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Newton-Multigrid Least-Squares FEM for V-V-P and S-V-P Formulations of the Navier-Stokes Equations

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Least-Squares FEM

Advantages

- The discrete linear system is symmetric and positive definite
- The FE spaces are not subject to the LBB stability condition
- No stabilization technique is required

Challenges

- Lack of local mass conservation
- Discontinuities of the pressure
- Imposing the Natural B.Cs. ("$\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0$" at outflow)
- Fast Newton-Multigrid solver

Navier-Stokes equations

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Conservation of mass and momentum

$$\begin{cases} \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p - \nu \triangle \boldsymbol{u} = f & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g}_D & \text{on } \Gamma_D \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{g}_N & \text{on } \Gamma_N \end{cases}$$

along with the zero mean pressure constraint

$$\int_\Omega p = 0$$

Contents

First-order systems of equations

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- Stress-Velocity-Pressure (S-V-P) system of equations

$$\begin{cases} \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nabla \cdot \boldsymbol{\sigma} = f & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{\sigma} + p\mathbf{I} - \nu \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T \right) = 0 & \text{in } \Omega \end{cases}$$

- Velocity-Vorticity-Pressure (V-V-P) system of equations

$$\begin{cases} \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p + \nu \nabla \times \omega = f & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \omega - \nabla \times \boldsymbol{u} = 0 & \text{in } \Omega \end{cases}$$

- B.Cs.

$$\begin{cases} \boldsymbol{u} = \boldsymbol{g}_D & \text{on } \Gamma_D \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{g}_N & \text{on } \Gamma_N \end{cases}$$

Continuous least-squares principle

Least-squares energy functional

$$\mathcal{J}(\tilde{\boldsymbol{v}}; f) = \frac{1}{2} \left\| \mathcal{R}(\tilde{\boldsymbol{v}}) \right\|_0^2$$

Find $\tilde{\boldsymbol{u}} \in \boldsymbol{V}$ such that

$$\tilde{\boldsymbol{u}} = \underset{\tilde{\boldsymbol{v}} \in \boldsymbol{V}}{\operatorname{argmin}} \mathcal{J}(\tilde{\boldsymbol{v}}; f)$$

- S-V-P least-squares energy functional

$$\mathcal{J}_{\text{S-V-P}}(\boldsymbol{\sigma}, \boldsymbol{u}, p; f) = \frac{1}{2} \left(\frac{1}{\nu} \left\| \boldsymbol{\sigma} + p\mathbf{I} - \nu \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T \right) \right\|_0^2 + \alpha \left\| \nabla \cdot \boldsymbol{u} \right\|_0^2 + \left\| \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \cdot \boldsymbol{\sigma} - f \right\|_0^2 \right)$$

- V-V-P least-squares energy functional

$$\mathcal{J}_{\text{V-V-P}}(\omega, \boldsymbol{u}, p; f) = \frac{1}{2} \left(\frac{1}{\nu} \left\| \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p + \nu \nabla \times \omega - f \right\|_0^2 + \alpha \left\| \nabla \cdot \boldsymbol{u} \right\|_0^2 + \left\| \omega - \nabla \times \boldsymbol{u} \right\|_0^2 \right)$$

Newton Linearization

$$\tilde{\boldsymbol{u}}^{k+1} = \tilde{\boldsymbol{u}}^{k} - \left(\left[\frac{\partial \mathcal{R}(\tilde{\boldsymbol{u}}^{k})}{\partial x} \right]^{*} \left[\frac{\partial \mathcal{R}(\tilde{\boldsymbol{u}}^{k})}{\partial x} \right] \right)^{-1} \left[\frac{\partial \mathcal{R}(\tilde{\boldsymbol{u}}^{k})}{\partial x} \right]^{*} \mathcal{R}(\tilde{\boldsymbol{u}}^{k})$$

Indeed,

$$\tilde{\boldsymbol{u}}^{k+1} = \tilde{\boldsymbol{u}}^{k} + \delta \tilde{\boldsymbol{u}}$$

Then,

$$\begin{aligned} \mathcal{R}(\tilde{\boldsymbol{u}}^{k+1}) &= \mathcal{R}(\tilde{\boldsymbol{u}}^{k} + \delta \tilde{\boldsymbol{u}}) \\ &\simeq \mathcal{R}(\tilde{\boldsymbol{u}}^{k}) + \left[\frac{\partial \mathcal{R}(\tilde{\boldsymbol{u}}^{k})}{\partial x} \right] \delta \tilde{\boldsymbol{u}} \end{aligned}$$

The operations are done in the variational formulation !

