Implicit Finite Element Discretizations based on the Flux-Corrected Transport Algorithm

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Design Criteria

Algebraic Constraint I (semi-discrete level)

Jameson (1993) showed that if all coefficients $\sigma_{ij} \geq 0$ then

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \sum_{j\neq i} \sigma_{ij}(u_j - u_i)$$

is **local extremum diminishing** due to the fact that

- maxima do not increase:
$$u_i = \max_j u_j \quad\Rightarrow\quad u_j - u_i \leq 0 \quad\Rightarrow\quad \frac{\mathrm{d}u_i}{\mathrm{d}t} \leq 0,$$
- minima do not decrease:
$$u_i = \min_j u_j \quad\Rightarrow\quad u_j - u_i \geq 0 \quad\Rightarrow\quad \frac{\mathrm{d}u_i}{\mathrm{d}t} \geq 0.$$

Algebraic Constraint II (fully discrete level)

A fully discrete scheme of the form

$$Au^{n+1} = Bu^n, \quad u^n \geq 0$$

is **positivity-preserving** if A is a so-called M-matrix and all entries of B are non-negative.

Tool: *Discrete diffusion operators* $D = \{d_{ij}\}$

are defined as symmetric matrices with zero row and column sums such that diffusive terms can be decomposed into a sum of antisymmetric fluxes $f_{ji} = -f_{ij}$

$$(Du)_i = \sum_{j\neq i} f_{ij} \quad \text{where} \quad f_{ij} = d_{ij}(u_j - u_i).$$

Discrete Upwinding

Consider a lumped-mass Galerkin discretization

$$M_L \frac{\mathrm{d}u}{\mathrm{d}t} = Ku \quad\Leftrightarrow\quad m_i \frac{\mathrm{d}u_i}{\mathrm{d}t} = \sum_{j\neq i} k_{ij}(u_j - u_i).$$

Compute optimal diffusion coefficients

$$d_{ii} = -\sum_{j\neq i} d_{ij}, \quad d_{ij} = \max\{0, -k_{ij}, -k_{ji}\} = d_{ji}.$$

Initialize the low-order operator $L := K$ and eliminate negative off-diagonal entries according to AC–I edge-by-edge

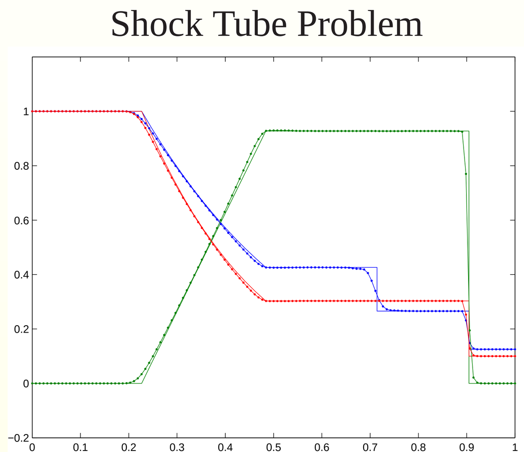
$$l_{ii} := l_{ii} - d_{ij}, \quad l_{ij} := l_{ij} + d_{ij},$$
$$l_{ji} := l_{ji} + d_{ij}, \quad l_{jj} := l_{jj} - d_{ij}.$$

The resulting linear low-order scheme is given by

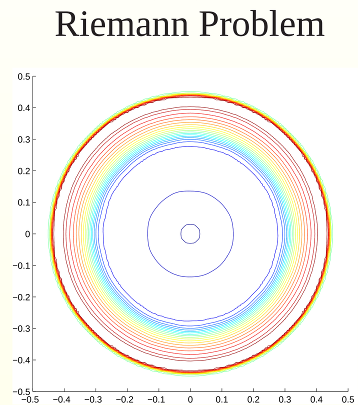
$$M_L \frac{\mathrm{d}u}{\mathrm{d}t} = Lu \quad \text{such that} \quad l_{ij} \geq 0, \forall j \neq i.$$

