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Implicit FEM-FCT algorithm for multidimensional conservation laws

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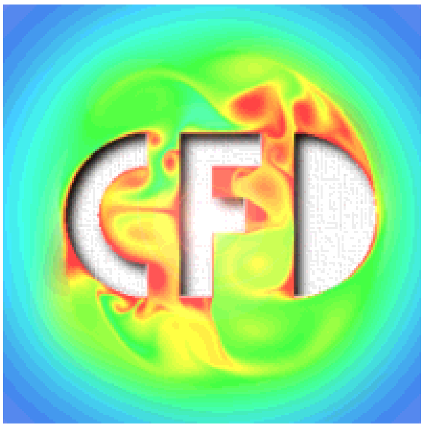


- Classical FCT
- Discrete upwinding
- Implicit FEM-FCT
- Unified limiting strategy
- Nonlinear hyperbolic systems

State of the art: scalar convection

Generic conservation law

$$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u - d\nabla u) = 0 \qquad \text{in } \Omega$$

Godunov Theorem (1959)

> *No linear discretization scheme of order higher than first is monotonicity preserving*

- High-order methods: oscillatory
- Low-order methods: overdiffusive
- High-resolution methods: **nonlinear**

Flux-Corrected Transport (FCT) algorithm

Boris and Book (1973), Zalesak (1979)

1. Compute a *transported and diffused* solution by a linear monotonicity-preserving scheme
2. Invoke *flux limiter* to determine the percentage of artificial diffusion which can be removed without generating oscillations
3. Add (limited) compensating *antidiffusion* to recover the high accuracy in smooth regions

Finite element FCT procedure

Löhner *et al.* (1987)

Given u^n at time t^n:

step 1: High-order finite element scheme

$$M_C u^H = M_C u^n + R$$

step 2: Low-order finite element scheme

$$M_L u^L = \left[(1-c_d)M_L + c_d M_C\right] u^n + R$$

step 3: Antidiffusive Element Contributions

$$F_e = M_L^{-1}\Big|_e \left[\hat{M}_L - \hat{M}_C\right]\left((c_d - 1)\hat{u}^n + \hat{u}^H\right)$$

step 4: Zalesak's limiter

$$F_e^* = \alpha_e F_e, \quad 0 \le \alpha_e \le 1$$

step 5: Element-by-element correction

$$u_i^{n+1} = u_i^L + \sum_e F_{e,i}^*$$

Remarks:

- CFL restriction due to explicit time stepping
- formulation in terms of AEC rather than fluxes
- arbitrarily chosen constant artificial diffusion

Finite element discretization

Generic conservation law

$$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u - d\nabla u) = 0 \qquad \text{in } \Omega$$

Galerkin discretization

$$\sum_j \left[\int_\Omega \varphi_i \varphi_j \, d\mathbf{x}\right] \frac{du_j}{dt} - \sum_j \left[\int_\Omega \nabla\varphi_i \cdot (\mathbf{v}_j \varphi_j - d\nabla\varphi_j) \, d\mathbf{x}\right] u_j$$

$$+ \sum_j \left[\int_{\Gamma_{\text{out}}} \varphi_i \varphi_j \mathbf{v}_j \cdot \mathbf{n} \, ds\right] u_j = \int_{\Gamma_{\text{in}}} \varphi_i g \, ds$$

Semi-discrete problem

$$M_C \frac{du}{dt} = (K - B)u + q$$

$$m_{ij} = \int_\Omega \varphi_i \varphi_j \, d\mathbf{x}, \qquad k_{ij} = \mathbf{v}_j \cdot \mathbf{c}_{ij} - d\, s_{ij}$$

$$\mathbf{c}_{ij} = \int_\Omega \nabla\varphi_i \varphi_j \, d\mathbf{x}, \qquad s_{ij} = \int_\Omega \nabla\varphi_i \nabla\varphi_j \, d\mathbf{x}$$

Column sum property

$$\sum_i k_{ij} = \mathbf{v}_j \cdot \sum_i \mathbf{c}_{ij} - d \sum_i s_{ij} = 0$$

Discrete positivity criteria

Semi-discrete problem

$$m_i \frac{du_i}{dt} = \sum_{j \neq i} k_{ij}(u_j - u_i) + r_i u_i + q_i$$

LED criterion: $k_{ij} \geq 0, \quad \forall j \neq i$

Jameson (1993)

$$u_k = \max_i u_i \quad \Rightarrow \quad u_j - u_i \leq 0 \quad \Rightarrow \quad \frac{du_k}{dt} \leq 0$$

Discrete diffusion operators

$$d_{ij} = d_{ji}, \quad \sum_j d_{ij} = \sum_i d_{ij} = 0$$

$$(Du)_i = \sum_j d_{ij} u_j = \sum_{j \neq i} d_{ij}(u_j - u_i) = \sum_{j \neq i} f_{ij}$$

$$f_{ij} = d_{ij}(u_j - u_i), \quad f_{ji} = -f_{ij}$$

Adaptive artificial diffusion $\quad L = K + D$

$$d_{ij} = d_{ji} = \max\{0, -k_{ij}, -k_{ji}\}, \quad d_{ii} = -\sum_{k \neq i} d_{ik}$$

LEMMA. A discrete scheme of the form

$$\mathcal{L}u^{n+1} = \mathcal{R}u^n, \quad u^n \geq 0$$

is positivity-preserving if $\mathcal{L}$ is an *M-matrix* and all entries of $\mathcal{R}$ are non-negative.

