An Iterative FEM-FCT Algorithm for the Compressible Euler Equations

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- State of the art: discretization techniques
- Discrete upwinding for scalar equations
- Iterative defect correction scheme
- Generalized FEM-FCT formulation
- Matrix assembly for the Euler equations
- Construction of artificial viscosities
- Solution strategies for coupled equations
- Implementation of boundary conditions

Algebraic Flux Correction of FCT-type

Scalar conservation law $\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u) = 0 \quad \text{in}\ \Omega$

1. Linear high-order scheme (e.g. Galerkin FEM)

$$M_C \frac{u^{n+1} - u^n}{\Delta t} = \theta K u^{n+1} + (1-\theta) K u^n, \qquad \exists\, j \neq i : \ k_{ij} < 0$$

2. Linear low-order scheme $L = K + D$

$$M_L \frac{u^{n+1} - u^n}{\Delta t} = \theta L u^{n+1} + (1-\theta) L u^n, \qquad l_{ij} \geq 0,\ \forall j \neq i$$

3. Nonlinear high-resolution scheme $f_i = \sum_{j \neq i} f_{ij}^a$

$$M_L \frac{u^{n+1} - u^n}{\Delta t} = \theta L u^{n+1} + (1-\theta) L u^n + f(u^{n+1}, u^n)$$

Equivalent representation $A u^{n+1} = B(\tilde{u})\tilde{u}$, where

$A = M_L - \theta \Delta t L$ is an *M-matrix* and $b_{ij} \geq 0,\ \forall i, j$

Algebraic Design Criteria

Algebraic Constraint I: (semi-discrete level)

$$\boxed{\frac{\mathrm{d}u_i}{\mathrm{d}t} = \sum_{j\neq i} \sigma_{ij}(u_j - u_i), \quad \sigma_{ij} \geq 0} \quad u_i = \left\{ \begin{matrix} \max_j \\ \min_j \end{matrix} \right\} u_j \quad \Rightarrow \quad u_j - u_i \left\{ \begin{matrix} \leq \\ \geq \end{matrix} \right\} 0 \quad \Rightarrow \quad \frac{\mathrm{d}u_i}{\mathrm{d}t} \left\{ \begin{matrix} \leq \\ \geq \end{matrix} \right\} 0$$

In 1D, Jameson's (1993) LED property is equivalent to Harten's (1983) TVD conditions.

Algebraic Constraint II: (fully discrete level)

A discrete scheme of the form $\boxed{Au^{n+1} = Bu^n, \quad u^n \geq 0}$ is positivity-preserving

if A is an *M-matrix* ($a_{ij} \leq 0, \forall j \neq i, \quad A^{-1} \geq 0$) and all entries of B are non-negative.

Tool: Discrete diffusion operators $D = \{d_{ij}\}$,

$$\text{where} \quad d_{ij} = d_{ji} \quad \text{and} \quad \sum_j d_{ij} = \sum_i d_{ij} = 0.$$

Flux decomposition of diffusive terms into antisymmetric fluxes

$$(Du)_i = \sum_j d_{ij} u_j = \sum_{j\neq i} d_{ij}(u_j - u_i) = \sum_{j\neq i} f_{ij}, \quad \text{where} \quad f_{ji} = -f_{ij}.$$

High- and Low-Order Schemes

Difference between high- and low-order scheme

$$P(u) = [M_L - M_C]\frac{\mathrm{d}u}{\mathrm{d}t} - \underbrace{[L - K]}_{D}u$$

Flux decomposition of raw antidiffusion

$$P_i = \sum_{j \neq i} f_{ij}, \qquad f_{ij} = -[m_{ij}\frac{\mathrm{d}}{\mathrm{d}t} + d_{ij}](u_j - u_i), \quad f_{ji} = -f_{ij}$$

Nonlinear system for an implicit time discretization (standard $\theta-$ scheme)

$$M_L \frac{u^{n+1} - u^n}{\Delta t} = \theta L u^{n+1} + (1-\theta) L u^n + P(u^{n+1}, u^n)$$

Successive approximation $\boxed{Au^{(m+1)} = b^{(m+1)}}$ $\quad u^{(0)} = u^n, \qquad m = 0, 1, 2, \ldots$

Preconditioner $\quad A = M_L - \theta \Delta t L \quad$ and load-vector $\quad b^{(m+1)} = b^n + P(u^{(m)}, u^n)$

Low-order contribution $\quad b^n = [M_L + (1-\theta)\Delta t L]u^n$

Raw antidiffusion $\quad P_i^{(m)} = \sum_{j \neq i} f_{ij}^{(m)} \quad$ with antisymmetric fluxes

$$f_{ij}^{(m)} = [m_{ij} - (1-\theta)\Delta t d_{ij}^n](u_j^n - u_i^n) - [m_{ij} + \theta \Delta t d_{ij}^{(m)}](u_j^{(m)} - u_i^{(m)}) = -f_{ji}^{(m)}$$

Basic FEM-FCT Algorithm

$$\boxed{Au^{(m+1)} = b^{(m+1)}}$$

By construction, $A = M_L - \theta \Delta t L$ is an M-matrix which is easy to 'invert' and satisfies *Algebraic Constraint II*.

Strategy: Multiply the antidiffusive fluxes by 'some' correction factors $\alpha_{ij} \in [0,1]$ so that there exists a matrix $B(\tilde{u}) \geq 0$ and a positivity-preserving solution $\tilde{u}$ such that $b^{(m+1)} = B(\tilde{u})\tilde{u}$.

Positivity transfer cycle $\boxed{u^n \geq 0 \quad \Rightarrow \quad \tilde{u} \geq 0 \quad \Rightarrow \quad u^{(m+1)} = A^{-1}B(\tilde{u})\tilde{u} \geq 0}$

Auxiliary solution to the explicit subproblem $\quad M_L \tilde{u}^n = b^n$

proves to be positivity-preserving for $\Delta t \leq \frac{1}{1-\theta} \min\limits_i \{-m_i/l_{ii} | l_{ii} < 0\}, \quad 0 \leq \theta < 1$

Correction factors $\quad \alpha_{ij}^{(m)} = \alpha_{ij}(\tilde{u}^n, f_{ij}^{(m)}), \quad 0 \leq \alpha_{ij} \leq 1 \quad$ Zalesak's limiter (1979)

Modified right-hand side $\quad \boxed{b_i^{(m+1)} = b_i^n + \sum_{j \neq i} \alpha_{ij}^{(m)} f_{ij}^{(m)}}$

Remark: The admissible percentage of the antidiffusive flux depends on the magnitude of the time step Δt.

Iterative FEM-FCT Algorithm

Strategy: Build accepted antidiffusion into $\tilde{u}$ and limit only the rejected portion.

