On the use of slope limiters for the design of recovery based error indicators

M. Möller and D. Kuzmin

matthias.moeller@math.uni-dortmund.de

Institute of Applied Mathematics (LS III)
University of Dortmund, Germany

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Overview

- Finite element discretization
- Edgewise definition of *a posteriori* error estimates
- Gradient reconstruction
  - Limited slope averaging
  - Classical gradient recovery techniques
    - variational projection schemes
    - discrete patch recovery
  - Edgewise slope limiting
- Numerical examples and conclusions

Finite element discretization

Partial differential equation

$$\mathcal{L}\mathrm{u} = \mathrm{f} \quad \text{in} \quad \Omega$$

Variational form

$$\int_\Omega w\,[\mathcal{L}\mathrm{u} - \mathrm{f}]\,\mathrm{d}\mathbf{x} = 0 \quad \forall w \in \mathcal{V}$$

Galerkin finite element discretization

$$\mathrm{u} \approx u_h = \sum_j u_j \varphi_j \qquad \text{where} \qquad \varphi_i \in \mathcal{V}_h \subset \mathcal{V}$$

Algebraic flux correction scheme, *Kuzmin et al.*

$$K^*(u_h)u_h = f_h \qquad \text{where} \qquad K^*(u_h) = K + D + F(u_h)$$

Finite element discretization

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Finite element error estimation

Let $\mathcal{T}_h$ be a regular triangulation of Ω_h then the space of piecewise linear finite elements is given by $\mathcal{V}_h = \{v \in C^0(\bar{\Omega}) : v|_K \in P_1(K)\ \forall K \in \mathcal{T}_h\}$

Exact gradient error

$$\mathbf{e} = \nabla \mathrm{u} - \nabla u_h$$

Consistent FE gradient

$$\nabla u_h = \sum_j u_j \nabla \varphi_j$$

A posteriori error estimation

Replace the exact gradient $\nabla \mathrm{u}$ by a smoothed gradient field $\hat{\nabla} u_h$

$$\mathbf{e} \approx \hat{\mathbf{e}} = \hat{\nabla} u_h - \nabla u_h$$

Edgewise error measures

L_2-norm of the gradient error

$$\|\hat{\mathbf{e}}\|_{L_2}^2 = \sum_{K\in\mathcal{T}_h} \|\hat{\mathbf{e}}\|_{L_2(K)}^2$$

Local L_2-norm for $K \in \mathcal{T}_h$

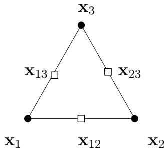
$$\|\hat{\mathbf{e}}\|_{L_2(K)} = \left[\int_K \hat{\mathbf{e}}^T\hat{\mathbf{e}}\,\mathrm{d}\mathbf{x}\right]^{1/2}$$

Consistent gradient $\nabla u_h = \left[\frac{\partial u_h}{\partial x_1}, \frac{\partial u_h}{\partial x_2}\right]$ is constant on each $K \in \mathcal{T}_h$

2^{nd}-order Newton-Cotes quadrature rule

$$\int_K \hat{\mathbf{e}}^T\hat{\mathbf{e}}\,\mathrm{d}\mathbf{x} = \frac{|K|}{3}\sum_{ij}\hat{\mathbf{e}}_{ij}^T\hat{\mathbf{e}}_{ij}$$

where $\hat{\mathbf{e}}_{ij} = \hat{\nabla} u_{ij} - \nabla u_{ij}$



$$\mathbf{x}_{ij} := \frac{\mathbf{x}_i + \mathbf{x}_j}{2}$$

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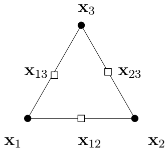


$$\mathbf{x}_{ij} := \frac{\mathbf{x}_i + \mathbf{x}_j}{2}$$

Useful ideas: *FVM/DGFEM*

Piecewise linear approximation

$$u_h(x) = \bar{u}_j + u_j'(x - x_j)$$

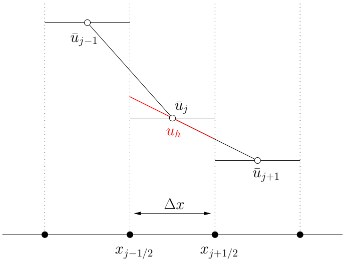
Limited average operators, *Jameson*

P1. $\mathcal{L}(a, b) = \mathcal{L}(b, a)$.