Flux-based FEM-FCT formulation

Galerkin θ-scheme

$$[M_L - \theta\Delta t\, L]\, u^H = [M_L + (1-\theta)\Delta t\, L]\, u^n + F(u^H, u^n)$$

Flux decomposition

$$\begin{aligned} F(u^H, u^n) &= -[(M_C - M_L) + \theta\Delta t\,(L-K)]\, u^H \\ &+ [(M_C - M_L) - (1-\theta)\Delta t\,(L-K)]\, u^n \\ F(u^H, u^n) &= \sum_{j\neq i} f_{ij}, \qquad F^*(u^H, u^n) = \sum_{j\neq i} \alpha_{ij} f_{ij} \end{aligned}$$

Antidiffusive fluxes

$$\begin{aligned} f_{ij} &= -(m_{ij} + \theta\Delta t d_{ij})\,(u_j^H - u_i^H) \\ &+ (m_{ij} - (1-\theta)\Delta t d_{ij})\,(u_j^n - u_i^n) \\ f_{ji} &= -f_{ij}, \qquad i \neq j \end{aligned}$$

Implicit FEM-FCT algorithm Kuzmin (2001)

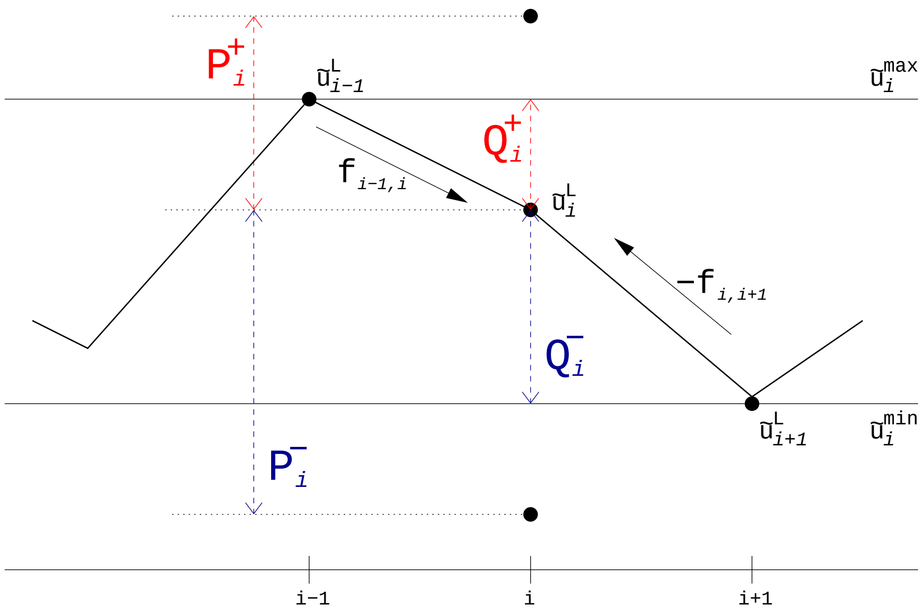
$$\boxed{[M_L - \theta\Delta t\, L] u^{n+1} = M_L \tilde{u} + F^*(u^H, u^n)}$$

Intermediate solution $\tilde{u} = u^L(t^{n+1-\theta})$

$$M_L \tilde{u} = [M_L + (1-\theta)\Delta t\, L] u^n$$

Zalesak's limiter

Prelimiting: $f_{ij} := 0, \quad \text{if} \quad f_{ij}(\tilde{u}_i - \tilde{u}_j) < 0$



Auxiliary quantities

$$P_i^{\pm} = \frac{1}{m_i} \sum_{j \neq i} \begin{matrix}\max \\ \min\end{matrix} \{0, f_{ij}\}, \qquad Q_i^{\pm} = \tilde{u}_i^{\substack{\max \\ \min}} - \tilde{u}_i$$

$$R_i^{\pm} = \begin{cases} \min\{1, Q_i^{\pm}/P_i^{\pm}\}, & \text{if } P^+ > 0 > P^- \\ 0, & \text{if } P_i^{\pm} = 0 \end{cases}$$