Variable auxiliary solution $M_L \tilde{u}^{(m)} = b^{(m)}, \qquad b^{(0)} = b^n$

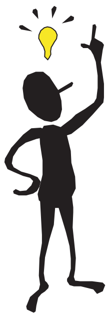
Rejected antidiffusive fluxes $\Delta f_{ij}^{(m)} = f_{ij}^{(m)} - g_{ij}^{(m)}, \qquad \Delta f_{ij}^{(0)} = f_{ij}^{(0)}$

Correction factors $\alpha_{ij}^{(m)} = \alpha_{ij}(\tilde{u}^{(m)}, \Delta f_{ij}^{(m)}), \qquad 0 \leq \alpha_{ij}^{(m)} \leq 1$

Accepted antidiffusive fluxes $g_{ij}^{(m+1)} = g_{ij}^{(m)} + \alpha_{ij}^{(m)} \Delta f_{ij}^{(m)}, \qquad g_{ij}^{(0)} = 0$

Modified right-hand side

$$\boxed{b_i^{(m+1)} = b_i^{(m)} + \sum_{j \neq i} \alpha_{ij}^{(m)} \Delta f_{ij}^{(m)}}$$



Consistency: $\alpha_{ij}^{(m)} \equiv 1 \quad \Rightarrow \quad b_i^{(m+1)} = b_i^n + \sum_{j \neq i} \left(g_{ij}^{(m)} + \Delta f_{ij}^{(m)} \right) = b_i^n + \sum_{j \neq i} f_{ij}^{(m)}$

Remark: The iterative flux limiter makes it possible to ‘recycle’ the rejected antidiffusion.

Zalesak's Flux Limiter

1. Positive/negative contributions $$P_i = P_i^+ + P_i^-, \quad \text{where} \quad P_i^\pm = \sum_{j\neq i} \begin{matrix}\max\\ \min\end{matrix}\{0, f_{ij}\}$$

2. Maximum/minimum increment $$Q_i^\pm = \begin{matrix}\max\\ \min\end{matrix} \Delta u_{ij}^\pm, \quad \text{where} \quad \Delta u_{ij}^\pm = \begin{matrix}\max\\ \min\end{matrix}\{0, \tilde{u}_j - \tilde{u}_i\}$$

3. Nodal correction factors $$R_i^\pm = \begin{cases} \min\{1, m_i Q_i^\pm / P_i^\pm\}, & \text{if } P_i^\pm \neq 0 \\ 1, & \text{if } P_i^\pm = 0 \end{cases}$$

4. Final correction factors $$\alpha_{ij} = \begin{cases} \min\{R_i^+, R_j^-\}, & \text{if } f_{ij} \geq 0 \\ \min\{R_j^+, R_i^-\}, & \text{if } f_{ij} < 0 \end{cases}$$

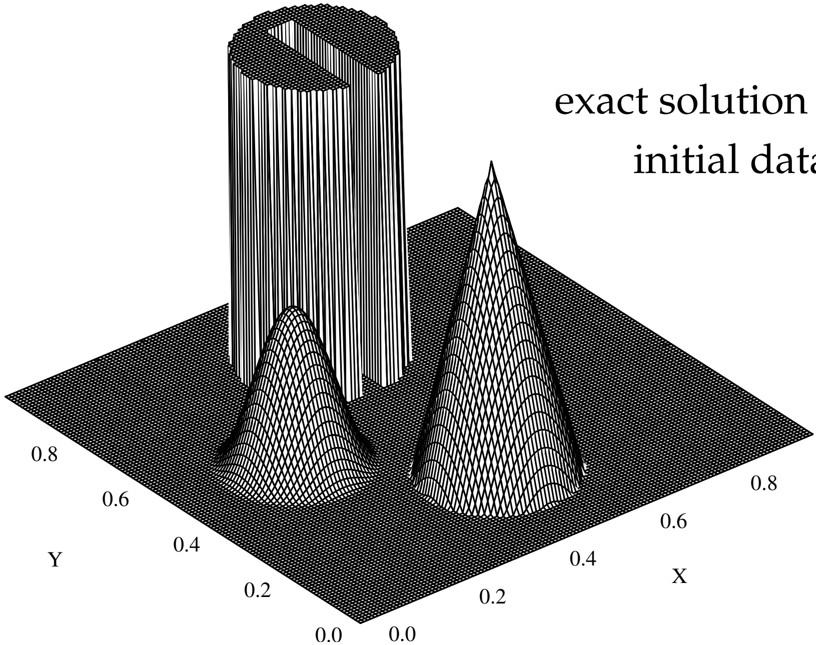
Remark: This choice of the correction factors guarantees that

$$\boxed{\tilde{u}_i^{(m)} + Q_i^- \;\leq\; \tilde{u}_i^{(m+1)} \;\leq\; \tilde{u}_i^{(m)} + Q_i^+}$$

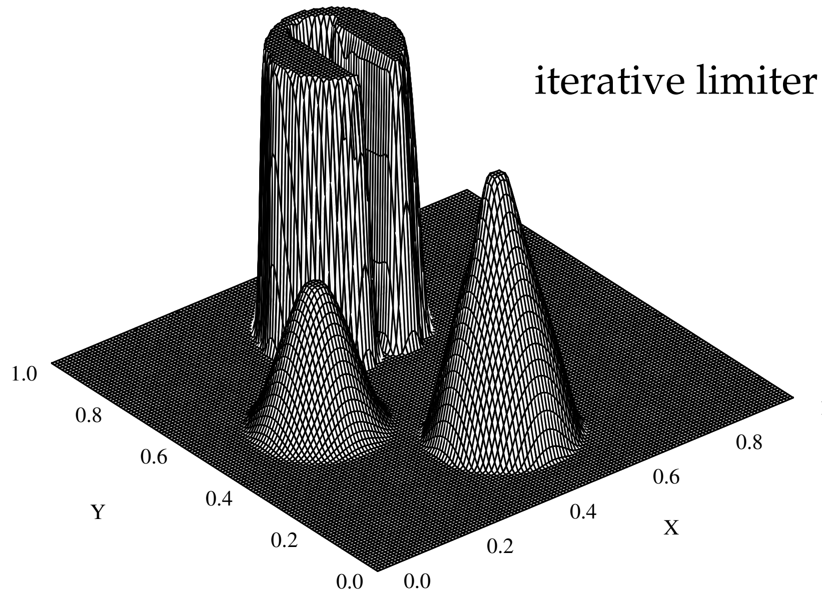
so that no enhancement of local extrema takes place.

Solid Body Rotation

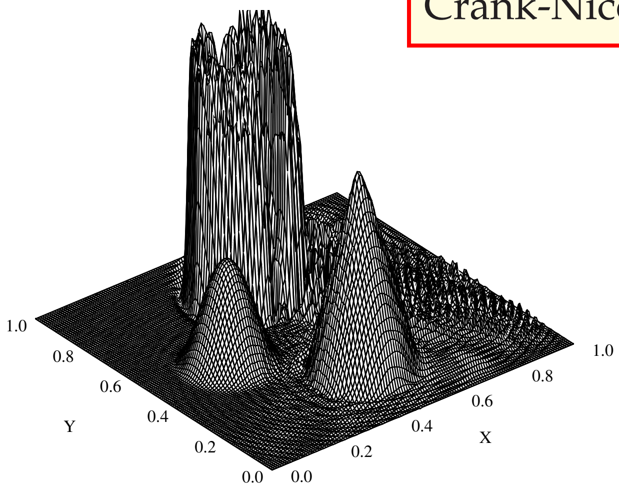
exact solution /
initial data



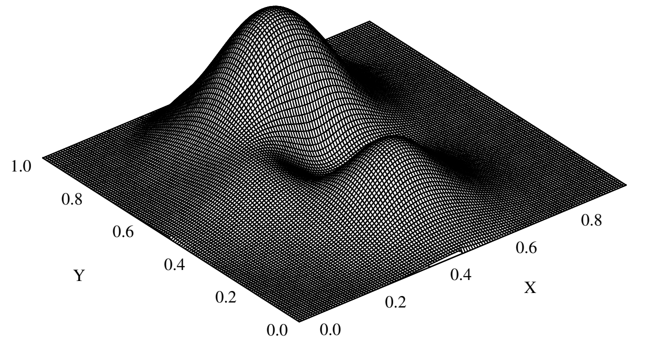
iterative limiter



Crank-Nicolson time-stepping, $\Delta t = 10^{-3}$, $16,384$ Q_1 elements, at $t = 2\pi$