P2. $\mathcal{L}(ca, cb) = c\mathcal{L}(a, b)$.

P3. $\mathcal{L}(a, a) = a$.

P4. $\mathcal{L}(a, b) = 0$ if $ab \leq 0$.



$$u_j' := \mathcal{L}\left(\frac{\bar{u}_{j-1} - \bar{u}_j}{\Delta x}, \frac{\bar{u}_{j+1} - \bar{u}_j}{\Delta x}\right)$$

Apply **limited averaging** to the constant gradient values ∇u_h

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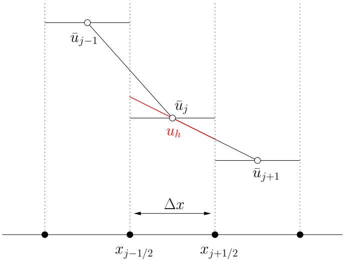

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Apply **limited averaging** to the constant gradient values ∇u_h

Limited gradient averaging

Limited slope average

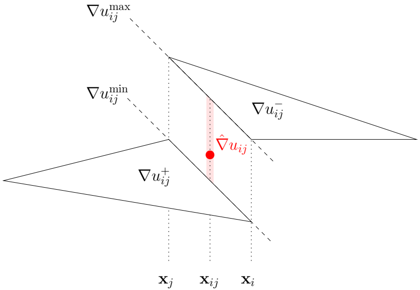
$$\hat{\nabla} u_{ij} := \mathcal{L}(\nabla u_{ij}^{+}, \nabla u_{ij}^{-})$$

Upper/lower bounds

$$\nabla u_{ij}^{\substack{\max \\ \min}} = \substack{\max \\ \min} \{\nabla u_{ij}^{+}, \nabla u_{ij}^{-}\}$$

Boundedness of edge gradient

$$\nabla u_{ij}^{\min} \leq \hat{\nabla} u_{ij} \leq \nabla u_{ij}^{\max}$$



P4. Discrete condition for local exterma

$$\text{If} \quad \frac{\partial u_{ij}^{+}}{\partial x_d} \frac{\partial u_{ij}^{-}}{\partial x_d} < 0 \quad \text{then} \quad \frac{\partial \hat{u}_{ij}}{\partial x_d} := 0.$$

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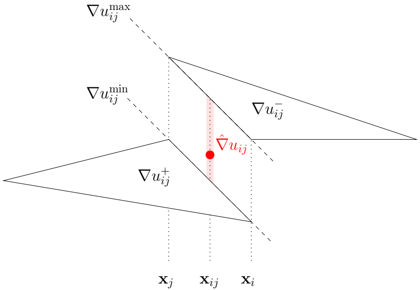

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TVD limiter functions

Standard slope limiters

1. minmod: $\mathcal{L}(a,b) = \mathcal{S}(a,b)\ \min\{|a|,|b|\}$
2. maxmod: $\mathcal{L}(a,b) = \mathcal{S}(a,b)\ \max\{|a|,|b|\}$
3. Van Leer: $\mathcal{L}(a,b) = \mathcal{S}(a,b)\ \frac{2|a||b|}{|a|+|b|}$
4. MC: $\mathcal{L}(a,b) = \mathcal{S}(a,b)\ \min\left\{\frac{|a+b|}{2}, 2|a|, 2|b|\right\}$
5. superbee: $\mathcal{L}(a,b) = \mathcal{S}(a,b)\ \max\{\min\{2|a|,|b|\}, \min\{|a|,2|b|\}\}$

where $\mathcal{S}(a,b) = \frac{\text{sign}(a)+\text{sign}(b)}{2} = \begin{cases} 1 & \text{if} \quad a > 0 \wedge b > 0 \\ -1 & \text{if} \quad a < 0 \wedge b < 0 \\ 0 & \text{otherwise} \end{cases}$

Roadmap: *Limited gradient reconstruction*

1. Compute the *low-order* gradient

$$\nabla u_h = \sum_j u_j \nabla \varphi_j$$

2. Compute the *high-order* gradient

- variational recovery $\hat{\nabla} u_h = \sum_j \hat{\nabla} u_j \phi_i, \quad \int_{\Omega_h} \phi_j (\hat{\nabla} u_h - \nabla u_h)\, \mathrm{d}\mathbf{x} = 0$
- discrete recovery $\hat{\nabla} u_h = p(\mathbf{x})\mathbf{a}, \quad \sum_j [p(\mathbf{x}_j)\mathbf{a} - \nabla u_h(\mathbf{x}_j)]^2 \to \min$

