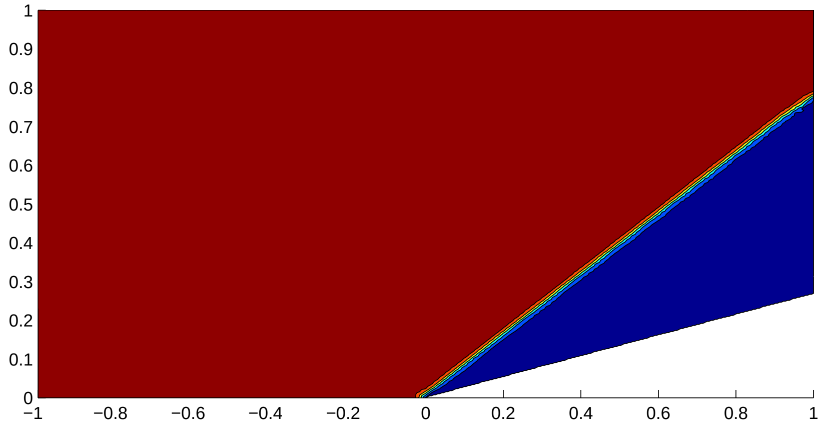
cutplane $y = 0$



Crank-Nicolson time-stepping, $\Delta t = 10^{-3}, \; 128 \times 128 \; Q_1$ elements

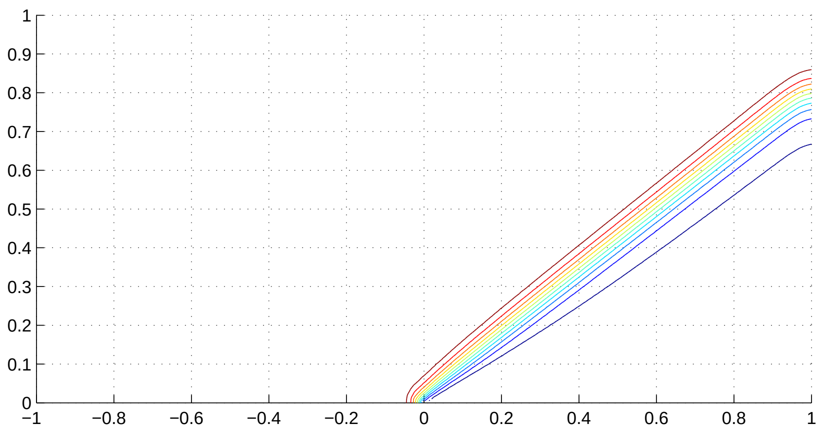
Compression corner $M = 2.5,\ \theta = 15^\circ$

