Implicit FEM-FCT algorithm for compressible flows

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- State of the art: discretisation techniques
- Discrete upwinding for scalar equations
- Iterative defect correction scheme
- Generalised FEM-FCT formulation
- Matrix assembly for the Euler equations
- Artificial viscosity, scalar dissipation
- Synchronised flux limiter for systems

Convection dominated flows

Generic conservation law

$$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u) - \nabla \cdot (\epsilon \nabla u) = 0$$

Compressible Euler equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\ \frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p &= 0 \\ \frac{\partial (\rho E)}{\partial t} + \nabla \cdot (\rho H \mathbf{v}) &= 0 \end{aligned}$$

Godunov's Theorem (1959)

No linear discretisation scheme of order higher than first is monotonicity preserving

- ⊖ High-order methods: oscillatory
- ⊖ Convergence to nonphysical weak solution
- ⊖ Low-order methods: overdiffusive
- ⊖ Non-symmetric, ill-conditioned matrices
- ⊕ High-resolution methods: **nonlinear**

High-resolution methods

Flux-Corrected Transport (FCT) algorithm
Boris & Book (1973), Zalesak (1979)

1. Compute a *transported and diffused* solution by a linear monotonicity preserving scheme
2. Invoke *flux limiter* to determine the percentage of artificial diffusion which can be removed without generating oscillations
3. Add (limited) compensating *antidiffusion* to recover the high accuracy in smooth regions

State of the art

- ⊖ finite differences/volumes
- ⊖ explicit FEM-FCT formulation — Löhner (1987)
- ⊖ 1D nature, cartesian/simplex meshes

Objective: a *methodology* which is

- ⊕ based on mathematical theory, parameter-free
- ⊕ applicable to arbitrary time/space discretisations
- ⊕ multidimensional, independent of underlying mesh

FEM-FCT or FEM-TVD

Discrete positivity criteria

LED: $\frac{\mathrm{d}u_i}{\mathrm{d}t} = \sum_{j \neq i} c_{ij}(u_j - u_i), \quad c_{ij} \geq 0$

Jameson (1993)

$$e.g. \quad u_i = \max_{j \in S_i} u_j \quad \Rightarrow \quad u_j - u_i \leq 0 \quad \Rightarrow \quad \frac{\mathrm{d}u_i}{\mathrm{d}t} \leq 0$$

Lemma. A discrete scheme of the form

$$Au^{n+1} = Bu^n, \quad u^n \geq 0$$

is positivity-preserving if A is an *M-matrix* and all entries of B are non-negative.

Discrete diffusion operators $D = \{d_{ij}\}$

$$d_{ij} = d_{ji}, \quad \sum_j d_{ij} = \sum_i d_{ij} = 0$$

Flux decomposition of diffusive terms

$$(Du)_i = \sum_j d_{ij}u_j = \sum_{j \neq i} d_{ij}(u_j - u_i) = \sum_{j \neq i} f_{ij}$$

$$f_{ij} = d_{ij}(u_j - u_i), \quad f_{ji} = -f_{ij}$$

Discrete upwinding

Weak form
$$\int_\Omega w\left[\frac{\partial u}{\partial t} + \nabla\cdot(\mathbf{v}u - \epsilon\nabla u)\right]\,\mathrm{d}\mathbf{x} = 0$$

Group FEM formulation Fletcher (1983)

$$u = \sum_j u_j\varphi_j, \qquad \mathbf{v}u = \sum_j (\mathbf{v}_j u_j)\,\varphi_j$$

Galerkin semi-discretisation with mass lumping

$$M_L\frac{\mathrm{d}u}{\mathrm{d}t} = Ku \qquad m_i\frac{\mathrm{d}u_i}{\mathrm{d}t} = \sum_{j\neq i} k_{ij}(u_j - u_i) + \delta_i u_i$$

$$k_{ij} = -\mathbf{v}_j\cdot\mathbf{c}_{ij} - \epsilon\, s_{ij} \qquad \delta_i = \sum_j k_{ij} \approx \nabla\cdot\mathbf{v}$$

$$\mathbf{c}_{ij} = \int_\Omega \varphi_i\nabla\varphi_j\,\mathrm{d}\mathbf{x} \qquad s_{ij} = \int_\Omega \nabla\varphi_i\cdot\nabla\varphi_j\,\mathrm{d}\mathrm{x}$$

Adaptive artificial diffusion $L = K + D$

$$d_{ij} = d_{ji} = \max\{0, -k_{ij}, -k_{ji}\}, \qquad d_{ii} = -\sum_{j\neq i} d_{ij}$$

Edge-based elimination of negative off-diagonal entries

$$L := K \quad \rightarrow \quad \begin{array}{ll} l_{ii} := l_{ii} - d_{ij}, & l_{ij} := l_{ij} + d_{ij} \\ l_{ji} := l_{ji} + d_{ij}, & l_{jj} := l_{jj} - d_{ij} \end{array}$$

FEM-FCT formulation

Galerkin θ-scheme $\quad$ $$Au^H = b^n + f(u^n, u^H)$$

$$A = M_L - \theta \Delta t\, L, \qquad b^n = M_L u^n + (1-\theta)\Delta t\, L u^n$$

Antidiffusive contribution

$$\begin{aligned} f(u^n, u^H) &= [(M_C - M_L) - (1-\theta)\Delta t\,(L-K)]\, u^n \\ &- [(M_C - M_L) + \theta \Delta t\,(L-K)]\, u^H \\ f(u^n, u^H) &= \sum_{j \neq i} f_{ij}, \qquad f_{ji} = -f_{ij}, \quad i \neq j \end{aligned}$$

Antidiffusive flux decomposition

$$\begin{aligned} f_{ij} &= (m_{ij} - (1-\theta)\Delta t d_{ij})\,(u_j^n - u_i^n) \\ &- (m_{ij} + \theta \Delta t d_{ij})\,(u_j^H - u_i^H) \end{aligned}$$