3. Predict the provisional edge gradient

$$\hat{\nabla} u_{ij} := \hat{\nabla} u_h(\mathbf{x}_{ij}), \quad \text{e.g.,} \quad \hat{\nabla} u_{ij} = \frac{1}{2}[\hat{\nabla} u_i + \hat{\nabla} u_j]$$

4. Apply **slope limiting** to correct edge gradient

$$\nabla u_{ij}^{\min} \leq \hat{\nabla} u_{ij}^{*} \leq \nabla u_{ij}^{\max}$$

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Variational recovery, *Zienkiewicz-Zhu (1987)*

Averaging projection

$$\hat{\nabla} u_h = \sum_j \hat{\nabla} u_j \phi_j \qquad \text{whereby} \qquad \int_{\Omega_h} \phi_i (\hat{\nabla} u_h - \nabla u_h)\, \mathrm{d}\mathbf{x} = 0 \quad \forall \phi_i \in \mathcal{W}_h$$

Linear algebraic system

$$M \hat{\nabla} u_h = \mathbf{C} u$$

Successive approximations

$$\hat{\nabla} u_h^{(m+1)} = \hat{\nabla} u_h^{(m)} + M_L^{-1} [\mathbf{C} u - M \hat{\nabla} u_h^{(m)}]$$

Edge-by-edge assembly

$$(\mathbf{C} u)_i = \sum_{j \neq i} \mathbf{c}_{ij} (u_j - u_i)$$

$$\mathbf{c}_{ij} = \int_{\Omega_h} \phi_i \nabla \varphi_j \, \mathrm{d}\mathbf{x} \qquad m_{ij} = \int_{\Omega_h} \phi_i \phi_j \, \mathrm{d}\mathbf{x}$$

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Lumped L_2-projection

$$\hat{\nabla} u_i = \frac{1}{m_i} \sum_{j \neq i} \mathbf{c}_{ij} (u_j - u_i)$$

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Discrete patch recovery, *Z-Z (1992)*

Least squares fit

$$\hat{\nabla} u_h = p(\mathbf{x})\mathbf{a} \quad \textit{whereby} \quad M\mathbf{a} = \mathbf{f}$$

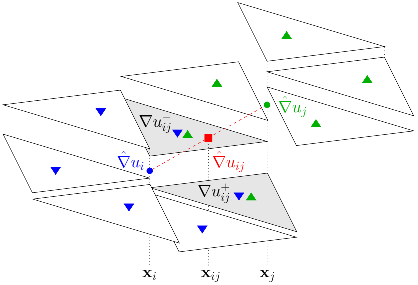
Vectors of monomials/coefficients

$$p(\mathbf{x}) = [1, x_1, x_2], \quad \mathbf{a} = [a_1, a_2, a_3]^T$$

Local matrix/vector

$$M = \sum_j p^T(\mathbf{x}_j)p(\mathbf{x}_j)$$

$$\mathbf{f} = \sum_j p^T(\mathbf{x}_j)\nabla u_h(\mathbf{x}_j)$$



Violates upper/lower bounds!

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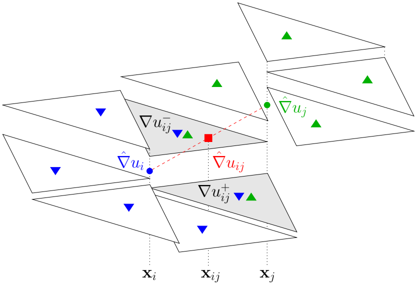

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Edgewise slope limited recovery

Predicted edge gradient

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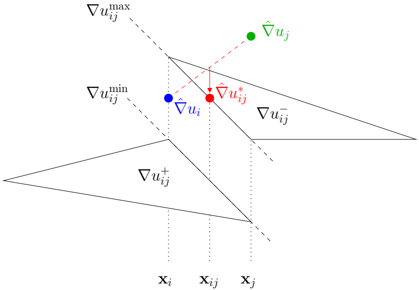
Upper/lower bounds

$$\nabla u_{ij}^{\max/\min} = \max/\min \{\nabla u_{ij}^{+}, \nabla u_{ij}^{-}\}$$