Benchmarks for Compressible Flows

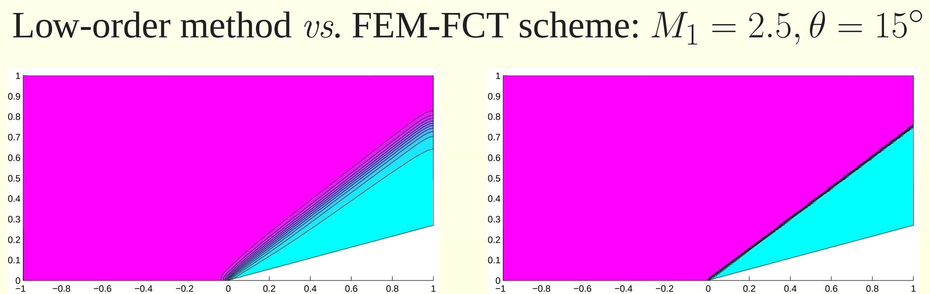
Shock Tube Problem



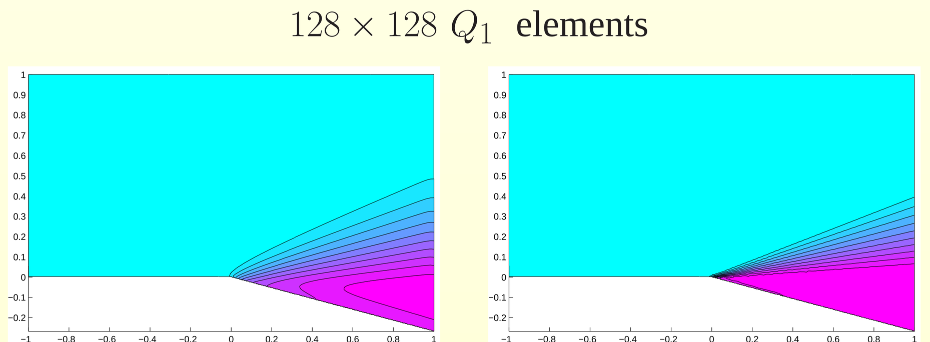
Radially symmetric Riemann Problem



Low-order method *vs.* FEM-FCT scheme: $M_1 = 2.5, \theta = 15°$



128×128 Q_1 elements



Algebraic Flux Correction of FCT Type

Scalar conservation law $\quad \frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u) = 0 \quad \text{in } \Omega.$

1. Linear high-order scheme (e.g. Galerkin FEM)
$$M_C \frac{u^{n+1} - u^n}{\Delta t} = \theta K u^{n+1} + (1-\theta) K u^n, \quad \exists\, j \neq i: \; k_{ij} < 0$$

2. Linear low-order scheme $\quad L = K + D$
$$M_L \frac{u^{n+1} - u^n}{\Delta t} = \theta L u^{n+1} + (1-\theta) L u^n, \quad l_{ij} \geq 0, \forall j \neq i$$

3. Nonlinear high-resolution scheme $\quad f_i = \sum_{j\neq i} f_{ij}^a$
$$M_L \frac{u^{n+1} - u^n}{\Delta t} = \theta L u^{n+1} + (1-\theta) L u^n + f(u^{n+1}, u^n)$$

Equivalent representation $\quad Au^{n+1} = B(\tilde{u})\tilde{u}, \quad$ where $A = M_L - \theta \Delta t L$ is an M-matrix and $b_{ij} \geq 0, \forall i, j$

FEM-FCT Algorithm for the Euler Equations

Conservative variables $U = [\rho, \rho\mathbf{v}, \rho E]^T$ and fluxes $\mathbf{F} = [\rho\mathbf{v}, \rho\mathbf{v} \otimes \mathbf{v} + p\mathcal{I}, \rho H \mathbf{v}]^T$

Divergence form
$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = \mathbf{0}$$

Quasi-linear form
$$\frac{\partial U}{\partial t} + \mathbf{A} \cdot \nabla U = \mathbf{0}$$

Group finite element formulation

$$M_C \frac{\mathrm{dU}}{\mathrm{d}t} = K\mathrm{U}, \text{ where } (K\mathrm{U})_i = -\sum_j \mathbf{c}_{ij} \cdot \mathbf{F}_j = -\sum_{j\neq i} \mathbf{c}_{ij} \cdot (\mathbf{F}_j - \mathbf{F}_i).$$