Implicit FEM-FCT algorithm $\qquad$ Kuzmin (2001)

$$Au^{n+1} = M_L \tilde{u}^n + f^*(u^n, u^H)$$

$$f^*(u^n, u^H) = \sum_{j \neq i} \alpha_{ij} f_{ij} \qquad \begin{aligned} &\alpha_{ij} = \alpha_{ij}(\tilde{u}^n, f_{ij}), \\ &0 \leq \alpha_{ij} \leq 1 \end{aligned}$$

Intermediate solution $\tilde{u}^n = u^L(t^{n+1-\theta}) : \quad M_L \tilde{u}^n = b^n$

Positivity constraint $\quad \Delta t \leq \frac{1}{1-\theta} \min_i \left\{ -\frac{m_i}{l_{ii}} \,\middle|\, l_{ii} < 0 \right\}$

Iterative defect correction

Successive approximations

$$u^{(m+1)} = u^{(m)} + [A(u^{(m)})]^{-1} r^{(m)}$$

Practical implementation

$$A(u^{(m)})\Delta u^{(m)} = r^{(m)}, \quad m = 0, 1, 2, \dots$$

$$u^{(m+1)} = u^{(m)} + \Delta u^{(m)}, \quad u^{(0)} = u^n$$

'Upwind' preconditioner $\quad A(u^{(m)}) = M_L - \theta \Delta t L(u^{(m)})$

Defect vector with antidiffusion and low-order RHS

$$\begin{aligned} r^{(m)} &= b^n - A(u^{(m)})u^{(m)} + f^*(u^n, u^{(m)}) \\ b^n &= M_L u^n + (1-\theta)\Delta t\, L(u^n)u^n \end{aligned}$$

Basic FEM-FCT algorithm $\quad A(u^{(m)})u^{(m+1)} = b^{(m+1)}$

$$b_i^{(m+1)} = b_i^n + \sum_{j \neq i} \alpha_{ij}^{(m)} f_{ij}^{(m)} \qquad \alpha_{ij}^{(m)} = \alpha_{ij}(\tilde{u}^n, f_{ij}^{(m)})$$

Principle: Let $b^{(m+1)} = B\tilde{u}$, $\quad B \geq 0$ and $\tilde{u}$ positivity preserving auxiliary solution. Then we have

$$u^n \geq 0 \quad \Rightarrow \quad \tilde{u} \geq 0 \quad \Rightarrow \quad u^{(m+1)} \geq 0$$

⊖ amount of accepted antidiffusion depends on the magnitude of the time step Δt

Iterative FEM-FCT

Strategy: build accepted antidiffusion into $\tilde{u}$

$$M_L \tilde{u}^{(m)} = b^{(m)}, \qquad b^{(0)} = b^n$$

Raw flux difference

$$\Delta f_{ij}^{(m)} = f_{ij}^{(m)} - g_{ij}^{(m)}, \qquad \Delta f_{ij}^{(0)} = f_{ij}^{(0)}$$

Correction factors (Zalesak's limiter)

$$\alpha_{ij}^{(m)} = \alpha_{ij}(\tilde{u}^{(m)}, \Delta f_{ij}^{(m)}), \qquad 0 \leq \alpha_{ij}^{(m)} \leq 1$$

Accepted antidiffusion

$$g_{ij}^{(m+1)} = g_{ij}^{(m)} + \alpha_{ij}^{(m)} \Delta f_{ij}^{(m)}, \qquad g_{ij}^{(0)} = 0$$

Update of the RHS

$$b_i^{(m+1)} = b_i^{(m)} + \sum_{j \neq i} \alpha_{ij}^{(m)} \Delta f_{ij}^{(m)}$$

- ⊕ rejected antidiffusion can be 'recycled'
- ⊕ no excessive flux limiting for large time steps

Zalesak's limiter

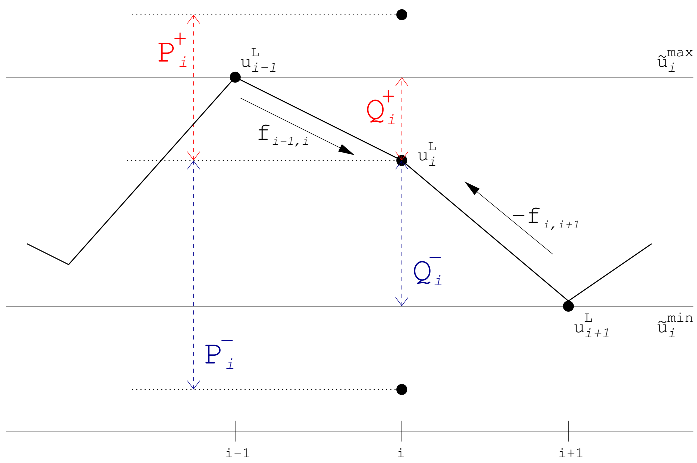
1. Prelimiting: $f_{ij} := 0$, if $f_{ij}(\tilde{u}_i - \tilde{u}_j) \leq 0$.