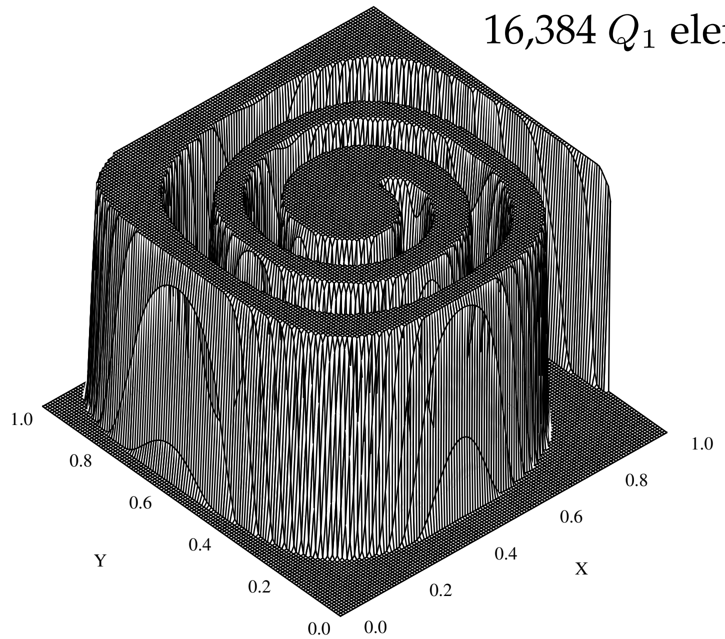


high-order *vs.*
low-order scheme

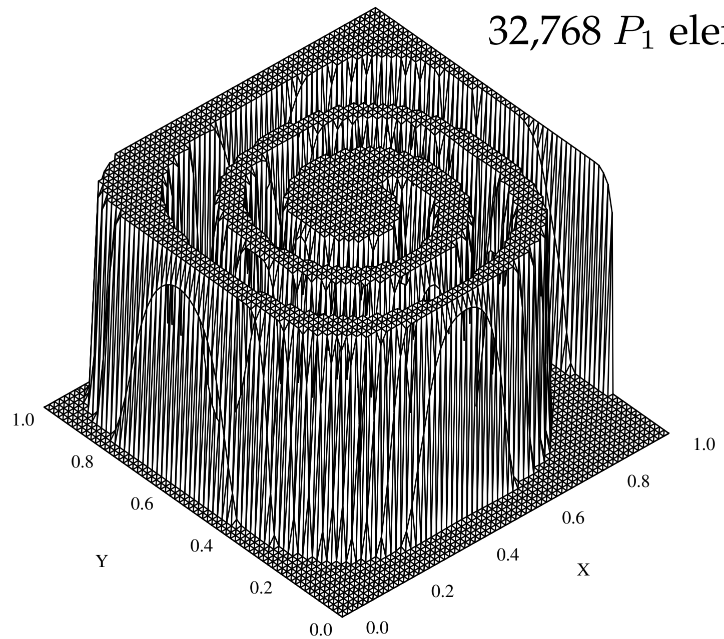


Swirling Flow Problem

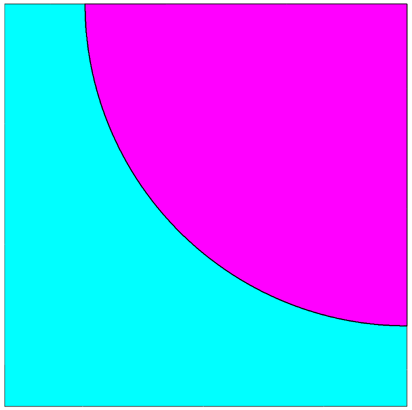
16,384 Q_1 elements



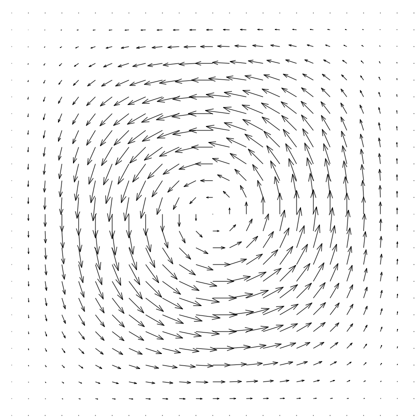
32,768 P_1 elements



Crank-Nicolson time-stepping, $\Delta t = 10^{-3}$, at $t = 2.5$



Initial data



Velocity field

Compressible Euler Equations

Divergence form $\boxed{\dfrac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = \mathbf{0}}$ where $\nabla \cdot \mathbf{F} = \sum_{d=1}^{3} \dfrac{\partial F^d}{\partial x_d}$

Conservative variables and fluxes

$$U = \begin{bmatrix} \rho \\ \rho \mathbf{v} \\ \rho E \end{bmatrix} \qquad \mathbf{F} = (F^1, F^2, F^3) = \begin{bmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \otimes \mathbf{v} + p\mathcal{I} \\ \rho H \mathbf{v} \end{bmatrix} \qquad \begin{array}{l} H = E + \dfrac{p}{\rho} \\ \gamma = c_p / c_v \end{array}$$

Equation of state $p = (\gamma - 1)\rho(E - 0.5|\mathbf{v}|^2)$ for a polytropic gas

Quasi-linear form $\boxed{\dfrac{\partial U}{\partial t} + \mathbf{A} \cdot \nabla U = \mathbf{0}}$ where $\mathbf{A} \cdot \nabla U = \sum_{d=1}^{3} A^d \dfrac{\partial U}{\partial x_d}$

Jacobian matrices $\mathbf{A} = (A^1, A^2, A^3)$ $\quad F^d = A^d U, \quad A^d = \dfrac{\partial F^d}{\partial U}, \quad d = 1, 2, 3$

High-Order Scheme

Group FEM formulation

$$\boxed{M_C \frac{\mathrm{dU}}{\mathrm{d}t} = K\mathrm{U}}$$

$$(K\mathrm{U})_i = -\sum_j \mathbf{c}_{ij} \cdot \mathbf{F}_j = -\sum_{j \neq i} \mathbf{c}_{ij} \cdot (\mathbf{F}_j - \mathbf{F}_i)$$

due to the fact that $\sum_j \varphi_j \equiv 1 \quad \Rightarrow \quad \mathbf{c}_{ii} = -\sum_{j \neq i} \mathbf{c}_{ij}$