Roe averaging $\mathbf{F}_j - \mathbf{F}_i = \hat{\mathbf{A}}_{ij}(\mathrm{U}_j - \mathrm{U}_i)$, where $\hat{\mathbf{A}}_{ij} = \mathbf{A}(\hat{\rho}_{ij}, \hat{\mathbf{v}}_{ij}, \hat{H}_{ij})$

$$\hat{\rho}_{ij} = \sqrt{\rho_i \rho_j}, \quad \mathbf{v}_{ij} = \frac{\sqrt{\rho_i}\mathbf{v}_i + \sqrt{\rho_j}\mathbf{v}_j}{\sqrt{\rho_i} + \sqrt{\rho_j}}, \quad \hat{H}_{ij} = \frac{\sqrt{\rho_i}H_i + \sqrt{\rho_j}H_j}{\sqrt{\rho_i} + \sqrt{\rho_j}}$$

Contribution of the edge $\overrightarrow{ij}$

$$(K\mathrm{U})_i \longleftarrow \mathbf{c}_{ij} \cdot (\mathbf{F}_i - \mathbf{F}_j) = \mathbf{c}_{ij} \cdot \hat{\mathbf{A}}_{ij}(\mathrm{U}_i - \mathrm{U}_j),$$
$$(K\mathrm{U})_j \longleftarrow \mathbf{c}_{ji} \cdot (\mathbf{F}_j - \mathbf{F}_i) = \mathbf{c}_{ji} \cdot \hat{\mathbf{A}}_{ij}(\mathrm{U}_j - \mathrm{U}_i).$$

Cumulative Roe matrices

$$\mathrm{A}_{ij} = \mathbf{a}_{ij} \cdot \hat{\mathbf{A}}_{ij}, \quad \mathbf{a}_{ij} = \tfrac{1}{2}(\mathbf{c}_{ij} - \mathbf{c}_{ji})$$
$$\mathrm{B}_{ij} = \mathbf{b}_{ij} \cdot \hat{\mathbf{A}}_{ij}, \quad \mathbf{b}_{ij} = \tfrac{1}{2}(\mathbf{c}_{ij} + \mathbf{c}_{ji})$$

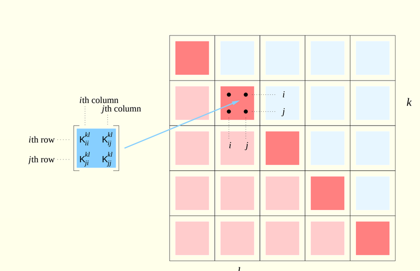
Contribution of the edge $\overrightarrow{ij}$

$$(K\mathrm{U})_i \longleftarrow (\mathrm{A}_{ij} + \mathrm{B}_{ij})(\mathrm{U}_i - \mathrm{U}_j),$$
$$(K\mathrm{U})_j \longleftarrow (\mathrm{A}_{ij} - \mathrm{B}_{ij})(\mathrm{U}_i - \mathrm{U}_j)$$

– Matrix Assembly –

Assembly of the operator K

$$\mathrm{K}_{ii} = \mathrm{A}_{ij} + \mathrm{B}_{ij}, \qquad \mathrm{K}_{ij} = -\mathrm{A}_{ij} - \mathrm{B}_{ij},$$
$$\mathrm{K}_{ji} = \mathrm{A}_{ij} - \mathrm{B}_{ij}, \qquad \mathrm{K}_{jj} = -\mathrm{A}_{ij} + \mathrm{B}_{ij}.$$



Assembly of the operator L

$$\mathrm{L}_{ii} = \mathrm{A}_{ij} - \mathrm{D}_{ij}, \qquad \mathrm{L}_{ij} = -\mathrm{A}_{ij} + \mathrm{D}_{ij},$$
$$\mathrm{L}_{ji} = \mathrm{A}_{ij} + \mathrm{D}_{ij}, \qquad \mathrm{L}_{jj} = -\mathrm{A}_{ij} - \mathrm{D}_{ij}.$$

Raw antidiffusive flux incorporating the symmetric boundary part

$$\mathrm{F}_{ij} = -\left(\mathrm{M}_{ij}\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{D}_{ij} + \mathrm{B}_{ij}\right)(\mathrm{U}_j - \mathrm{U}_i), \qquad \mathrm{F}_{ji} = -\mathrm{F}_{ij}.$$

– Artificial Viscosities –

LED principle for systems (semi-discrete level)

Render all off-diagonal blocks L_{ij} positive semi-definite.