2. Positive/negative fluxes and upper/lower bounds

$$P_i^{\pm} = \frac{1}{m_i} \sum_{j \neq i} \begin{matrix} \max \\ \min \end{matrix} \{0, f_{ij}\}, \qquad Q_i^{\pm} = \tilde{u}_i^{\substack{\max \\ \min}} - \tilde{u}_i$$

3. Nodal correction factors

$$R_i^{\pm} = \begin{cases} \min\{1, Q_i^{\pm}/P_i^{\pm}\}, & \text{if } P_i^{+} > 0 > P_i^{-} \\ 0, & \text{if } P_i^{\pm} = 0 \end{cases}$$



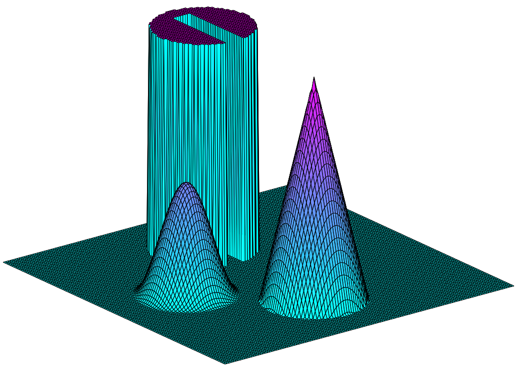
4. Postlimiting: $R_i^{\pm} := 1$ at in-/outlet 

5. Final correction factors

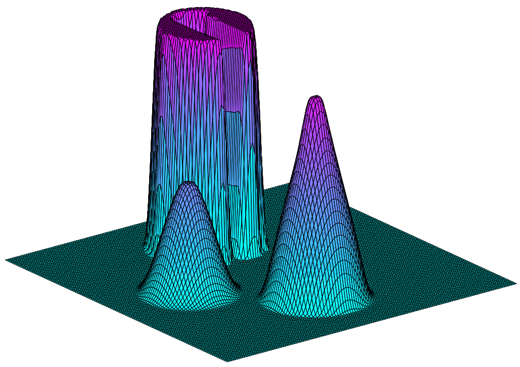
$$\alpha_{ij} = \begin{cases} \min\{R_i^{+}, R_j^{-}\}, & \text{if } f_{ij} \geq 0 \\ \min\{R_j^{+}, R_i^{-}\}, & \text{if } f_{ij} < 0 \end{cases}$$

Solid body rotation

exact solution



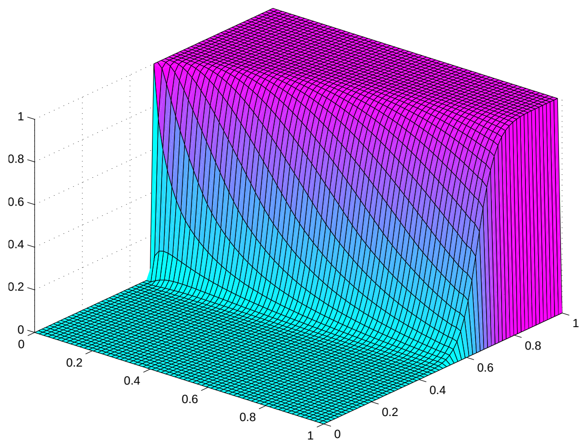
FEM-FCT



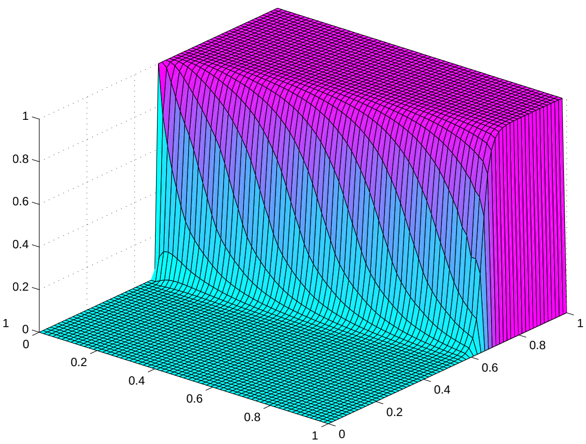
Crank-Nicolson, $\Delta t = 10^{-3}$, 128×128 Q_1 elements

Steady-state convection-diffusion

basic FEM-FCT



iterative FEM-FCT



Backward Euler, $\Delta t = 10^{-1}$, $\epsilon = 10^{-3}$, 64×64 Q_1 elements

Compressible Euler equations

Divergence form

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = 0$$

$$U = (\rho, \rho\mathbf{v}, \rho E)^T$$

$$\mathbf{F} = (F^1, F^2, F^3)$$

$$F^1 = \begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ \rho v_1 v_3 \\ \rho H v_1 \end{bmatrix}, \quad F^2 = \begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ \rho v_2 v_3 \\ \rho H v_2 \end{bmatrix}, \quad F^3 = \begin{bmatrix} \rho v_3 \\ \rho v_1 v_3 \\ \rho v_2 v_3 \\ \rho v_3^2 + p \\ \rho H v_3 \end{bmatrix}$$

where $H = E + p/\rho$ and $p = (\gamma - 1)\rho\left(E - |\mathbf{v}|^2/2\right)$

Quasi-linear formulation

$$\frac{\partial U}{\partial t} + \mathbf{A} \cdot \nabla U = 0$$

Jacobian matrices

$$\mathbf{A} = (A^1, A^2, A^3)$$

$$F^d = A^d U, \qquad A^d = \frac{\partial F^d}{\partial U}, \qquad d = 1, 2, 3.$$

Mathematical challenges:

- hyperbolicity
- nonlinearity
- strong coupling

Galerkin discretisation

Group FEM formulation $\sum_j \varphi_j \equiv 1 \Rightarrow \mathbf{c}_{ii} = -\sum_{j\neq i} \mathbf{c}_{ij}$

$$M_C \frac{\mathrm{d}U}{\mathrm{d}t} = KU \qquad (KU)_i = -\sum_{j\neq i} \mathbf{c}_{ij} \cdot (\mathbf{F}_j - \mathbf{F}_i)$$

$$\mathbf{F}_j - \mathbf{F}_i = \hat{\mathbf{A}}_{ij}(U_j - U_i) \quad \text{where} \quad \hat{\mathbf{A}}_{ij} = \mathbf{A}(\hat{\rho}_{ij}, \hat{\mathbf{v}}_{ij}, \hat{H}_{ij})$$