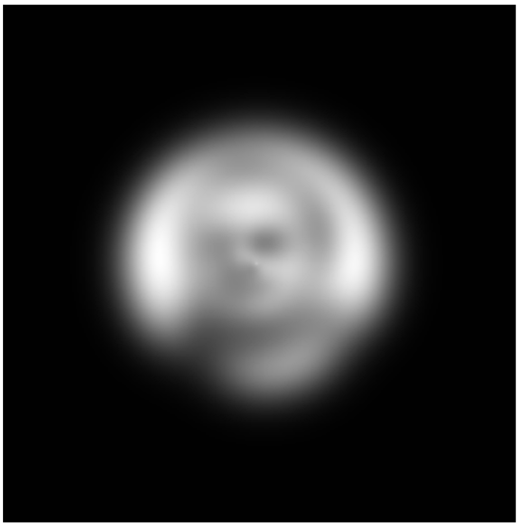
Correction factors

$$\alpha_{ij} = \begin{cases} \min\{R_i^+, R_j^-\}, & \text{if } f_{ij} \geq 0 \\ \min\{R_j^+, R_i^-\}, & \text{if } f_{ij} < 0 \end{cases}, \quad \alpha_{ji} = \alpha_{ij}$$

Rotation of a human face

exact solution

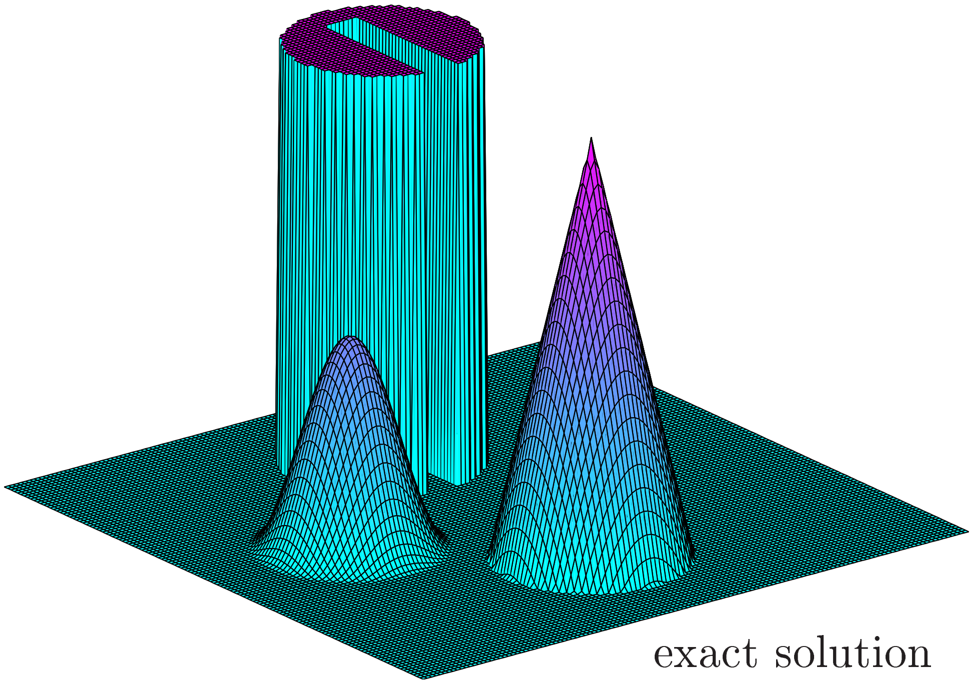
FEM-FCT



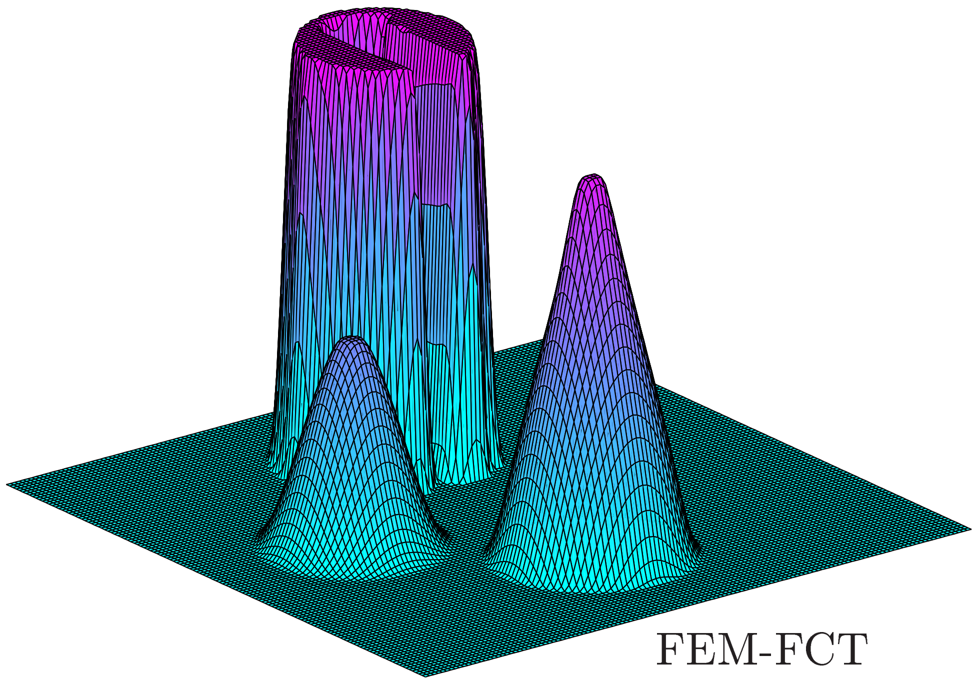
upwind method

Galerkin method

Solid body rotation



exact solution



FEM-FCT

Compressible Euler equations

Divergence form

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = 0$$

$$U = (\rho, \rho\mathbf{v}, E)^T, \quad \mathbf{F} = (F_1, F_2, F_3)^T$$

$$F_1 = \left| \begin{array}{c} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ \rho v_1 v_3 \\ v_1(E+p) \end{array} \right|, \quad F_2 = \left| \begin{array}{c} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ \rho v_2 v_3 \\ v_2(E+p) \end{array} \right|, \quad F_3 = \left| \begin{array}{c} \rho v_3 \\ \rho v_1 v_3 \\ \rho v_2 v_3 \\ \rho v_3^2 + p \\ v_3(E+p) \end{array} \right|$$

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho|\mathbf{v}|^2 \qquad \text{equation of state}$$

Quasi-linear formulation

$$\frac{\partial U}{\partial t} + \mathbf{A} \cdot \nabla U, \qquad \text{where} \quad \mathbf{A} = \frac{\partial \mathbf{F}}{\partial U}$$

Properties of the Jacobian

$$\mathbf{F} = \mathbf{A}U \qquad \text{homogeneous}$$

$$\mathbf{e} \cdot \mathbf{A} = R\Lambda R^{-1} \qquad \text{diagonalizable}$$