Roe averaging $\quad \mathbf{F}_j - \mathbf{F}_i = \hat{\mathbf{A}}_{ij}(\mathrm{U}_j - \mathrm{U}_i)$ $\quad$ where $\quad \hat{\mathbf{A}}_{ij} = \mathbf{A}(\hat{\rho}_{ij}, \hat{\mathbf{v}}_{ij}, \hat{H}_{ij})$

$$\hat{\rho}_{ij} = \sqrt{\rho_i \rho_j}, \qquad \hat{\mathbf{v}}_{ij} = \frac{\sqrt{\rho_i}\mathbf{v}_i + \sqrt{\rho_j}\mathbf{v}_j}{\sqrt{\rho_i} + \sqrt{\rho_j}}, \qquad \hat{H}_{ij} = \frac{\sqrt{\rho_i}H_i + \sqrt{\rho_j}H_j}{\sqrt{\rho_i} + \sqrt{\rho_j}}$$

Quasi-linear formulation

$$(K\mathrm{U})_i = -\sum_{j \neq i} \mathbf{c}_{ij} \cdot \hat{\mathbf{A}}_{ij}(\mathrm{U}_j - \mathrm{U}_i) = \sum_{j \neq i} (\mathrm{A}_{ij} + \mathrm{B}_{ij})(\mathrm{U}_j - \mathrm{U}_i)$$

Cumulative Roe matrices

$$\mathrm{A}_{ij} = \mathbf{a}_{ij} \cdot \hat{\mathbf{A}}_{ij} = -\mathrm{A}_{ji}, \qquad \mathbf{a}_{ij} = 0.5\,(\mathbf{c}_{ij} - \mathbf{c}_{ji})$$

$$\mathrm{B}_{ij} = \mathbf{b}_{ij} \cdot \hat{\mathbf{A}}_{ij} = \quad \mathrm{B}_{ji}, \qquad \mathbf{b}_{ij} = 0.5\,(\mathbf{c}_{ij} + \mathbf{c}_{ji})$$

Contribution of edge $\vec{ij}$

$$(\mathrm{A}_{ij} + \mathrm{B}_{ij})(\mathrm{U}_i - \mathrm{U}_j) \quad \longrightarrow \quad (K\mathrm{U})_i$$

$$(\mathrm{A}_{ij} - \mathrm{B}_{ij})(\mathrm{U}_i - \mathrm{U}_j) \quad \longrightarrow \quad (K\mathrm{U})_j$$

Galerkin Matrix Assembly

Edge contribution to the operator K

$$\mathrm{K}_{ii} = \mathrm{A}_{ij} + \mathrm{B}_{ij} \qquad \mathrm{K}_{ij} = -\mathrm{A}_{ij} - \mathrm{B}_{ij}$$

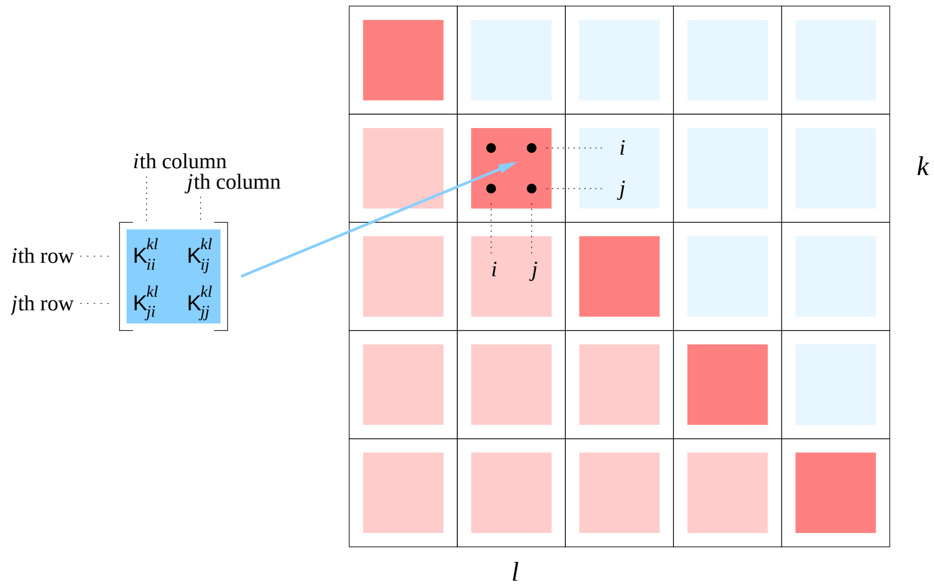
$$\mathrm{K}_{ji} = \mathrm{A}_{ij} - \mathrm{B}_{ij} \qquad \mathrm{K}_{jj} = -\mathrm{A}_{ij} + \mathrm{B}_{ij}$$

Edge contribution to the operator L

$$\mathrm{L}_{ii} = \mathrm{A}_{ij} - \mathrm{D}_{ij} \qquad \mathrm{L}_{ij} = -\mathrm{A}_{ij} + \mathrm{D}_{ij}$$

$$\mathrm{L}_{ji} = \mathrm{A}_{ij} + \mathrm{D}_{ij} \qquad \mathrm{L}_{jj} = -\mathrm{A}_{ij} - \mathrm{D}_{ij}$$

Structure of the global matrix



Raw antidiffusive flux

$$\mathrm{F}_{ij} = -[\mathrm{M}_{ij}\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{D}_{ij} + \mathrm{B}_{ij}](\mathrm{U}_j - \mathrm{U}_i), \qquad \mathrm{F}_{ji} = -\mathrm{F}_{ij}$$

where $\mathrm{M}_{ij} = m_{ij}\mathrm{I}$ is a block of M_C and D_{ij} is the tensorial artificial diffusion.

Remark: Depending on the solution strategy (segregated/coupled),
only ‘a few’ blocks of the global matrix need to be assembled and stored.

Design of Artificial Viscosities

LED principle for systems: (semi-discrete level)

Render all off-diagonal matrix blocks L_{ij} positive semi-definite

Characteristic decomposition $\boxed{\mathrm{A}_{ij} = |\mathbf{a}_{ij}|\, R_{ij}\Lambda_{ij}R_{ij}^{-1}}$ $\qquad |\mathbf{a}_{ij}| = \sqrt{\mathbf{a}_{ij}\cdot\mathbf{a}_{ij}}$

where R_{ij} is the matrix of right eigenvectors and the eigenvalues of A_{ij}

are given by $\Lambda_{ij} = \mathrm{diag}\{\hat{v}_{ij} - \hat{c}_{ij},\ \hat{v}_{ij},\ \hat{v}_{ij},\ \hat{v}_{ij},\ \hat{v}_{ij} + \hat{c}_{ij}\}$

Characteristic velocities $\quad \hat{v}_{ij} = \dfrac{\mathbf{a}_{ij}}{|\mathbf{a}_{ij}|}\cdot\hat{\mathbf{v}}_{ij}, \qquad \hat{c}_{ij} = \sqrt{(\gamma-1)(\hat{H}_{ij} - 0.5\,|\hat{\mathbf{v}}_{ij}|^2)}$

System upwinding (expensive)

$$\mathrm{D}_{ij} = |\mathrm{A}_{ij}| = |\mathbf{a}_{ij}|\, R_{ij}|\Lambda_{ij}|R_{ij}^{-1}$$

Generalization of Roe's approximate Riemann solver (1981) to multidimensions

Scalar dissipation (efficient)

$$\mathrm{D}_{ij} = d_{ij}\mathrm{I} \quad \text{where} \quad d_{ij} = |\mathbf{a}_{ij}|\max_i |\lambda_i|$$