Roe mean values $\hat{\rho}_{ij} = \sqrt{\rho_i \rho_j}$

$$\hat{\mathbf{v}}_{ij} = \frac{\sqrt{\rho_i}\mathbf{v}_i + \sqrt{\rho_j}\mathbf{v}_j}{\sqrt{\rho_i} + \sqrt{\rho_j}}, \qquad \hat{H}_{ij} = \frac{\sqrt{\rho_i}H_i + \sqrt{\rho_j}H_j}{\sqrt{\rho_i} + \sqrt{\rho_j}}$$

Quasi-linear formulation

$$(KU)_i = -\sum_{j\neq i} \mathbf{c}_{ij} \cdot \hat{\mathbf{A}}_{ij}(U_j - U_i) = \sum_{j\neq i} (\mathrm{A}_{ij} + \mathrm{B}_{ij})(U_j - U_i)$$

Cumulative Roe matrices

$$\mathrm{A}_{ij} = \mathbf{a}_{ij} \cdot \hat{\mathbf{A}}_{ij}, \qquad \mathbf{a}_{ij} = -\frac{\mathbf{c}_{ij} - \mathbf{c}_{ji}}{2}$$

$$\mathrm{B}_{ij} = \mathbf{b}_{ij} \cdot \hat{\mathbf{A}}_{ij}, \qquad \mathbf{b}_{ij} = -\frac{\mathbf{c}_{ij} + \mathbf{c}_{ji}}{2}$$

Contribution of edge $\vec{ij}$

$$(\mathrm{A}_{ij} + \mathrm{B}_{ij})(U_j - U_i) \quad \longrightarrow \quad (KU)_i$$

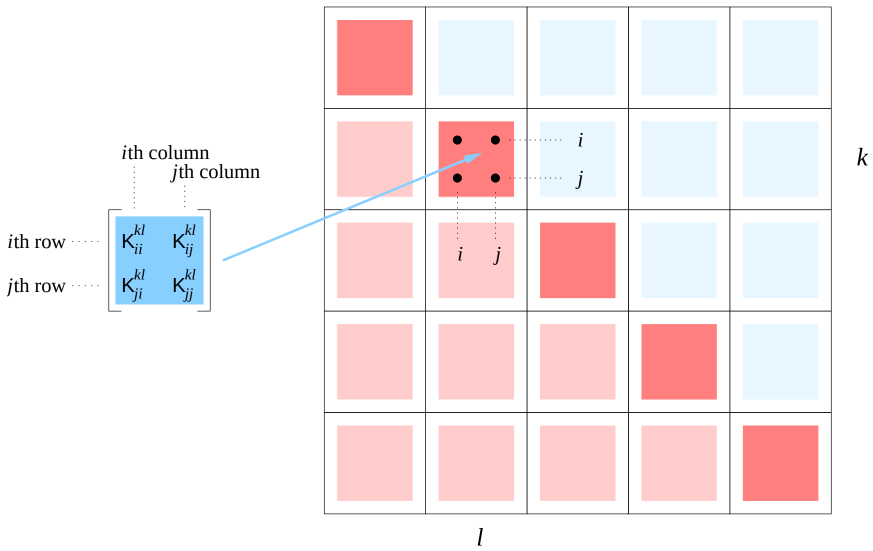
$$(\mathrm{A}_{ij} - \mathrm{B}_{ij})(U_j - U_i) \quad \longrightarrow \quad (KU)_j$$

Galerkin matrix assembly

Edge contribution to the operator K

$$\mathrm{K}_{ii} = -\mathrm{A}_{ij} - \mathrm{B}_{ij} \qquad \mathrm{K}_{ij} = \mathrm{A}_{ij} + \mathrm{B}_{ij}$$

$$\mathrm{K}_{ji} = -\mathrm{A}_{ij} + \mathrm{B}_{ij} \qquad \mathrm{K}_{jj} = \mathrm{A}_{ij} - \mathrm{B}_{ij}$$



Edge contribution to the operator L

$$\mathrm{L}_{ii} = -\mathrm{A}_{ij} - \mathrm{D}_{ij} \qquad \mathrm{L}_{ij} = \mathrm{A}_{ij} + \mathrm{D}_{ij}$$

$$\mathrm{L}_{ji} = -\mathrm{A}_{ij} + \mathrm{D}_{ij} \qquad \mathrm{L}_{jj} = \mathrm{A}_{ij} - \mathrm{D}_{ij}$$

Raw antidiffusive flux

$$\mathrm{F}_{ij} = -\left(\mathrm{M}_{ij}\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{D}_{ij} - \mathrm{B}_{ij}\right)(U_j - U_i), \quad \mathrm{F}_{ji} = -\mathrm{F}_{ij}$$

where $\mathrm{M}_{ij} = m_{ij}I$ and D_{ij} is the tensorial dissipation

Artificial viscosity

LED-principle for systems: *all off-diagonal blocks of the global Jacobian matrix should be positive definite*

Characteristic decomposition

$$\mathrm{A}_{ij} = R_{ij}\Lambda_{ij}R_{ij}^{-1}$$

$$\Lambda_{ij} = |\mathbf{a}_{ij}|\mathrm{diag}\,\{\lambda_1,\dots,\lambda_5\}, \quad |\mathbf{a}_{ij}| = \sqrt{\mathbf{a}_{ij}\cdot\mathbf{a}_{ij}}$$

Eigenvalues of the cumulative Roe matrix A_{ij}

$$\lambda_1 = \hat{v}_{ij} - \hat{c}_{ij}, \quad \lambda_2 = \lambda_3 = \lambda_4 = \hat{v}_{ij}, \quad \lambda_5 = \hat{v}_{ij} + \hat{c}_{ij}$$

$$\hat{v}_{ij} = \frac{\mathbf{a}_{ij}\cdot\hat{\mathbf{v}}_{ij}}{|\mathbf{a}_{ij}|}, \qquad \hat{c}_{ij} = \sqrt{(\gamma-1)\left(\hat{H}_{ij} - \frac{|\hat{\mathbf{v}}_{ij}|^2}{2}\right)}$$