Corrected edge gradient

$$\hat{\nabla} u_{ij}^{*} := s_{ij} \left| \max\{\nabla u_{ij}^{\min}, \min\{\hat{\nabla} u_{ij}, \nabla u_{ij}^{\max}\}\} \right|, \qquad s_{ij} = \mathcal{S}(\nabla u_{ij}^{\min}, \nabla u_{ij}^{\max})$$

Edgewise slope limited recovery

Predicted edge gradient

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Upper/lower bounds

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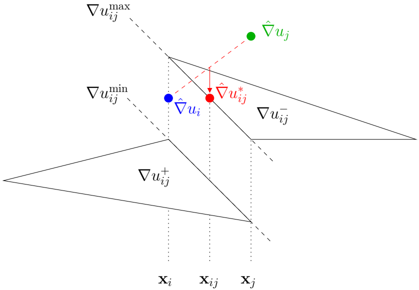

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Grid adaptivity

Relative percentage error

$$\frac{\|\nabla \mathrm{u} - \nabla u_h\|_{L_2}}{\|\nabla \mathrm{u}\|_{L_2}} \leq \eta_{\mathrm{tol}}$$

Approximate gradient error

$$\frac{\|\hat{\mathbf{e}}\|_{L_2}}{[\,\|\nabla u_h\|^2_{L_2} + \|\hat{\mathbf{e}}\|^2_{L_2}\,]^{1/2}} \leq \eta_{\mathrm{tol}}$$

Equal error distribution

$$\xi = \left[\frac{\|\nabla u_h\|^2_{L_2} + \|\hat{\mathbf{e}}\|^2_{L_2}}{|\mathcal{T}_h|}\right]^{1/2} \quad \Rightarrow \quad \frac{\|\hat{\mathbf{e}}\|_{L_2(K)}}{\xi} \begin{cases} > \eta_{\mathrm{ref}} & \text{refine} \quad K \\ < \eta_{\mathrm{crs}} & \text{coarsen} \quad K \end{cases}$$

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Approximate gradient error

$$\frac{\left[\sum_{K\in\mathcal{T}_h} \|\hat{\mathbf{e}}\|^2_{L_2(K)}\right]^{1/2}}{\left[\,\|\nabla u_h\|^2_{L_2} + \|\hat{\mathbf{e}}\|^2_{L_2}\,\right]^{1/2}} \leq \eta_{\mathrm{tol}}$$

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Approximate gradient error

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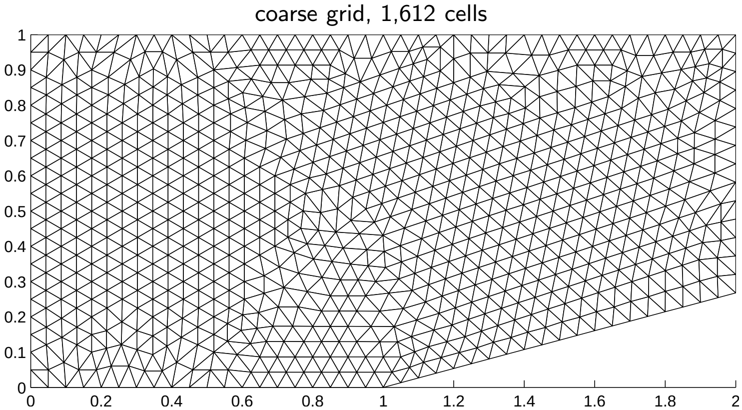
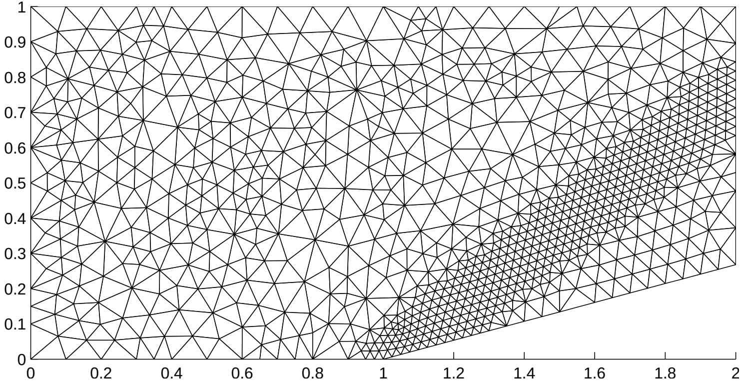
15° Compression corner at $M_\infty = 2.5$