Mathematical challenges:

- hyperbolicity
- nonlinearity
- strong coupling

Euler equations: FEM discretization

Galerkin flux decomposition

$$m_i \frac{dU_i}{dt} = \sum_{j \neq i} \mathbf{k}_{ij} \cdot (\mathbf{F}_j - \mathbf{F}_i) = \sum_{j \neq i} (\mathrm{A}_{ij} + \mathrm{B}_{ij})(U_j - U_i)$$

$$m_j \frac{dU_j}{dt} = \sum_{j \neq i} \mathbf{k}_{ji} \cdot (\mathbf{F}_i - \mathbf{F}_j) = \sum_{j \neq i} (\mathrm{A}_{ij} - \mathrm{B}_{ij})(U_j - U_i)$$

Cumulative Jacobian matrices

$$\mathrm{A}_{ij} := \mathbf{a}_{ij} \cdot \hat{\mathbf{A}}_{ij}, \qquad \mathbf{a}_{ij} := \frac{\mathbf{k}_{ij} - \mathbf{k}_{ji}}{2}$$

$$\mathrm{B}_{ij} := \mathbf{b}_{ij} \cdot \hat{\mathbf{A}}_{ij}, \qquad \mathbf{b}_{ij} := \frac{\mathbf{k}_{ij} + \mathbf{k}_{ji}}{2}$$

Roe mean values

$$\hat{\mathbf{A}}_{ij} = \mathbf{A}(\hat{\rho}_{ij}, \hat{\mathbf{v}}_{ij}, \hat{h}_{ij}), \qquad \hat{\rho}_{ij} = \sqrt{\rho_i \rho_j}$$

$$\hat{\mathbf{v}}_{ij} = \frac{\sqrt{\rho_i}\mathbf{v}_i + \sqrt{\rho_j}\mathbf{v}_j}{\sqrt{\rho_i} + \sqrt{\rho_j}}, \qquad \hat{h}_{ij} = \frac{\sqrt{\rho_i}h_i + \sqrt{\rho_j}h_j}{\sqrt{\rho_i} + \sqrt{\rho_j}}$$

Discrete upwinding

$$\mathrm{K}_{ii} = -\mathrm{A}_{ij} - \mathrm{D}_{ij}, \quad \mathrm{K}_{ij} = \mathrm{A}_{ij} + \mathrm{D}_{ij}$$

$$\mathrm{K}_{ji} = -\mathrm{A}_{ij} + \mathrm{D}_{ij}, \quad \mathrm{K}_{jj} = \mathrm{A}_{ij} - \mathrm{D}_{ij}$$