Variational formulation

Least-squares principle

$$\delta\tilde{\boldsymbol{u}} = \underset{\delta\tilde{\boldsymbol{v}}\in\boldsymbol{V}}{\operatorname{argmin}} \frac{1}{2}\left\| \mathcal{R}(\tilde{\boldsymbol{u}}^k) + \left[\frac{\partial\mathcal{R}(\tilde{\boldsymbol{u}}^k)}{\partial x}\right]\delta\tilde{\boldsymbol{v}} \right\|_0^2$$

Find $\delta\tilde{\boldsymbol{u}} \in \boldsymbol{V}$ such that

$$\left(\mathcal{R}(\tilde{\boldsymbol{u}}^k) + \left[\frac{\partial\mathcal{R}(\tilde{\boldsymbol{u}}^k)}{\partial x}\right]\delta\tilde{\boldsymbol{u}}, \left[\frac{\partial\mathcal{R}(\tilde{\boldsymbol{u}}^k)}{\partial x}\right]\tilde{\boldsymbol{v}}\right) = 0 \quad \forall\tilde{\boldsymbol{v}}$$

Operator form

$$\mathcal{A}(\tilde{\boldsymbol{u}}^k)\delta\tilde{\boldsymbol{u}} = \mathcal{F}(\tilde{\boldsymbol{u}}^k)$$

where

$$\mathcal{A}(\tilde{\boldsymbol{u}}^k) := \left[\frac{\partial\mathcal{R}(\tilde{\boldsymbol{u}}^k)}{\partial x}\right]^* \left[\frac{\partial\mathcal{R}(\tilde{\boldsymbol{u}}^k)}{\partial x}\right], \quad \mathcal{F}(\tilde{\boldsymbol{u}}^k) := -\left[\frac{\partial\mathcal{R}(\tilde{\boldsymbol{u}}^k)}{\partial x}\right]^* \mathcal{R}(\tilde{\boldsymbol{u}}^k)$$

S-V-P formulation

- S-V-P bilinear form $(\mathcal{A}_{\text{S-V-P}}(\boldsymbol{\sigma}, \boldsymbol{u}, p)(\delta\boldsymbol{\sigma}, \delta\boldsymbol{u}, \delta p), (\boldsymbol{\tau}, \boldsymbol{v}, q)) =$

$$\int_\Omega \left(\delta\boldsymbol{\sigma} + \delta p\mathbf{I} - \nu\left(\nabla\delta\boldsymbol{u} + \nabla\delta\boldsymbol{u}^T\right)\right)\left(\boldsymbol{\tau} + q\mathbf{I} - \nu\left(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T\right)\right) d\Omega$$

$$+ \int_\Omega (\nabla\cdot\delta\boldsymbol{u})\,(\nabla\cdot\boldsymbol{v})\; d\Omega$$

$$+ \int_\Omega (\delta\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\delta\boldsymbol{u} + \nabla\cdot\boldsymbol{\sigma})\,(\underbrace{\boldsymbol{v}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{v}}_{C(\boldsymbol{u})\boldsymbol{v}} + \nabla\cdot\boldsymbol{\tau})\, d\Omega$$

- S-V-P operator

$$\mathcal{A}_{\text{S-V-P}}(\boldsymbol{\sigma}, \boldsymbol{u}, p) = \left[\frac{\partial\mathcal{R}_{\text{S-V-P}}(\boldsymbol{\sigma}, \boldsymbol{u}, p)}{\partial x}\right]^* \left[\frac{\partial\mathcal{R}_{\text{S-V-P}}(\boldsymbol{\sigma}, \boldsymbol{u}, p)}{\partial x}\right] =$$

$$\begin{pmatrix} \mathbf{I} - \nabla\nabla\cdot & -\nu(\nabla + \nabla^T) - \nabla C(\boldsymbol{u}) & I \\ \nu\nabla\cdot + C^*(\boldsymbol{u})\nabla\cdot & -\nu^2\nabla\cdot(\nabla + \nabla^T) - \nabla\nabla\cdot + C^*(\boldsymbol{u})C(\boldsymbol{u}) & \nu\nabla\cdot \\ \mathbf{I} & -\nu(\nabla + \nabla^T) & I \end{pmatrix}$$