$$EL - AP, \ \eta_{\rm ref} = 1\%, \quad \eta_{\rm crs} = 0.1\%$$

15° Compression corner at $M_\infty = 2.5$

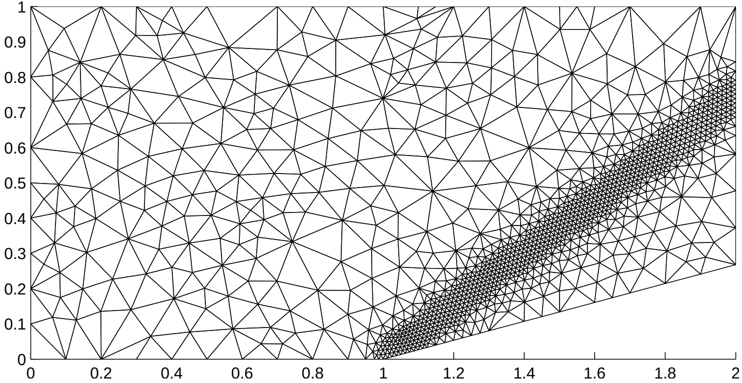
AMR(1) grid, 1,705 cells



$$EL - AP, \ \eta_{\mathrm{ref}} = 1\%, \quad \eta_{\mathrm{crs}} = 0.1\%$$

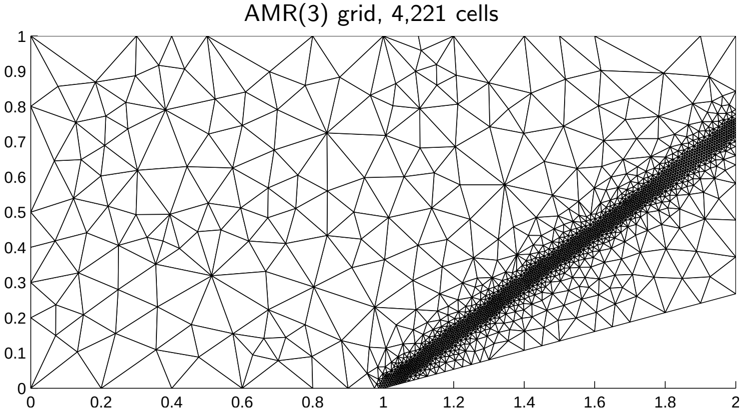
15° Compression corner at $M_\infty = 2.5$

AMR(2) grid, 2,351 cells



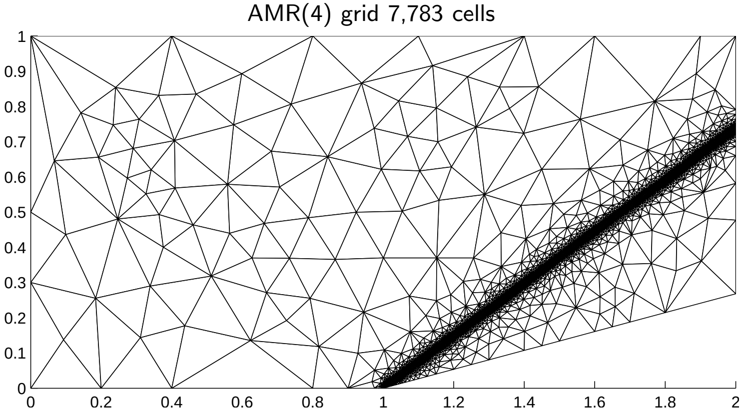
$$EL - AP, \ \eta_{\rm ref} = 1\%, \quad \eta_{\rm crs} = 0.1\%$$

15° Compression corner at $M_\infty = 2.5$



$$EL - AP, \ \eta_{\mathrm{ref}} = 1\%, \quad \eta_{\mathrm{crs}} = 0.1\%$$

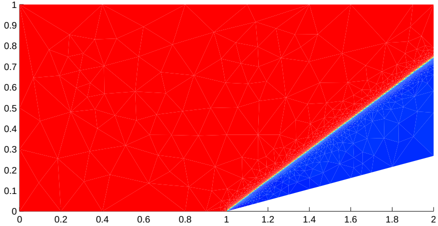
15° Compression corner at $M_\infty = 2.5$



$$EL - AP, \ \eta_{\text{ref}} = 1\%, \quad \eta_{\text{crs}} = 0.1\%$$

15° Compression corner at $M_\infty = 2.5$, contd.