Factorization $\mathrm{A}_{ij} = |\mathbf{a}_{ij}|\,\mathrm{R}_{ij}\Lambda_{ij}\mathrm{R}_{ij}^{-1}$, $\quad |\mathbf{a}_{ij}| = \sqrt{\mathbf{a}_{ij} \cdot \mathbf{a}_{ij}}$ where $\Lambda_{ij} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_5\}$ and R_{ij} is the matrix of right eigenvectors.

System upwinding $\quad \mathrm{D}_{ij} = |\mathrm{A}_{ij}| = |\mathbf{a}_{ij}|\,\mathrm{R}_{ij}|\Lambda_{ij}|\mathrm{R}_{ij}^{-1}$
Generalization of Roe's approximate Riemann solver

Scalar dissipation $\quad \mathrm{D}_{ij} = d_{ij}\mathrm{I}, \quad$ where $\quad d_{ij} = |\mathbf{a}_{ij}| \max_i |\lambda_i|$
Optimal for FCT as excessive artificial diffusion is removed by the limiter.

– Flux Limiting for Systems –

Synchronization of correction factors (Löhner, 1987)

- Evaluate correction factor for a single 'indicator variable'.
- Use the minimum of correction factors for a group of variables.
- Perform conservative limiting making use of local variable transformations.

FEM-FCT Algorithm

Nonlinear system for an implicit time discretization

$$M_L \frac{u^{n+1} - u^n}{\Delta t} = \theta L u^{n+1} + (1-\theta) L u^n + f(u^{n+1}, u^n)$$

Successive approximations

$$Au^{(m+1)} = b^{(m+1)} \quad \text{where} \quad b^{(m+1)} = b^n + f(u^{(m)}, u^n)$$

Preconditioner *(M-matrix)*
$$A = M_L - \theta \Delta t L$$

Low-order contribution
$$b^n = [M_L + (1-\theta)\Delta t L]u^n$$

Raw antidiffusive flux

$$f_{ij}^{(m)} = [m_{ij} - (1-\theta)\Delta t d_{ij}^n](u_j^n - u_i^n) - [m_{ij} + \theta \Delta t d_{ij}^{(m)}](u_j^{(m)} - u_i^{(m)})$$

Limited antidiffusion

$$f_i = \sum_{j\neq i} \alpha_{ij}^{(m)} f_{ij}^{(m)}$$

Basic FEM-FCT Algorithm

Start with $u^n \geq 0$ and compute $\tilde{u}^n = M_L^{-1} b^n$ which proves to be positivity-preserving for $0 \leq \theta < 1$ provided that

$$\Delta t \leq \frac{1}{1-\theta} \min_i \{-m_i / l_{ii} | l_{ii} < 0\}.$$

The flux limiter should guarantee the existence of $B = \{b_{ij}\}$ such that $b^{(m+1)} = B(\tilde{u}^n)\tilde{u}^n$ and $b_{ij} \geq 0$, $\forall i, j$.

Modify right-hand side $\quad \boxed{b_i^{(m+1)} = b_i^n + \sum_{j\neq i} \alpha_{ij}^{(m)} f_{ij}^{(m)}}$

Iterative FEM-FCT Algorithm

Build the already accepted antidiffusion into $\tilde{u}^{(m)} = M_L^{-1} b^{(m)}$ and limit only the rejected portion of the antidiffusive flux