Optimal choice for FCT since artificial diffusion is removed by the flux limiter

Iterative Defect Correction

(Preconditioned) defect correction

$$\mathrm{U}^{(m+1)} = \mathrm{U}^{(m)} + [A(\mathrm{U}^{(m)})]^{-1}\mathrm{R}^{(m)}, \quad m = 0, 1, \ldots$$

$$\mathrm{R}^{(m)} = \mathrm{B}^n - A(\mathrm{U}^{(m)})\,\mathrm{U}^{(m)} + \mathrm{F}(\mathrm{U}^{(m)}, \mathrm{U}^n)$$

Practical implementation

$$A(\mathrm{U}^{(m)})\,\Delta\mathrm{U}^{(m+1)} = \mathrm{R}^{(m)}, \quad m = 0, 1, \ldots$$

$$\mathrm{U}^{(m+1)} = \mathrm{U}^{(m)} + \Delta\mathrm{U}^{(m+1)}, \quad \mathrm{U}^{(0)} = \mathrm{U}^n$$

How to solve this system?

$$\begin{bmatrix} A_{11}^{(m)} & A_{12}^{(m)} & A_{13}^{(m)} & A_{14}^{(m)} & A_{15}^{(m)} \\ A_{21}^{(m)} & A_{22}^{(m)} & A_{23}^{(m)} & A_{24}^{(m)} & A_{25}^{(m)} \\ A_{31}^{(m)} & A_{32}^{(m)} & A_{33}^{(m)} & A_{34}^{(m)} & A_{35}^{(m)} \\ A_{41}^{(m)} & A_{42}^{(m)} & A_{43}^{(m)} & A_{44}^{(m)} & A_{45}^{(m)} \\ A_{51}^{(m)} & A_{52}^{(m)} & A_{53}^{(m)} & A_{54}^{(m)} & A_{55}^{(m)} \end{bmatrix} \begin{bmatrix} \Delta\mathrm{U}_1^{(m+1)} \\ \Delta\mathrm{U}_2^{(m+1)} \\ \Delta\mathrm{U}_3^{(m+1)} \\ \Delta\mathrm{U}_4^{(m+1)} \\ \Delta\mathrm{U}_5^{(m+1)} \end{bmatrix} = \begin{bmatrix} \mathrm{R}_1^{(m)} \\ \mathrm{R}_2^{(m)} \\ \mathrm{R}_3^{(m)} \\ \mathrm{R}_4^{(m)} \\ \mathrm{R}_5^{(m)} \end{bmatrix}$$

Block-Jacobi method $\quad A_{kl}^{(m)} := \delta_{kl} A_{kl}^{(m)}$

$$A_{kk}^{(m)}\Delta\mathrm{U}_k^{(m+1)} = \mathrm{R}_k^{(m)}, \qquad k = 1, \ldots, 5$$

$$\mathrm{U}_k^{(m+1)} = \mathrm{U}_k^{(m)} + \Delta\mathrm{U}_k^{(m+1)}, \quad \mathrm{U}_k^{(0)} = \mathrm{U}_k^n$$

- $\oplus$ only 5 blocks need to be assembled and stored
- $\oplus$ equations can be solved separately or in parallel
- $\ominus$ poor/no convergence for large time steps due to increasing stiffness of the equations for Mach numbers near 0 or 1 $\longrightarrow$ local preconditioning