Euler equations: artificial diffusion

Characteristic decomposition

$$\mathrm{A}_{ij} = R(\mathbf{a}_{ij})\Lambda(\mathbf{a}_{ij})R(\mathbf{a}_{ij})^{-1}$$

$$\Lambda(\mathbf{a}_{ij}) = |\mathbf{a}_{ij}|\mathrm{diag}\left\{\hat{v}_{ij} - \hat{c}_{ij}, \hat{v}_{ij}, \hat{v}_{ij}, \hat{v}_{ij}, \hat{v}_{ij} + \hat{c}_{ij}\right\}$$

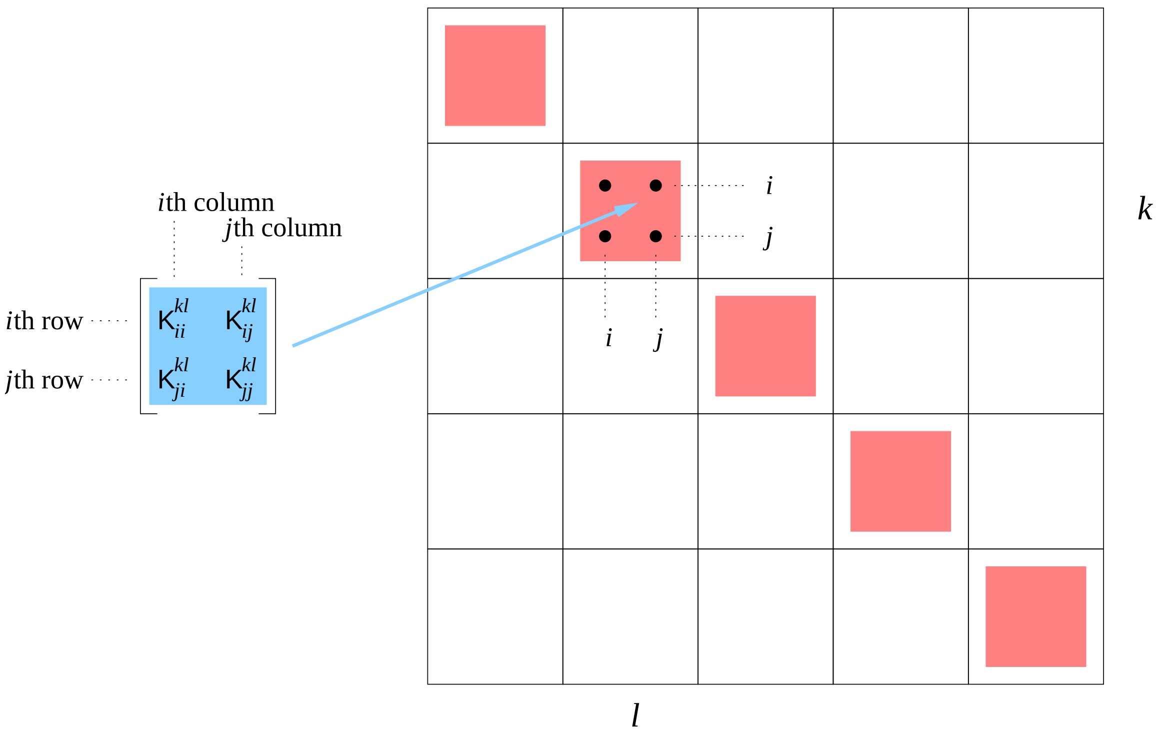
$$\hat{v}_{ij} = \frac{\mathbf{a}_{ij} \cdot \hat{\mathbf{v}}_{ij}}{|\mathbf{a}_{ij}|}, \qquad \hat{c}_{ij} = \sqrt{(\gamma - 1)(\hat{h}_{ij} - \frac{|\hat{\mathbf{v}}_{ij}|^2}{2})}$$

Approximate Riemann solver

$$\mathrm{D}_{ij} = |\mathrm{A}_{ij}|, \qquad |\mathrm{A}_{ij}| = R(\mathbf{a}_{ij})\,|\Lambda(\mathbf{a}_{ij})|\,R(\mathbf{a}_{ij})^{-1}$$

Scalar artificial dissipation

$$\mathrm{D}_{ij} = d_{ij} I, \qquad d_{ij} = |\lambda(\mathbf{a}_{ij})|$$



Euler equations: FEM-FCT algorithm

Fully discretized Euler equations $SU = G$

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \end{bmatrix}$$

Defect correction loop

$$\begin{aligned} U^{(m+1)} &= U^{(m)} + [C^{(m)}]^{-1}(G - S^{(m)}U^{(m)}) \\ C_k^{(m)} &= M_L - \theta\Delta t\, L_{kk}^{(m)}, \qquad k = 1, \dots, 5 \end{aligned}$$