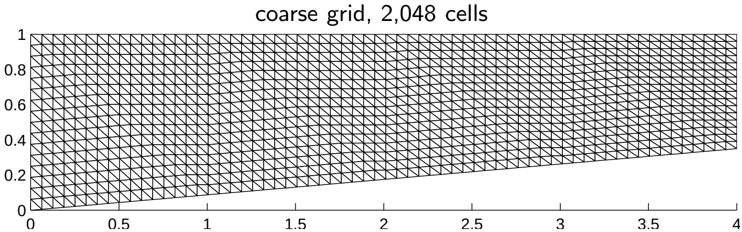
numerical solution



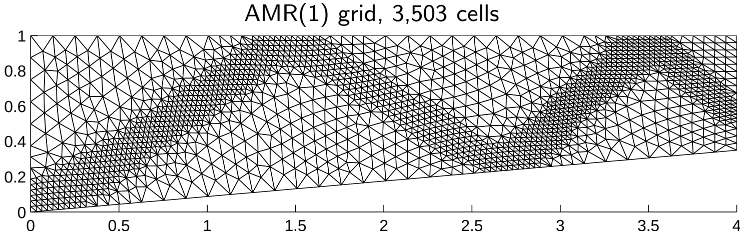
grid statistics

	grid 1	*grid 2*	*grid 3*	*grid 4*
SPR	1,877	2,869	5,000	9,329
EL-SPR	1,830	2,743	4,815	8,963
(lumped) AP	1,874	2,831	4,950	9,242
(lumped) EL-AP	1,827	2,738	4,781	8,888
AP	1,699	2,448	4,238	7,918
EL-AP	1,705	2,451	4,221	7,783

5° Converging channel at $M_\infty = 2$, *Shapiro*

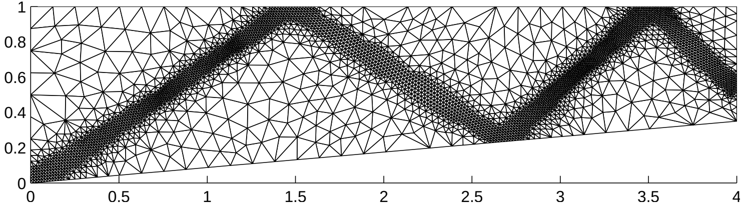


5° Converging channel at $M_\infty = 2$, *Shapiro*



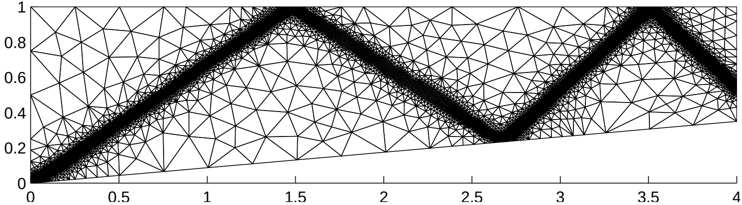
5° Converging channel at $M_\infty = 2$, *Shapiro*

AMR(2) grid, 7,194 cells

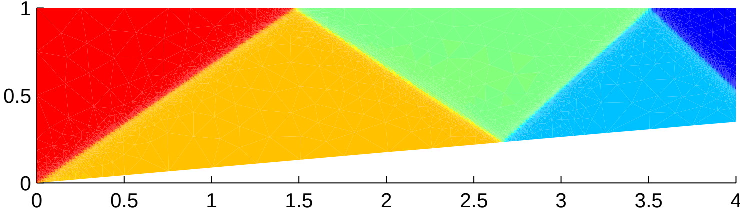


5° Converging channel at $M_\infty = 2$, *Shapiro*

AMR(3) grid, 15,664 cells

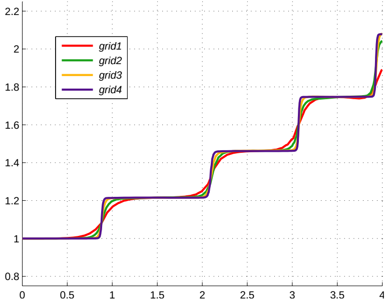


density distribution



5° Converging channel at $M_\infty = 2$, contd.