V-V-P formulation

- V-V-P bilinear form $(\mathcal{A}_{\text{V-V-P}}(\omega, \boldsymbol{u}, p)(\delta\omega, \delta\boldsymbol{u}, \delta p), (\xi, \boldsymbol{v}, q)) =$

$$\int_\Omega (\delta\omega - \nabla\times\delta\boldsymbol{u})\,(\xi - \nabla\times\boldsymbol{v})\,d\Omega + \int_\Omega (\nabla\cdot\delta\boldsymbol{u})\,(\nabla\cdot\boldsymbol{v})\,d\Omega$$

$$+ \int_\Omega (\delta\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\delta\boldsymbol{u} + \nu\nabla\times\delta\omega + \nabla\delta p)$$

$$(\underbrace{\boldsymbol{v}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{v}}_{C(\boldsymbol{u})\boldsymbol{v}} + \nu\nabla\times\xi + \nabla q)\,d\Omega$$

- V-V-P operator

$$\mathcal{A}_{\text{V-V-P}}(\omega, \boldsymbol{u}, p) = \left[\frac{\partial\mathcal{R}_{\text{V-V-P}}(\omega, \boldsymbol{u}, p)}{\partial x}\right]^* \left[\frac{\partial\mathcal{R}_{\text{V-V-P}}(\omega, \boldsymbol{u}, p)}{\partial x}\right] =$$

$$\begin{pmatrix} I - \nu^2\nabla^\perp\nabla\times & -\nabla\times - \nu\nabla^\perp C(\boldsymbol{u}) & -\nu\nabla^\perp\nabla \\ \nabla^\perp + C^*(\boldsymbol{u})\nabla\times & -\nabla^\perp\nabla\times - \nabla\nabla\cdot + C^*(\boldsymbol{u})C(\boldsymbol{u}) & C^*(\boldsymbol{u})\nabla \\ -\nu\nabla\cdot\nabla\times & -\nabla\cdot C(\boldsymbol{u}) & -\nabla\cdot\nabla \end{pmatrix}$$

where $\nabla^\perp = (\nabla\times)^*$ and $\nabla^\perp\nabla = 0$

Variational formulations

- S-V-P least squares variational formulation

$$V_{\text{S-V-P}} := \mathbf{H}_{0,N}(\mathrm{div},\Omega) \times \mathbf{H}^1_{0,D}(\Omega) \times L^2_0(\Omega)$$

$$\begin{cases} \textit{Find } (\boldsymbol{\sigma}, \boldsymbol{u}, p) \in V_{\text{S-V-P}} \quad s.t. \\ (\mathcal{A}_{\text{S-V-P}}(\boldsymbol{\sigma}, \boldsymbol{u}, p), (\boldsymbol{\tau}, \boldsymbol{v}, q)) = \mathcal{F}_{\text{S-V-P}}(\boldsymbol{\tau}, \boldsymbol{v}, q) \end{cases}$$

- V-V-P least squares variational formulation

$$V_{\text{V-V-P}} := H_{0,D}(\mathrm{curl},\Omega) \times \mathbf{H}^1_{0,D}(\Omega) \times H^1(\Omega) \cap L^2_0(\Omega)$$

$$\begin{cases} \textit{Find } (\omega, \boldsymbol{u}, p) \in V_{\text{V-V-P}} \quad s.t. \\ (\mathcal{A}_{\text{V-V-P}}(\omega, \boldsymbol{u}, p), (\xi, \boldsymbol{v}, q)) = \mathcal{F}_{\text{V-V-P}}(\xi, \boldsymbol{v}, q) \end{cases}$$

The S-V-P and V-V-P formulations are continuous and coercive

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- S-V-P least squares variational formulation

$$V^h_{\text{S-V-P}} := \mathbf{H}^{\mathrm{div},h}_{0,N}(\Omega) \times \mathbf{H}^{1,h}_{0,D}(\Omega) \times L^2_0(\Omega)$$

$$\begin{cases} \textit{Find } (\boldsymbol{\sigma}, \boldsymbol{u}, p) \in V_{\text{S-V-P}} \quad s.t. \\ \left(\mathcal{A}^h_{\text{S-V-P}}(\boldsymbol{\sigma}, \boldsymbol{u}, p), (\boldsymbol{\tau}, \boldsymbol{v}, q)\right) = \mathcal{F}^h_{\text{S-V-P}}(\boldsymbol{\tau}, \boldsymbol{v}, q) \end{cases}$$