Scalar subproblems

$$\begin{aligned} C_k^{(m)}\Delta u_k^{(m)} = M_L u_k^n &+ (1-\theta)\Delta t \sum\nolimits_l L_{kl}^n\, u_l^n \\ &- M_L u_k^{(m)} + \theta\Delta t \sum\nolimits_l L_{kl}^{(m)}\, u_l^{(m)} + F^*(u_k^{(m)}, u_k^n) \end{aligned}$$

$$u_k^{(m+1)} = u_k^{(m)} + \Delta u_k^{(m)}, \qquad u_k^{(0)} = u_k^n$$

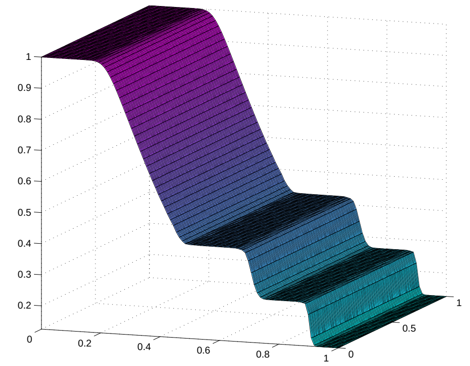
Raw antidiffusion

$$\begin{aligned} F(u_k^{(m)}, u_k^n) &= -[(M_C - M_L) + \theta\Delta t D_{kk}^{(m)}]u_k^{(m)} \\ &+ [(M_C - M_L) - (1-\theta)\Delta t D_{kk}^n]u_k^n \end{aligned}$$

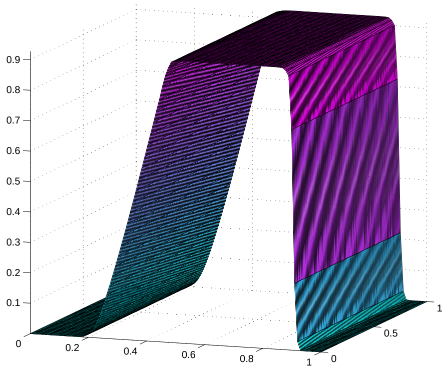
Synchronized limiter: $\alpha_{ij} = \mathcal{L}(\alpha_{ij}^1, \dots, \alpha_{ij}^z)$

Shock tube, compression corner $(M = 2.5)$

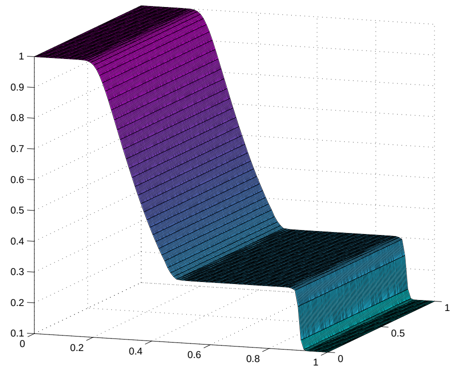
density



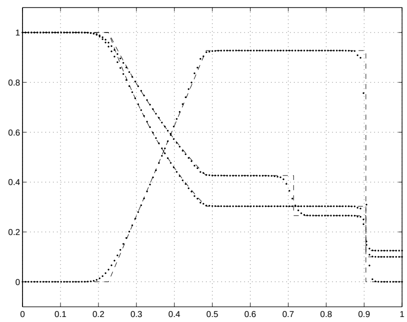
velocity



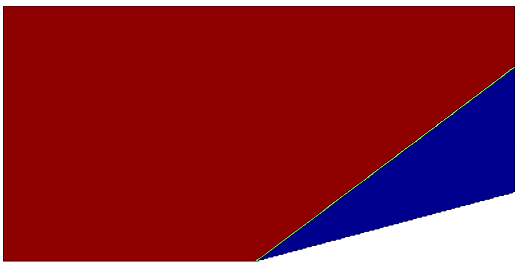
pressure

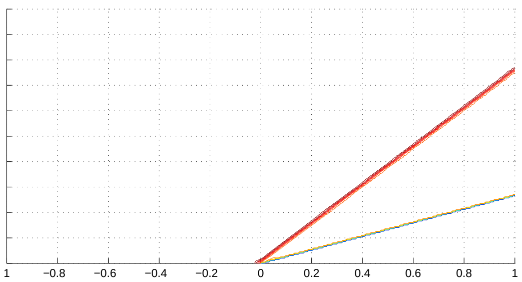


cutplane $y = 0.5$

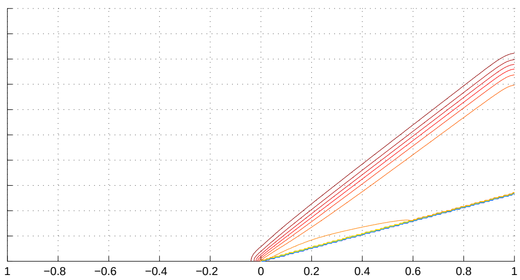


CN/FCT, $\Delta t = 10^{-3}$, $t = 0.231$.