System upwinding (expensive)

$$\mathrm{D}_{ij} = |\mathrm{A}_{ij}| = R_{ij}|\Lambda_{ij}|R_{ij}^{-1}$$

Generalisation of Roe's approximate Riemann solver

Scalar dissipation (efficient)

$$\mathrm{D}_{ij} = d_{ij}I \quad \text{where} \quad d_{ij} = |\mathbf{a}_{ij}|\max_i|\lambda_i|$$

is optimal for FEM-FCT since excessive artificial diffusion is removed by the flux limiter

Iterative defect correction

Successive approximations

$$U^{(m+1)} = U^{(m)} + [A(U^{(m)})]^{-1} R^{(m)}$$

Defect vector

$$R^{(m)} = B^{(m+1)} - A(U^{(m)})U^{(m)}$$

Right-hand side

$$\begin{aligned} B^{(m+1)} &= B^n + F^*(U^n, U^{(m)}) \\ \text{or} \quad B^{(m+1)} &= B^{(m)} + \Delta F^*(U^n, U^{(m)}) \end{aligned}$$

Initialisation

$$B^n = B^{(0)} = M_L U^n + (1-\theta)\Delta t L(U^n)U^n$$

Raw antidiffusion

$$\begin{aligned} F(U^n, U^{(m)}) &= [(M_C - M_L) - (1-\theta)\Delta t D(U^n)]U^n \\ &- [(M_C - M_L) + \theta \Delta t D(U^{(m)})]U^{(m)} \end{aligned}$$

$$\begin{aligned} \Delta F(U^n, U^{(m)}) &= F(U^n, U^{(m)}) - G^{(m)} \\ G^{(m+1)} &= G^{(m)} + \Delta F^*(U^n, U^{(m)}), \quad G^{(0)} = 0 \end{aligned}$$

Synchronisation of correction factors

$$\mathrm{F}^*_{ij} = f(\alpha^1_{ij}, \ldots, \alpha^5_{ij})\mathrm{F}_{ij} \qquad \text{Löhner (1987)}$$

Block-Jacobi preconditioner

Global linear system

$$\begin{bmatrix} \boxed{A_{11}^{(m)}} & A_{12}^{(m)} & A_{13}^{(m)} & A_{14}^{(m)} & A_{15}^{(m)} \\ A_{21}^{(m)} & \boxed{A_{22}^{(m)}} & A_{23}^{(m)} & A_{24}^{(m)} & A_{25}^{(m)} \\ A_{31}^{(m)} & A_{32}^{(m)} & \boxed{A_{33}^{(m)}} & A_{34}^{(m)} & A_{35}^{(m)} \\ A_{41}^{(m)} & A_{42}^{(m)} & A_{43}^{(m)} & \boxed{A_{44}^{(m)}} & A_{45}^{(m)} \\ A_{51}^{(m)} & A_{52}^{(m)} & A_{53}^{(m)} & A_{54}^{(m)} & \boxed{A_{55}^{(m)}} \end{bmatrix} \begin{bmatrix} \Delta u_1^{(m+1)} \\ \Delta u_2^{(m+1)} \\ \Delta u_3^{(m+1)} \\ \Delta u_4^{(m+1)} \\ \Delta u_5^{(m+1)} \end{bmatrix} = \begin{bmatrix} r_1^{(m)} \\ r_2^{(m)} \\ r_3^{(m)} \\ r_4^{(m)} \\ r_5^{(m)} \end{bmatrix}$$

Block-diagonal preconditioner

$$A_{kk}^{(m)} = M_L - \theta \Delta t L_{kk}^{(m)}, \quad A_{kl}^{(m)} = 0, \quad \forall l \neq k$$

Sequence of scalar subproblems

$$A_{kk}^{(m)} \Delta u_k^{(m)} = r_k^{(m)}, \qquad k = 1, \ldots, 5$$

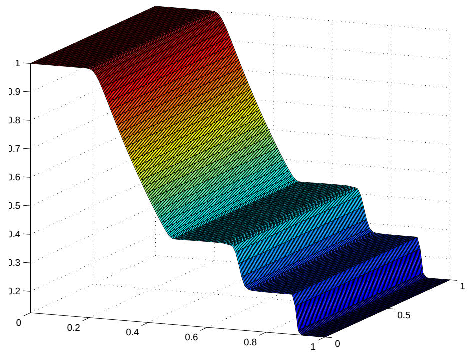
$$u_k^{(m+1)} = u_k^{(m)} + \Delta u_k^{(m)}, \quad u_k^{(0)} = u_k^n$$

- $\oplus$ only 5 blocks need to be assembled/stored
- $\oplus$ equations can be solved separately/in parallel
- $\ominus$ poor convergence for large time steps

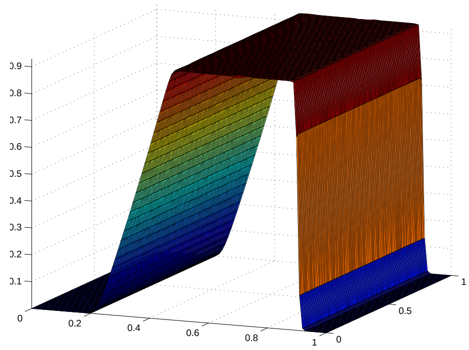
coupled solver (multigrid/BiCGSTAB with block-Gauss-Seidel smoother/preconditioner)