Implementation of Characteristic Boundary Conditions

Numbers of PBC *vs.* NBC $N_v = N_p + N_n$ depend on the local Mach number $M = |v_n|/c$

```
For all x_i ∈ Γ do
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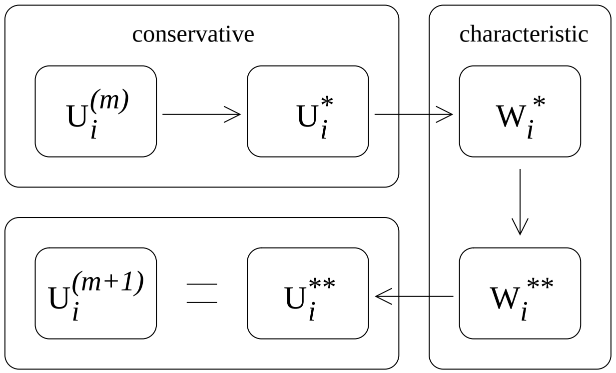
For all $\mathbf{x}_i \in \Gamma$ do

1. Nullify $a_{ij}^{kl} := 0 \quad \forall j \neq i, \ \forall l \neq k$
   and update $\mathrm{U}_i^* = \mathrm{U}_i^{(m)} + \mathrm{diag}\{\mathrm{A}_{ii}^{-1}\}\mathrm{R}_i^{(m)}$
2. Transform U_i^* into W_i^* and apply PBC
   for the incoming Riemann invariants
3. Transform W_i^{**} to U_i^{**} and nullify $\mathrm{R}_i^{(m)} := 0$

N_p/N_n	1D	2D	3D
subsonic inflow	2/1	3/1	4/1
subsonic outflow	3/0	4/0	5/0
supersonic inflow	1/2	1/4	1/4
supersonic outflow	0/3	0/4	0/5

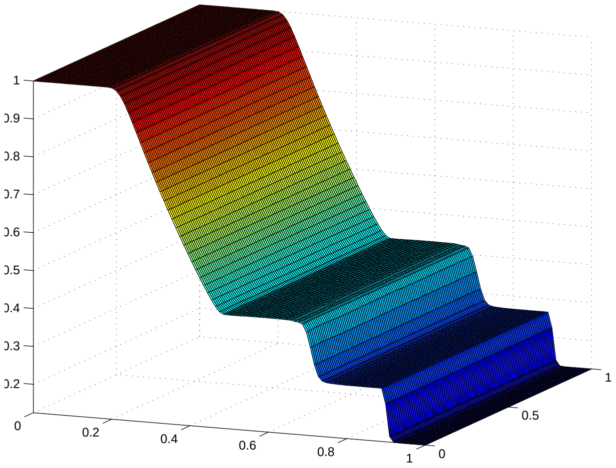
Variable transformations



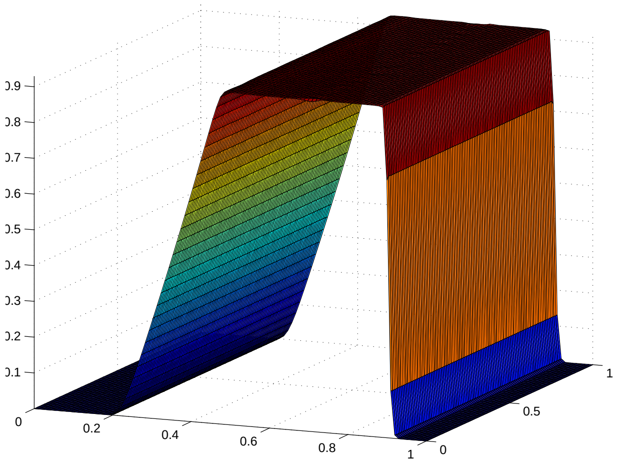
- Values U_i^{**} represent Dirichlet boundary conditions for the end-of-step solution $\mathrm{U}_i^{(m+1)}$
- No *ad hoc* extrapolation of data from the interior
- Easy to implement as a 'black-box' module

Shock Tube Problem

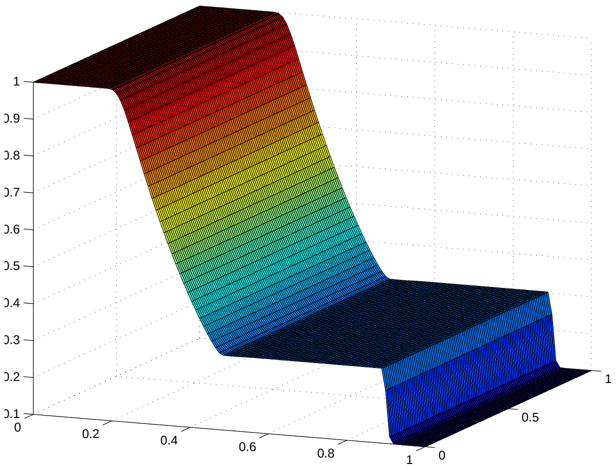
density