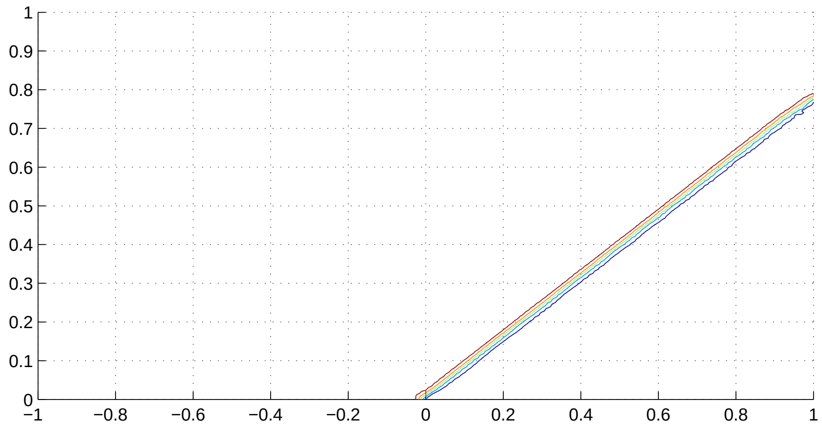




low-order method

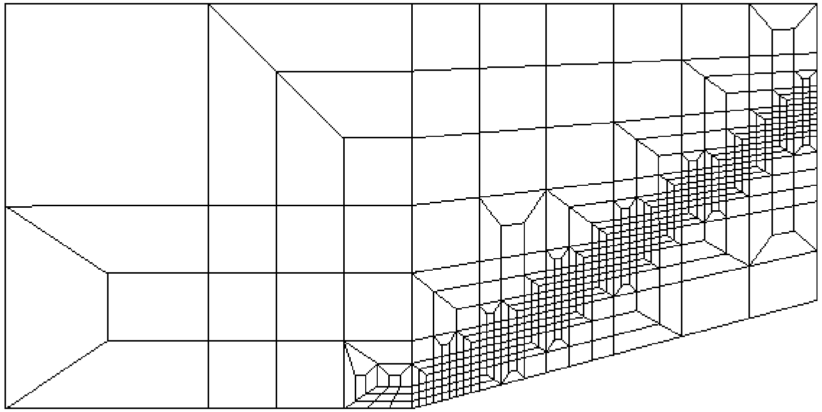
iterative FEM-FCT



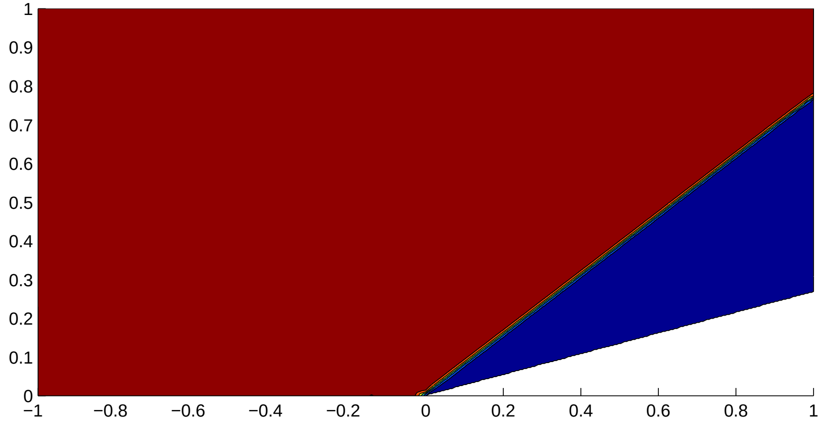


Backward Euler time-stepping, $\Delta t = 10^{-2},\ 128 \times 128\ Q_1$ elements

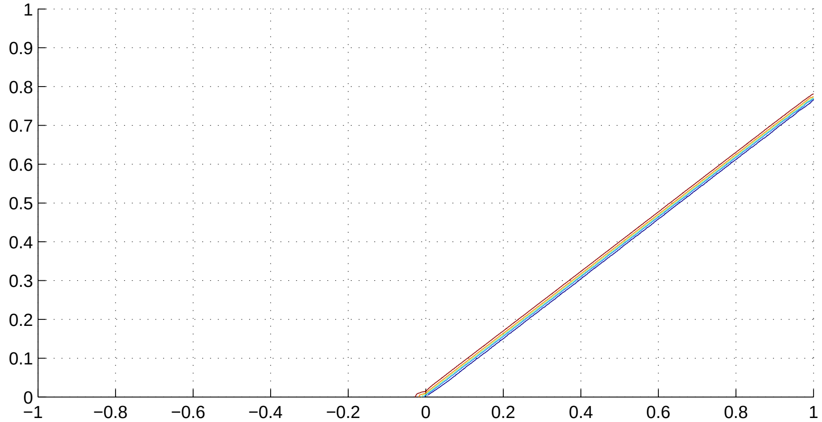
Compression corner $M = 2.5,\ \theta = 15^\circ$