Shock tube problem

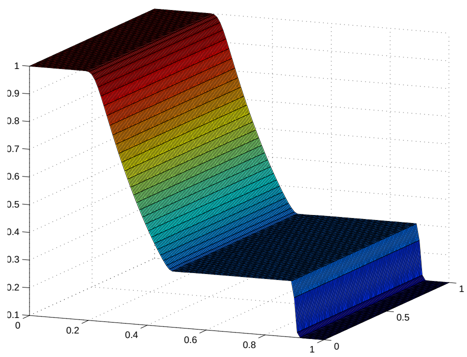
density



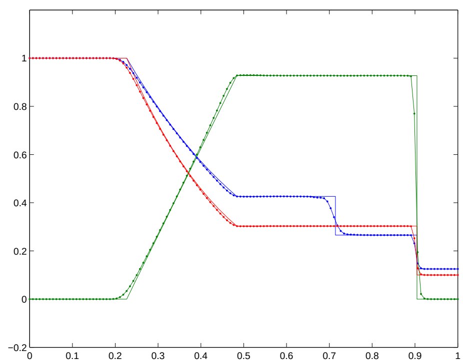
velocity



pressure

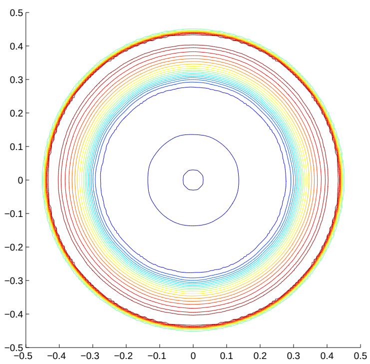


cutplane $y = 0.5$

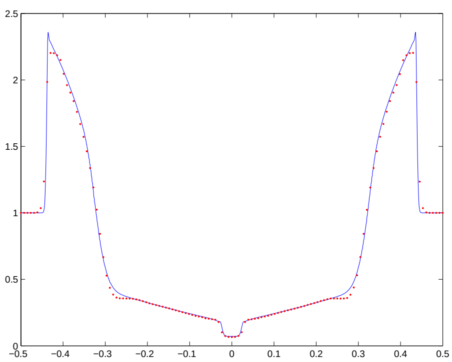


Radially symmetric Riemann problem

density



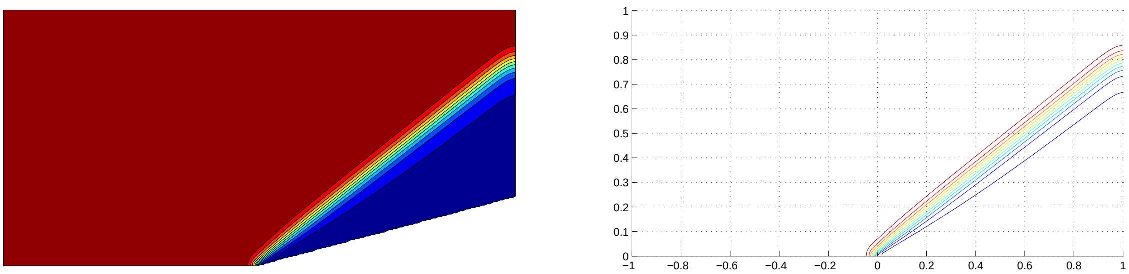
cutplane $y = 0$



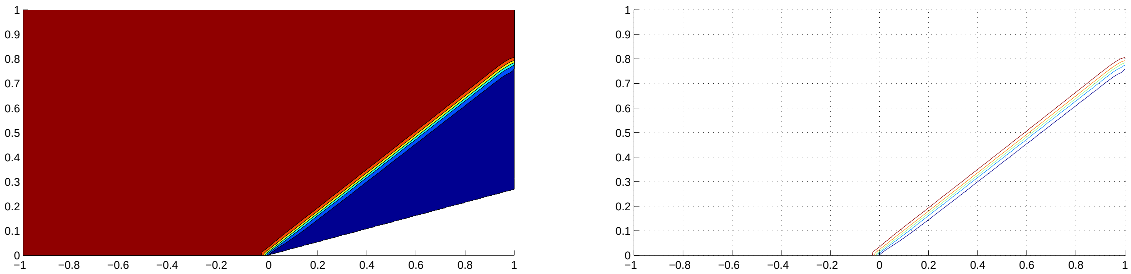
Crank-Nicolson, $\Delta t = 10^{-3}$, 128×128 Q_1 elements