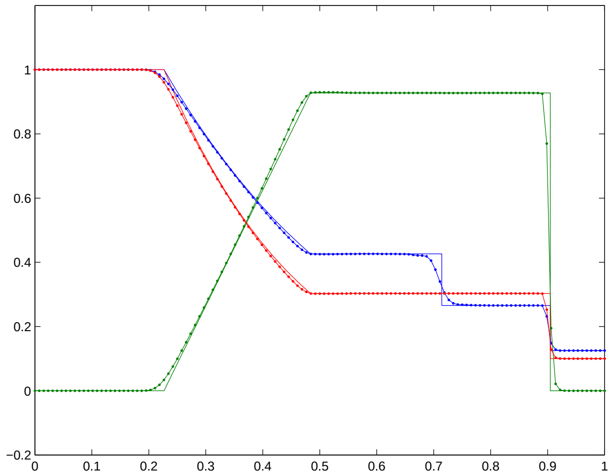
velocity



pressure



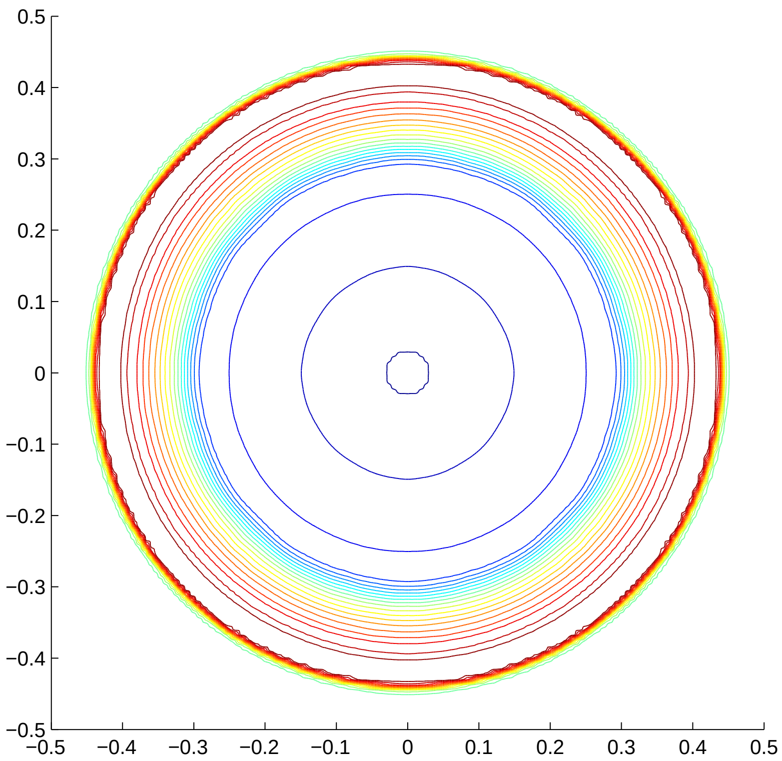
cutplane $y = 0.5$



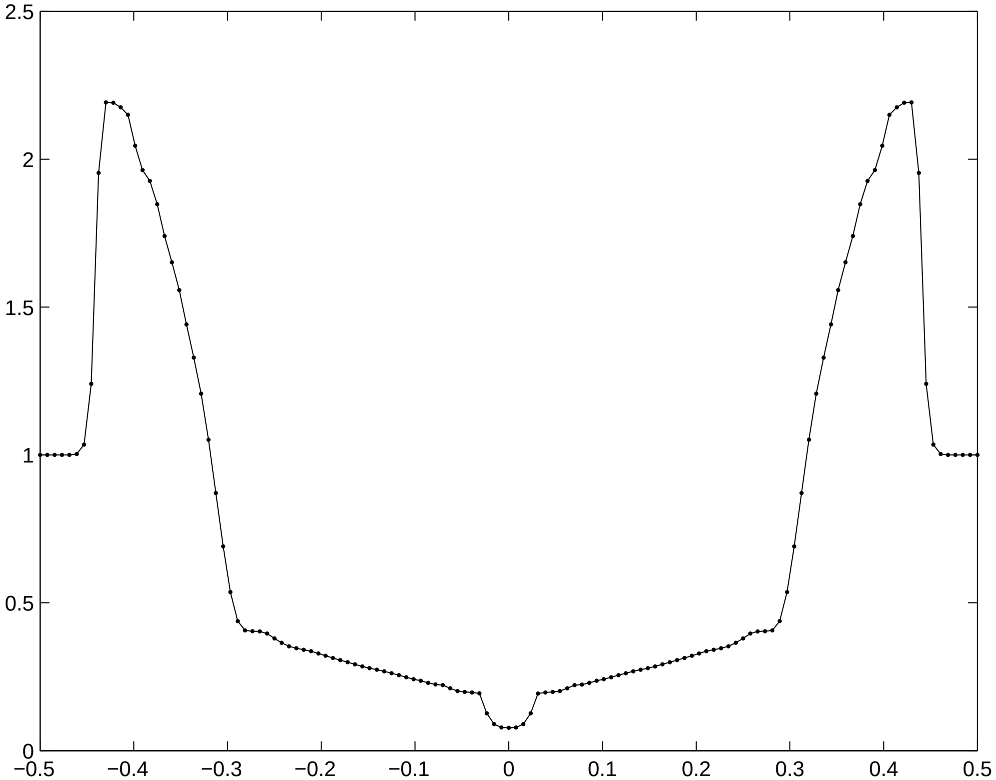
Crank-Nicolson time-stepping, $\Delta t = 10^{-3}$, $16,384$ Q_1 elements at $t = 0.231$

Radially Symmetric Riemann Problem

density

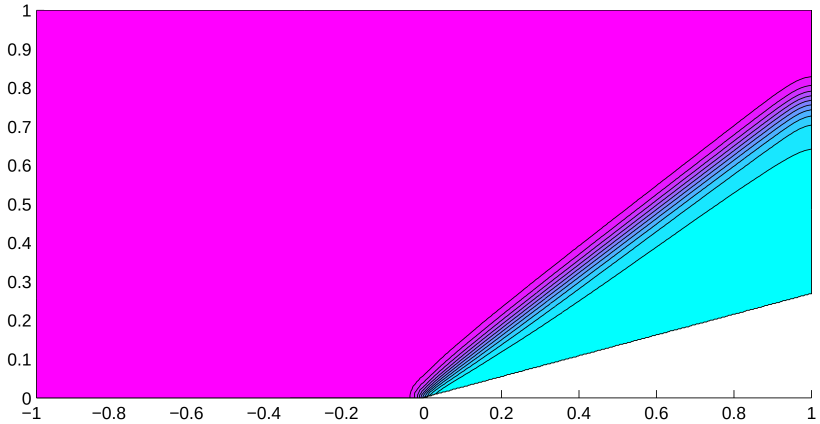


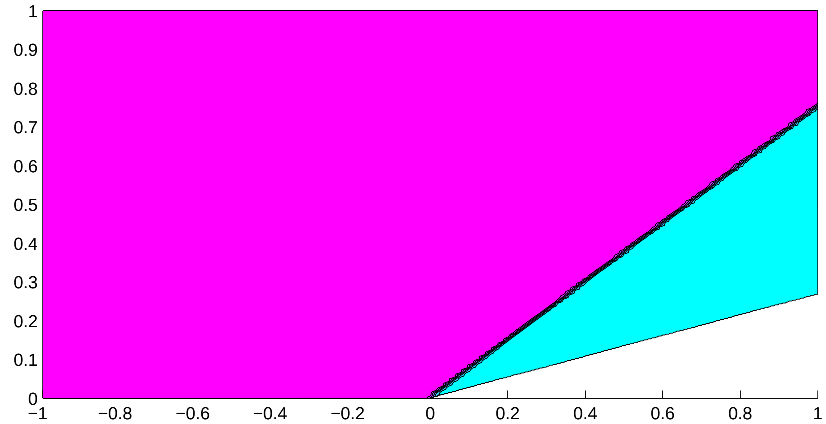
cutplane $y = 0$



Crank-Nicolson time-stepping, $\Delta t = 10^{-3}$, $16,384$ Q_1 elements at $t = 0.13$

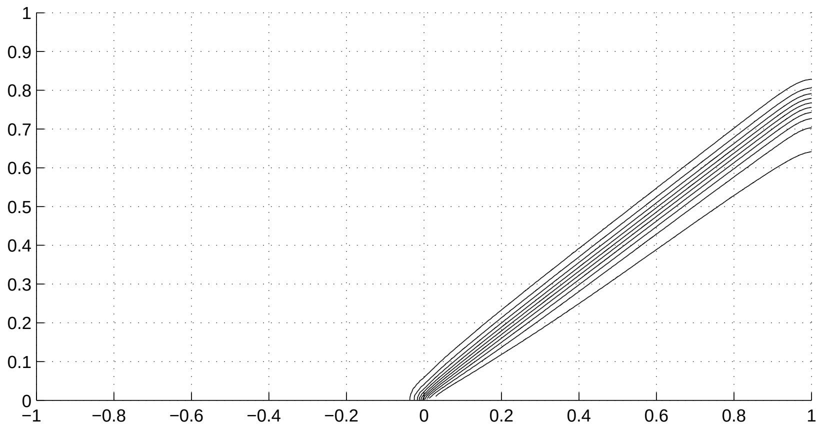
Compression Corner $M_\infty = 2.5,\ \theta = 15^\circ$

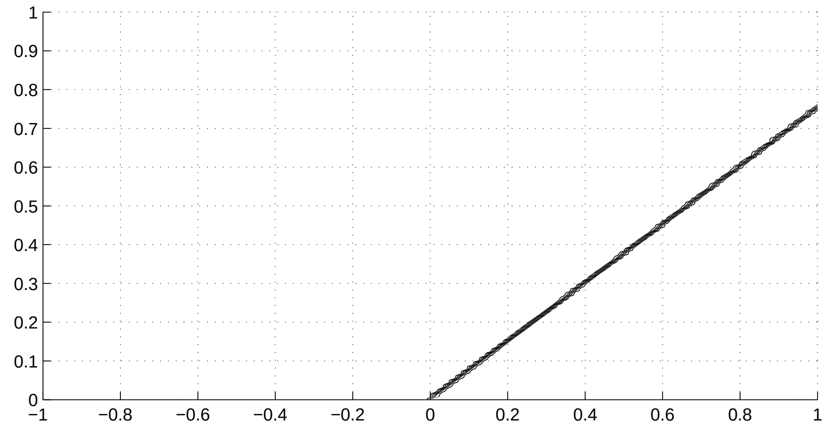




Low-order method , scalar dissipation

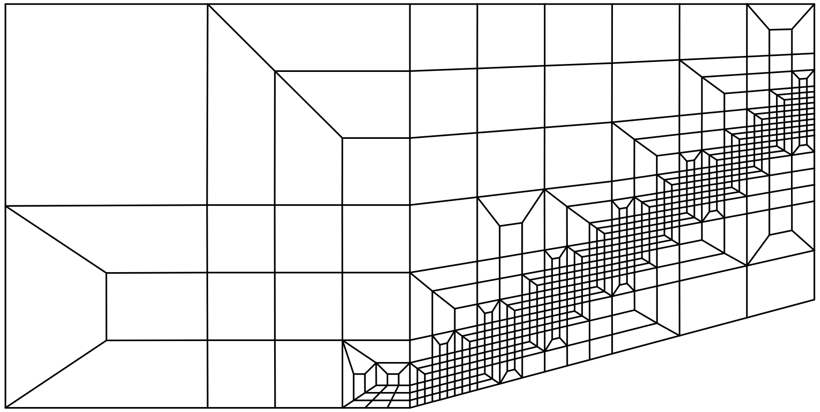
FEM-FCT, $\alpha = \min\{\alpha_\rho, \alpha_E\}$

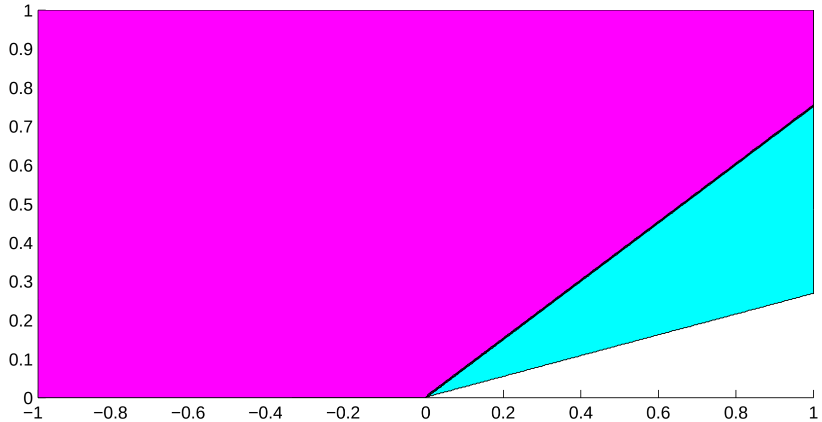




Backward Euler time-stepping, $\Delta t = 10^{-2}$, $16,384$ Q_1 elements

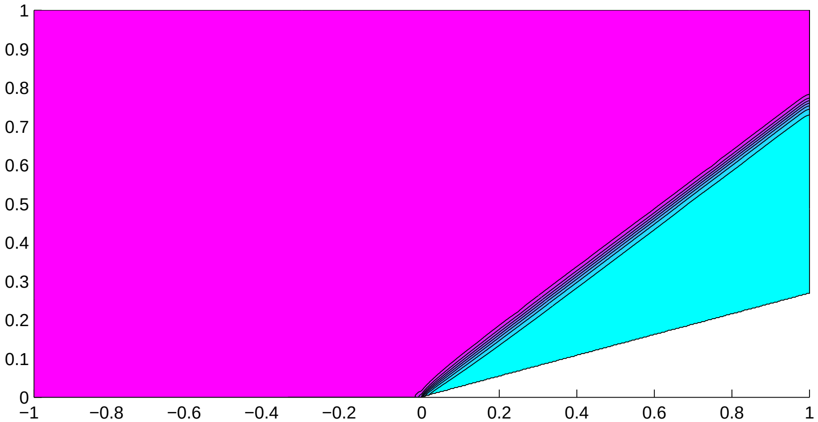
Compression Corner $M_\infty = 2.5,\ \theta = 15^\circ$

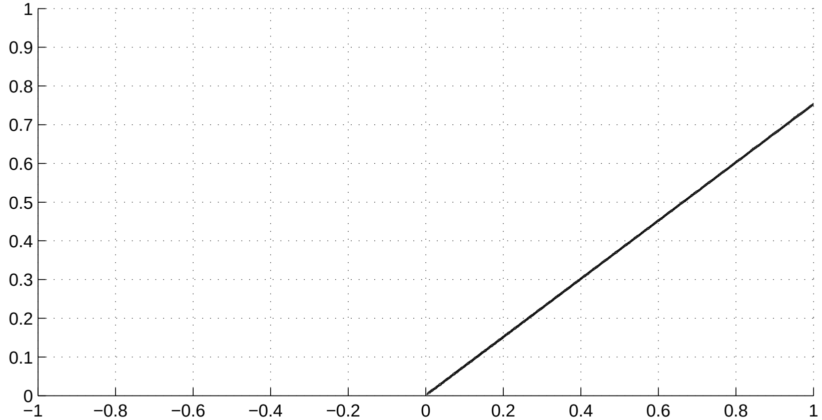




Low-order method, scalar dissipation

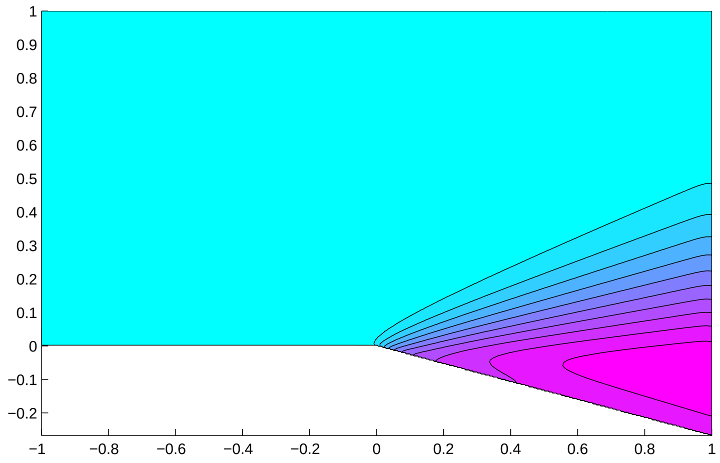
FEM-FCT, $\alpha = \min\{\alpha_\rho, \alpha_E\}$

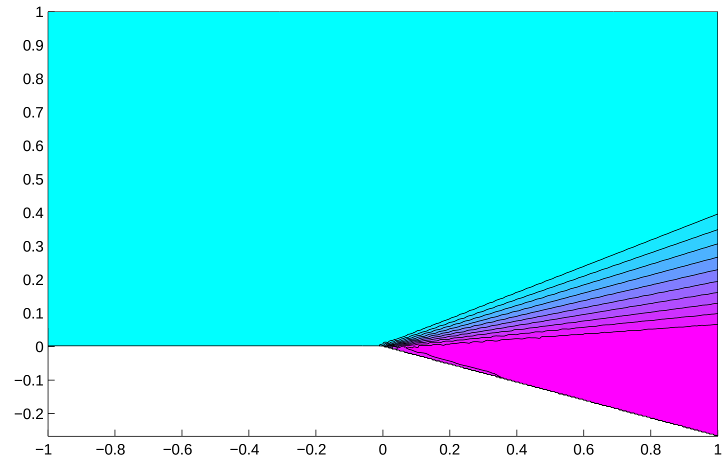