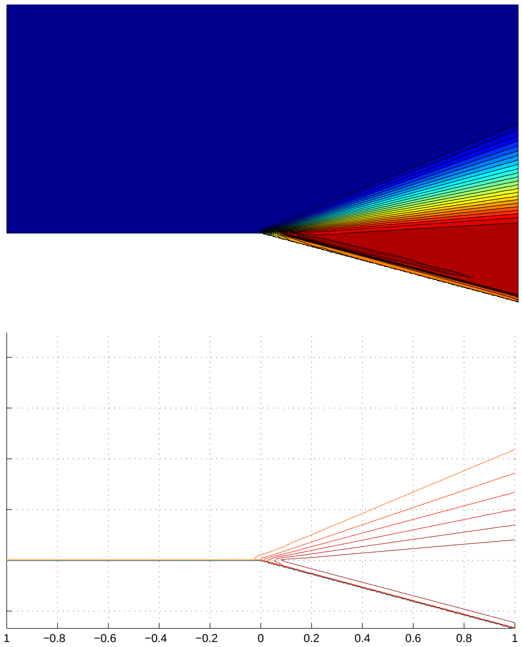


BE/FCT

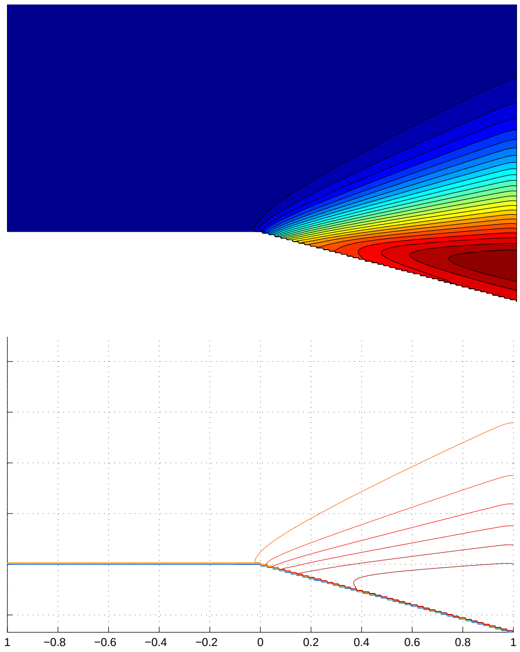


Low-order

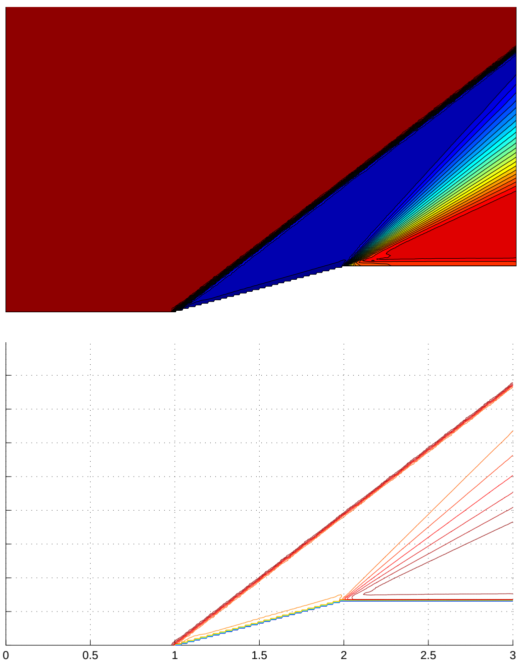
Compression / expansion ($M = 2.5$)