Compression corner $M = 2.5, \ \theta = 15^\circ$

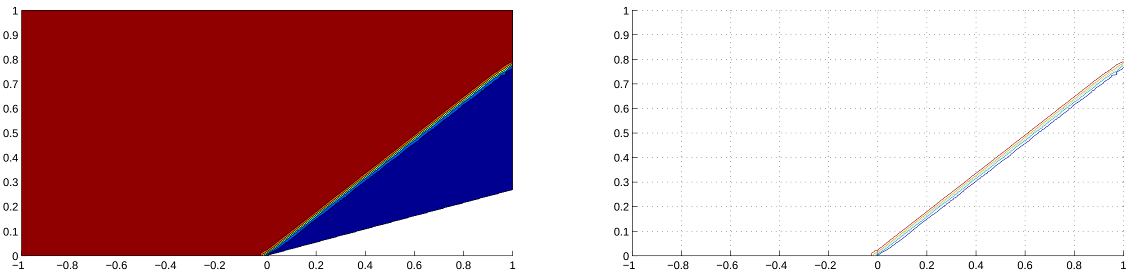
low-order method



FEM-FCT, basic limiter

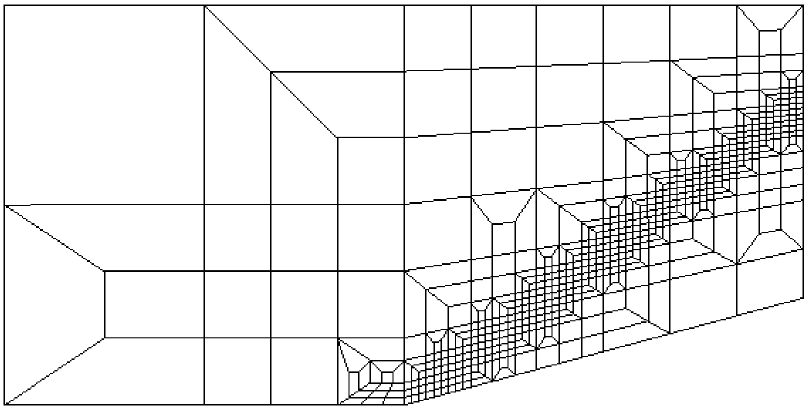


FEM-FCT, iterative limiter

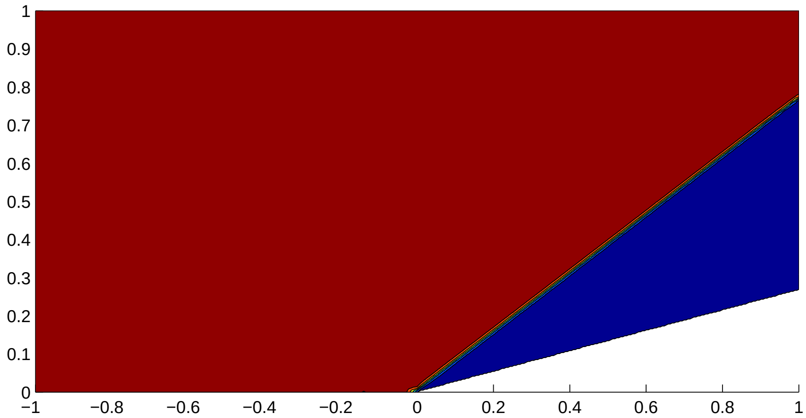


Backward Euler, $\Delta t = 10^{-2}$, 128×128 Q_1 elements

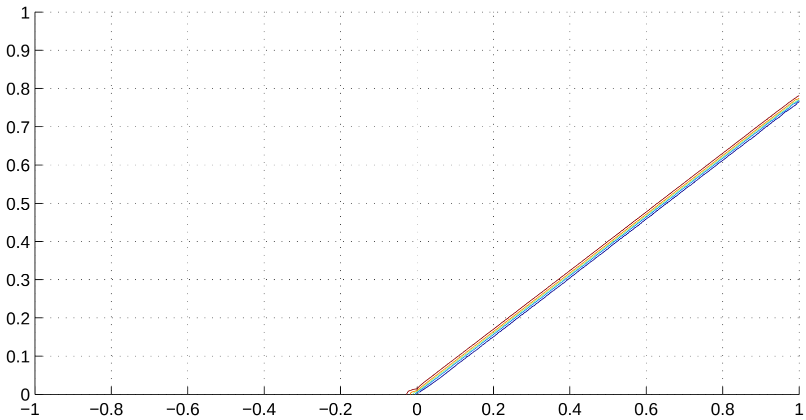
Compression corner $M = 2.5,\ \theta = 15^\circ$



coarse grid with local refinement



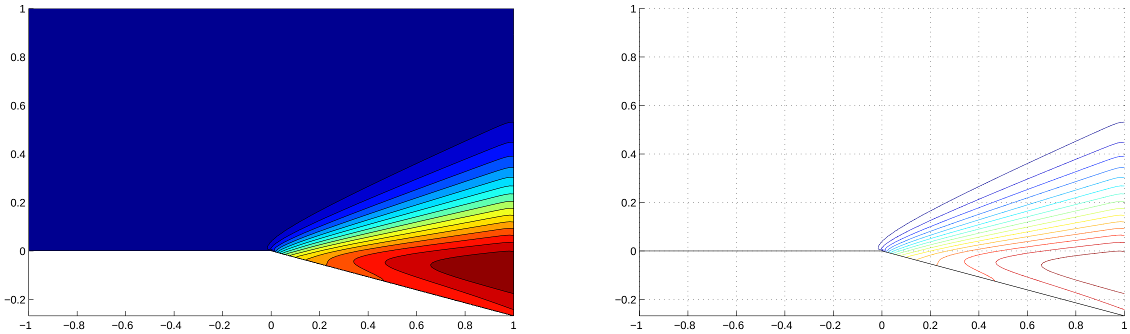
FEM-FCT, steady-state mach number distribution



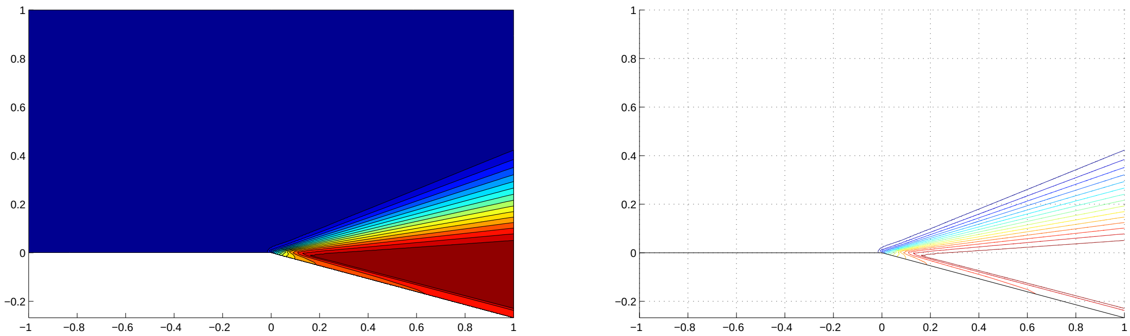
Backward Euler, $\Delta t = 10^{-2}$, iterative limiter