Backward Euler time-stepping, $\Delta t = 10^{-2}$, $10,016$ Q_1 elements

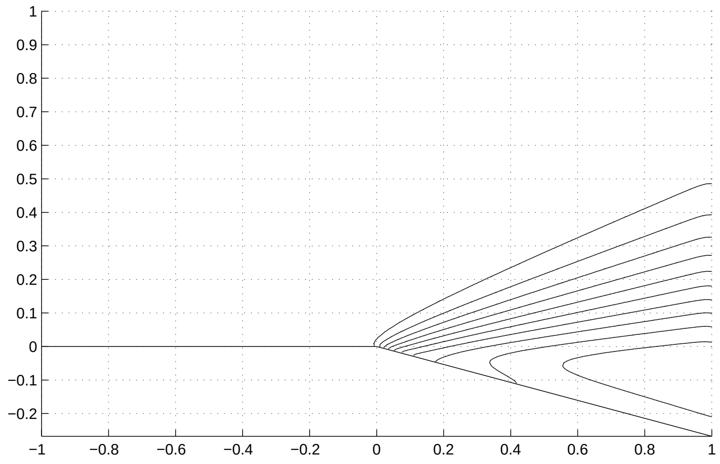
Prandtl-Meyer Expansion $M_\infty = 2.5,\ \theta = 15^\circ$

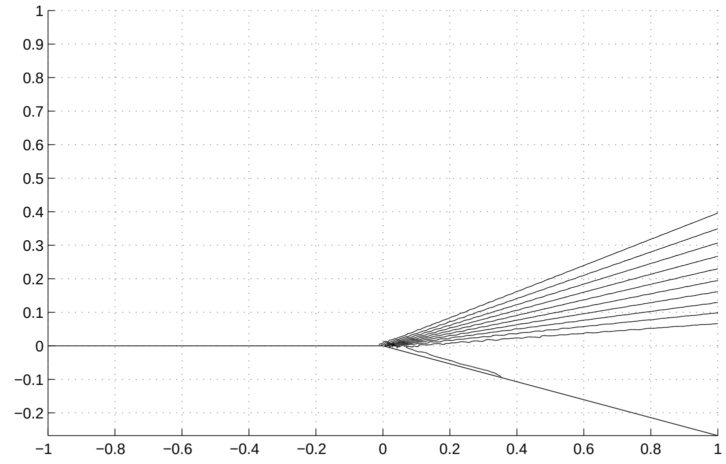




Low-order method, scalar dissipation

FEM-FCT, $\alpha = \min\{\alpha_\rho, \alpha_E\}$





Backward Euler time-stepping, $\Delta t = 10^{-2}$, $16,384$ Q_1 elements

Algebraic Flux Correction: FCT *vs.* TVD

- Both node-oriented flux limiters act at the algebraic level which makes them applicable to 'arbitrary' discretizations in time (explicit/implicit) and space (FD/FV/FE)

- Fully discrete FCT-type schemes, which are positivity preserving by construction, are to be recommended for time dependent problems

- TVD-type methods are preferable for the treatment of stationary flows. They are derived at the semi-discrete level but preserve positivity only upon convergence

Outstanding tasks

- Extension of the methodology to higher-order finite elements

- Robust and efficient iterative solvers for nonsymmetric algebraic systems

- Understanding of clipping and terracing phenomina

- Adaptive grid refinement for flux limiting schemes