density distribution



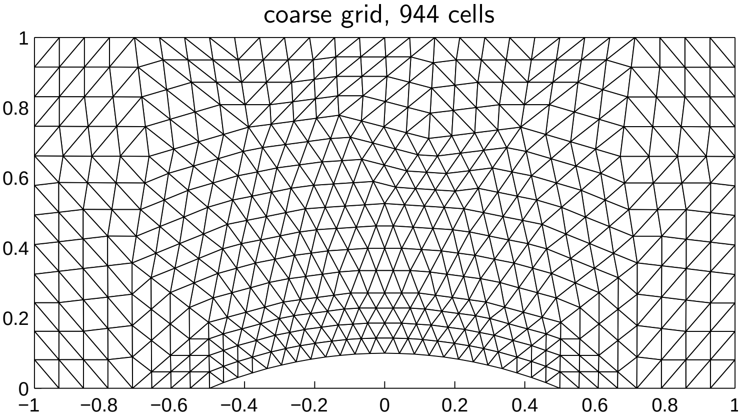
cutline at $y = 0.6$

exact *vs.* numerical solution

region	ρ_{exact}	ρ	Ma_{exact}	Ma
I	1.000	1.000	2.000	2.000
II	1.216	1.216	1.821	1.821
III	1.463	1.462	1.649	1.651
IV	1.747	1.747	1.478	1.479
V	2.081	2.079	1.302	1.304

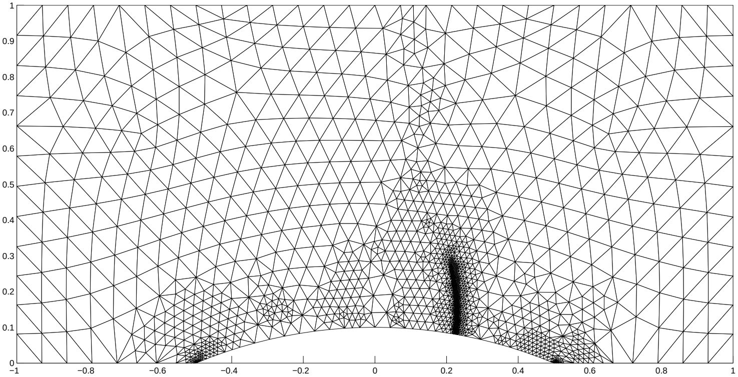
Reference: *R.A. Shapiro, 1991*

10% circular bump at $M_\infty = 0.67$

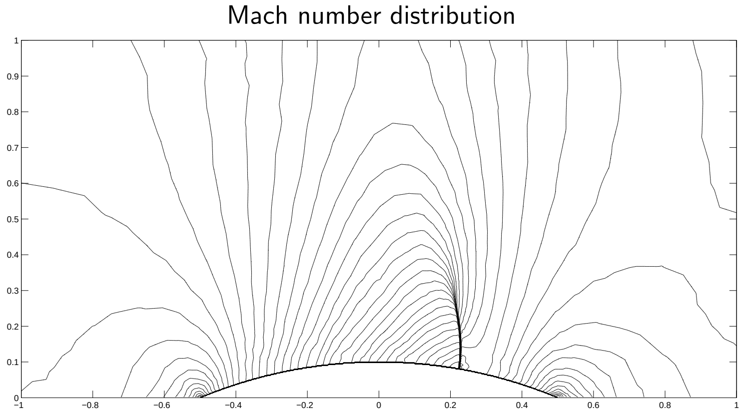


10% circular bump at $M_\infty = 0.67$

AMR(3) grid, 4260 cells

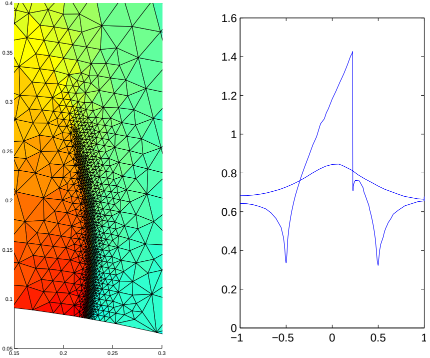


10% circular bump at $M_\infty = 0.67$



10% circular bump at $M_\infty = 0.67$

Close-up and Mach number at the wall



Conclusions

- *A posteriori* error measures for linear finite element approximations are expressed in terms of edge contributions
- Smoothed gradient values at the midpoints of edges are directly constructed as the limited average of the adjacent slopes
- Standard gradient recovery techniques – averaging projection, SPR, PPR – are used to predict high-order gradients
- Edgewise slope limiting corrects provisional edge gradients on the basis of natural bounds given by the constant slopes

Conclusions

Thank you for your attention.