Prandtl-Meyer expansion $M = 2.5, \ \theta = 15^\circ$

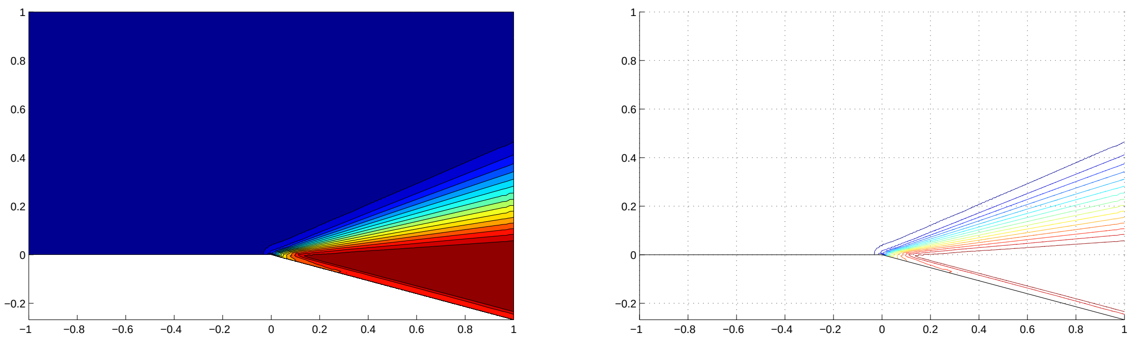
low-order method



FEM-FCT, basic limiter



FEM-FCT, iterative limiter



Backward Euler, $\Delta t = 10^{-2}$, 128×128 Q_1 elements

Conclusions

- Implicit FEM-FCT schemes can be derived on the basis of rigorous positivity criteria
- Discrete upwinding is performed by adding artificial diffusion so as to eliminate negative off-diagonal entries of the high-order operator
- In the case of hyperbolic systems scalar artificial dissipation proportional to the spectral radius of the Roe matrix can be utilised
- Flux correction can be performed within a defect correction preconditioned by the low-order operator
- Iterative limiting strategy prevents Zalesak's limiter from getting overly diffusive at large time steps

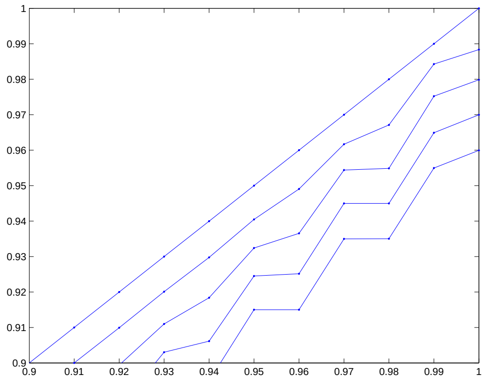
Workshop *High-resolution schemes for convection-dominated flows: 30 years of FCT*

University of Dortmund, September 29-30, 2003

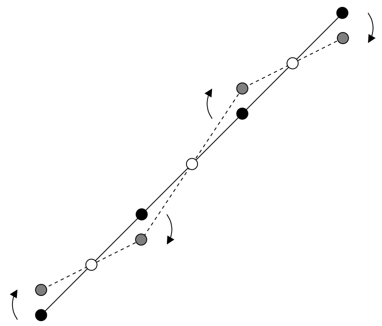
http://www.mathematik.uni-dortmund.de/lsiii/conf/fct30.html

Why postlimiting ?

Pathological behaviour of Zalesak's limiter



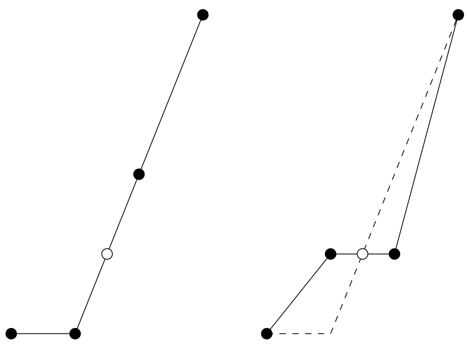
Lever model



Boundary adjustment at in-/outlet

$$R_i^{\pm} := 1$$

Prelimiting effect



$$f_{ij}(\tilde{u}_i - \tilde{u}_j) \leq 0$$

diffusive flux

$$\Longrightarrow f_{ij} := 0$$

Positivity proof for FEM-FCT

Representation of RHS

$$b_i = m_i \tilde{u}_i + \sum_{j \neq i} \alpha_{ij} f_{ij} \geq 0, \quad \text{if} \quad \tilde{u} \geq 0$$

Trivial case: $\sum_{j \neq i} \alpha_{ij} f_{ij} = 0 \;\Rightarrow\; b_i = m_i \tilde{u}_i \geq 0$

Nontrivial case: let $c_i = \frac{1}{Q_i} \sum_{j \neq i} \alpha_{ij} f_{ij}$ where

$$Q_i = \begin{cases} Q_i^+ = \tilde{u}_i^{\max} - \tilde{u}_i, & \text{if } \sum_{j \neq i} \alpha_{ij} f_{ij} > 0 \\ Q_i^- = \tilde{u}_i^{\min} - \tilde{u}_i, & \text{if } \sum_{j \neq i} \alpha_{ij} f_{ij} < 0 \end{cases}$$

$$\begin{aligned} b_i &= m_i \tilde{u}_i + c_i(\tilde{u}_k - \tilde{u}_i) \\ &= (m_i - c_i)\tilde{u}_i + c_i \tilde{u}_k \end{aligned}$$

Requirements

1. $m_i \geq c_i \geq 0$
2. $\alpha_{ij} \equiv 0$, if $Q_i = 0$

Zalesak's limiter: $Q_i^{\pm} = 0 \;\Rightarrow\; R_i^{\pm} = 0 \;\Rightarrow\; \alpha_{ij} = 0$

$$\sum_{j \neq i} \alpha_{ij} f_{ij} \leq \sum_{j \neq i} \alpha_{ij} \max\{0, f_{ij}\} \leq m_i R_i^+ P_i^+ \leq m_i Q_i^+$$

$$\sum_{j \neq i} \alpha_{ij} f_{ij} \geq \sum_{j \neq i} \alpha_{ij} \min\{0, f_{ij}\} \geq m_i R_i^- P_i^- \geq m_i Q_i^-$$

This implies $m_i \geq c_i$