coarse grid with local refinement

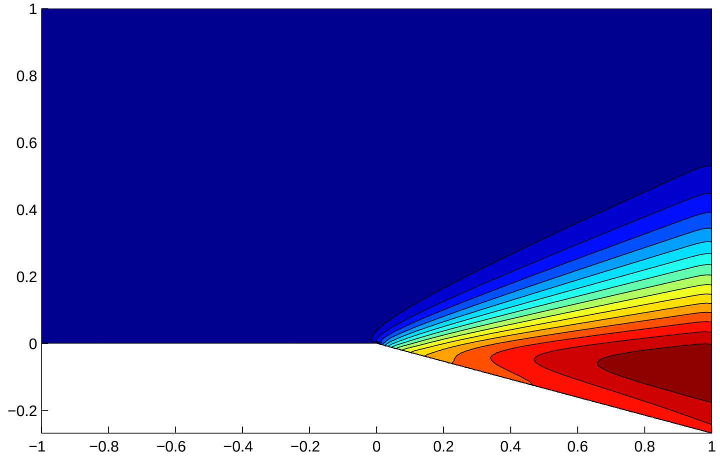


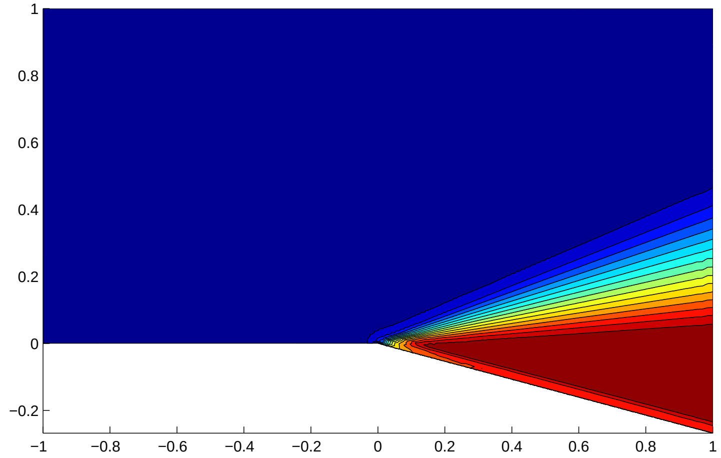
iterative FEM-FCT



Backward Euler time-stepping, $\Delta t = 10^{-2},\ 128 \times 128\ Q_1$ elements

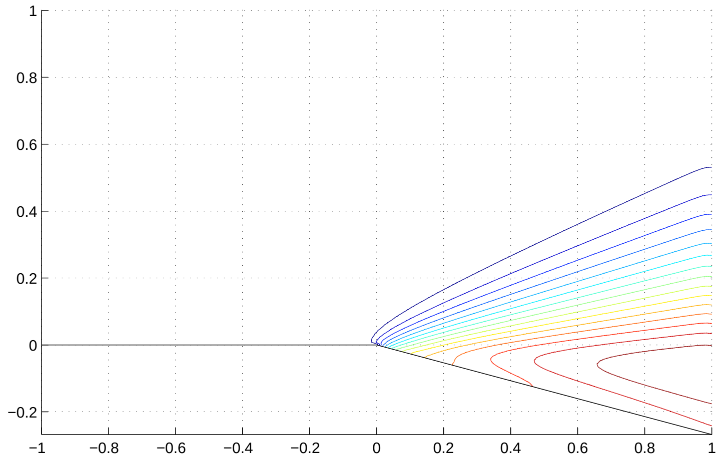
Prandtl-Meyer expansion $M = 2.5,\ \theta = 15^\circ$

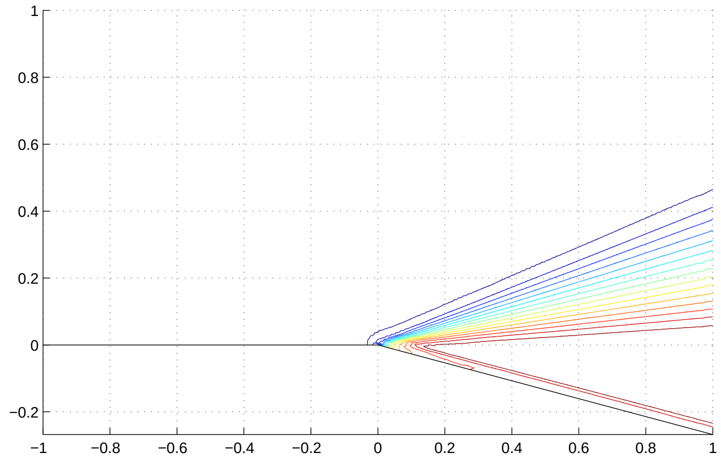




low-order method

iterative FEM-FCT





Backward Euler time-stepping, $\Delta t = 10^{-2},\ 128 \times 128\ Q_1$ elements

Conclusions

- Implicit FEM-FCT schemes can be derived on the basis of rigorous positivity criteria
- Discrete upwinding is performed by adding artificial diffusion so as to eliminate negative off-diagonal entries of the high-order operator
- In the case of hyperbolic systems scalar artificial dissipation proportional to the spectral radius of the Roe matrix can be utilized
- Flux correction can be performed within a defect correction preconditioned by the low-order operator
- Iterative limiting strategy prevents the flux limiter from getting overly diffusive at large time steps