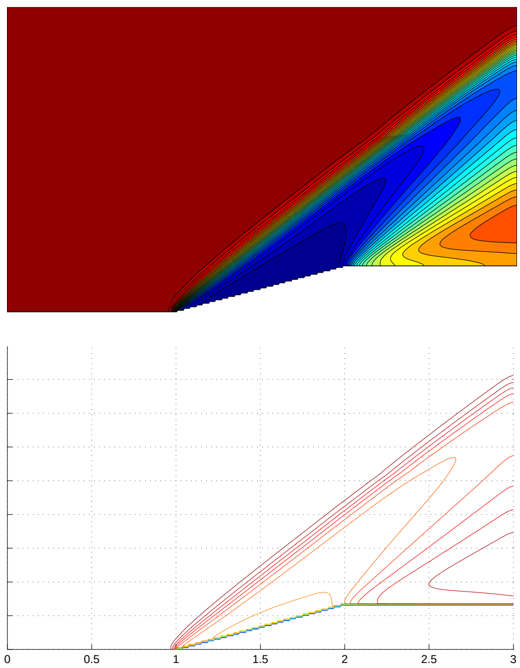
BE/FCT



Low-order



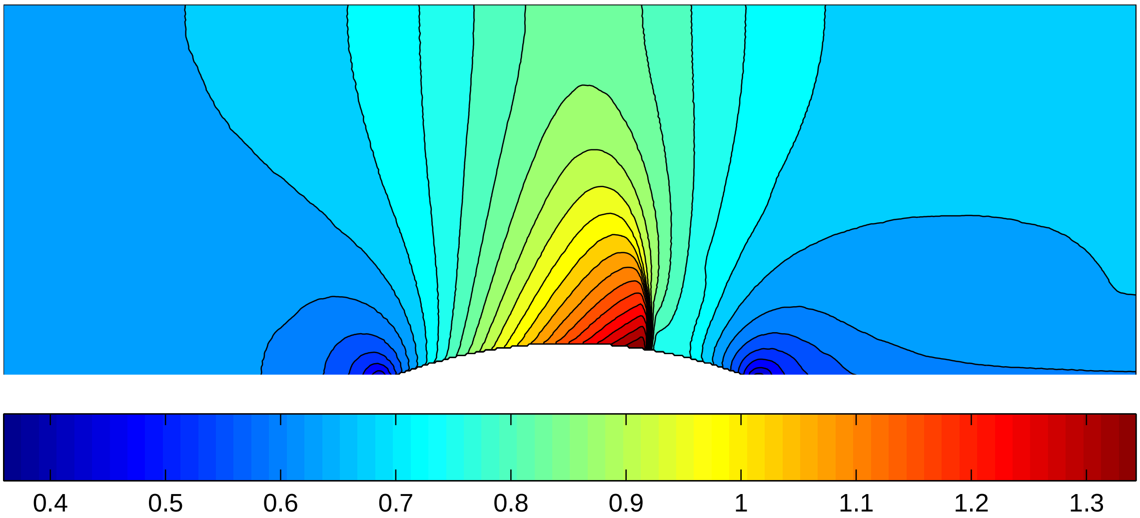
BE/FCT



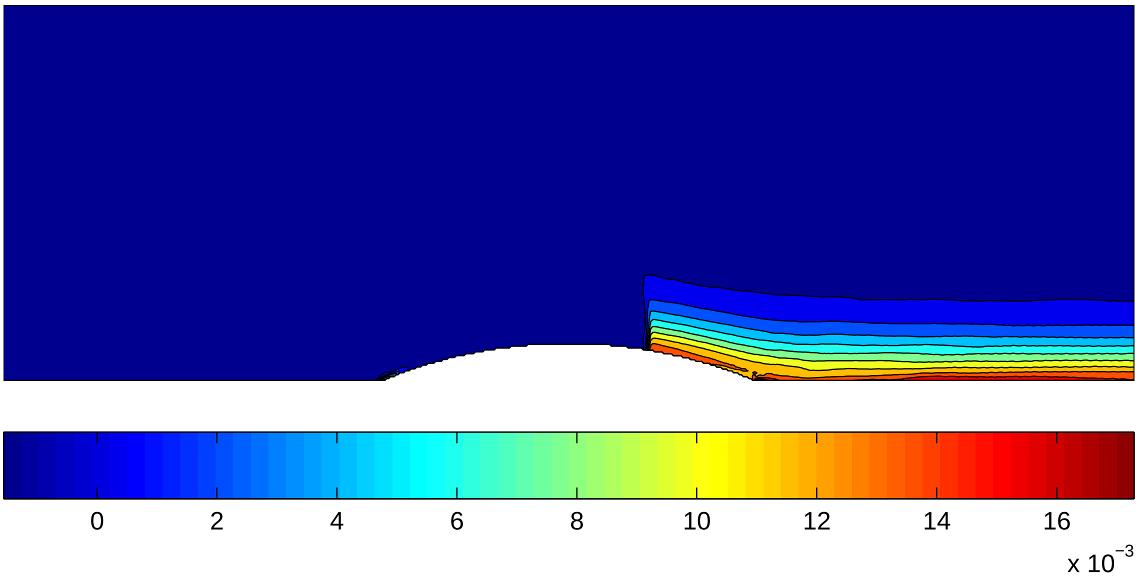
Low-order

Transonic flow over Ni's bump

geometry:	parallel channel with a 10.0% thick circular bump
grid:	49152 cells, 49667 nodes
boundary cond.:	inlet: $v = 0$, $p = 73340\,\rho^{\gamma}\,\frac{N}{m^2}$, $h = 278850\ \frac{m^2}{s^2}$
	outlet: $p = 72218\ \frac{N}{m^2}$



Isolines of the Mach number (M_{max}=1.38, M_{out}=0.675)



Isolines of the entropy

A. Kuz'min, 2002

Conclusions

- Implicit FEM-FCT schemes can be derived on the basis of rigorous positivity criteria.
- Low-order method can be constructed by adding discrete diffusion so as to eliminate negative off-diagonal entries of the high-order operator.
- A generalization to hyperbolic systems involves scalar artificial diffusion proportional to the spectral radius of the Roe matrix.
- Antidiffusive terms can be decomposed into a sum of internodal fluxes, which can be limited in an essentially one-dimensional fashion.
- Flux correction can be performed within a defect correction loop using a synchronized limiter.