$$\Delta f_{ij}^{(m)} = f_{ij}^{(m)} - g_{ij}^{(m)}, \quad g_{ij}^{(0)} = 0.$$

Update the amount of accepted antidiffusion

$$g_{ij}^{(m+1)} = g_{ij}^{(m)} + \alpha_{ij}^{(m)} \Delta f_{ij}^{(m)}.$$

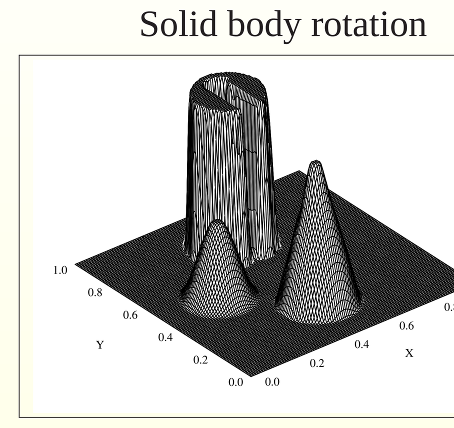
Modify right-hand side $\quad \boxed{b_i^{(m+1)} = b_i^{(m)} + \sum_{j\neq i} \alpha_{ij}^{(m)} \Delta f_{ij}^{(m)}}$

Zalesak's Multidimensional Limiter

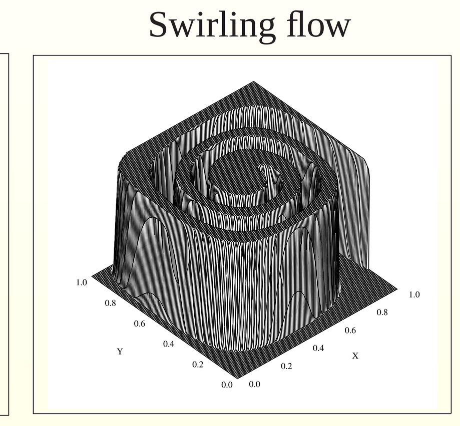
1. Sum of positive / negative antidiffusive fluxes:
$$P_i = P_i^+ + P_i^-, \quad P_i^{\pm} = \frac{1}{m_i} \sum_{j\neq i} \max_{\min}\{0, f_{ij}\}.$$
2. Maximum / minimum admissible increment:
$$Q_i^{\pm} = \max_{\min} \Delta u_{ij}^{\pm}, \quad \text{where} \quad \Delta u_{ij}^{\pm} = \max_{\min}\{0, \tilde{u}_j - \tilde{u}_i\}.$$
3. Nodal correction factors:
$$R_i^{\pm} = \begin{cases} \min\{1, m_i Q_i^{\pm}/P_i^{\pm}\}, & \text{if } P_i^{\pm} \neq 0, \\ 0, & \text{if } P_i^{\pm} = 0. \end{cases}$$
4. Final correction factors:
$$\alpha_{ji} = \alpha_{ij} = \begin{cases} \min\{R_i^+, R_j^-\}, & \text{if } f_{ij} \geq 0, \\ \min\{R_i^+, R_j^-\}, & \text{if } f_{ij} < 0. \end{cases}$$

Scalar Benchmarks

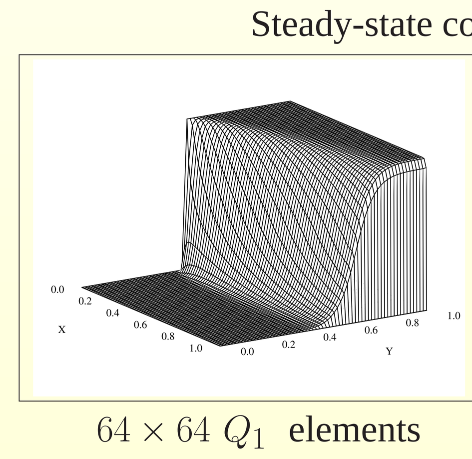
Solid body rotation



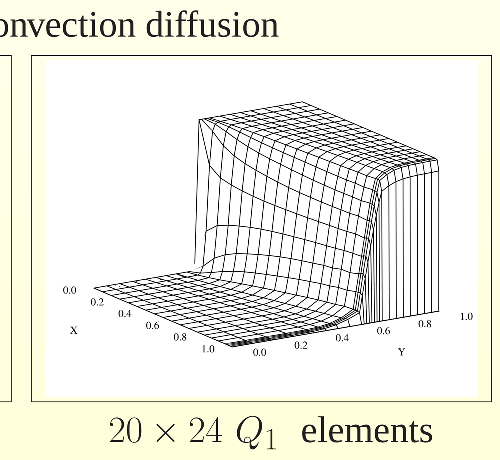
Swirling flow



Steady-state convection diffusion



64×64 Q_1 elements



20×24 Q_1 elements