- V-V-P least squares variational formulation

$$V^h_{\text{V-V-P}} := H^{\mathrm{curl},h}_{0,D}(\Omega) \times \mathbf{H}^{1,h}_{0,D}(\Omega) \times H^{1,h}(\Omega) \cap L^2_0(\Omega)$$

$$\begin{cases} \textit{Find } (\omega, \boldsymbol{u}, p) \in V^h_{\text{V-V-P}} \quad s.t. \\ \left(\mathcal{A}^h_{\text{V-V-P}}(\omega, \boldsymbol{u}, p), (\xi, \boldsymbol{v}, q)\right) = \mathcal{F}^h_{\text{V-V-P}}(\xi, \boldsymbol{v}, q) \end{cases}$$

where $\mathbf{H}^{1,h}(\Omega), \mathbf{H}^{\mathrm{div},h}(\Omega)$ and $H^{\mathrm{curl},h}(\Omega)$ are spaces of element wise H^1, $H(\mathrm{div})$ and $H(\mathrm{curl})$ resp. $\mathcal{A}^h$ is the discrete counterpart of $\mathcal{A}$.

The discrete S-V-P and V-V-P formulations are continuous and coercive

Finite Element approximations

- Least squares formulations allow a free choice of FE spaces
- Use different combinations of FE approximations
  - discontinuous P_0^{dc}, P_1^{dc}
  - H^1-nonconforming $\tilde{Q}_1$ and $\tilde{Q}_2$
  - H^1-conforming Q_1 and Q_2
  - $H(\mathrm{div})$-conforming RT_k

Linear Solver

Least squares variational problems in matrix form

$$\mathcal{A}_{\text{S-V-P}} = \begin{pmatrix} A_{\boldsymbol{\sigma\sigma}} & A_{\boldsymbol{\sigma u}} & A_{\boldsymbol{\sigma} p} \\ A_{\boldsymbol{u\sigma}} & A_{\boldsymbol{uu}} & A_{\boldsymbol{u}p} \\ A_{p\boldsymbol{\sigma}} & A_{p\boldsymbol{u}} & A_{pp} \end{pmatrix}, \quad \mathcal{A}_{\text{S-V-P}} = \begin{pmatrix} A_{\omega\omega} & A_{\omega\boldsymbol{u}} & A_{\omega p} \\ A_{\boldsymbol{u}\omega} & A_{\boldsymbol{uu}} & A_{\boldsymbol{u}p} \\ A_{p\omega} & A_{p\boldsymbol{u}} & A_{pp} \end{pmatrix}$$

- $A_{\boldsymbol{uu}}$ Operator defined in $\mathbf{H}^1(\Omega)$
- $A_{\boldsymbol{\sigma\sigma}}$ Operator defined in $\mathbf{H}(\mathrm{div}, \Omega)$
- $A_{\omega\omega}$ Operator defined in $H(\mathrm{curl}, \Omega)$
- A_{pp} Operator defined in $L^2(\Omega)$ for S-V-P formulation and in $H^1(\Omega)$ for V-V-P formulation

The multigrid method as a preconditioner for the CG (MPCG)

MPCG

The geometric multigrid solver as a preconditioner for the CG (MPCG)

- direct Gauss elimination as coarse-grid solver
- preconditioned CG smoothers (e.g. SSOR preconditioner)
- F-cycle multigrid
- intergrid transfer and coarse grid correction based on the underlying mesh hierarchy and the used finite elements

Parameter free smoothing & higher numerical stability of CG

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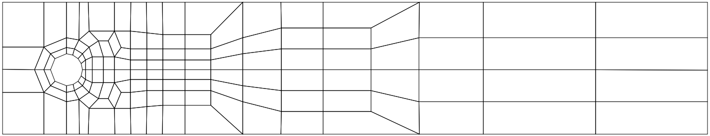
Flow around cylinder



Level	NEL	d.o.f. V-V-P		d.o.f. S-V-P	
		Q_1	Q_2	Q_1	Q_2
1	346	1,564	5,896	2,346	8,844
2	1,384	5,896	22,864	8,844	34,296
3	5,536	22,864	90,016	34,296	135,024
4	22,144	90,016	357,184	135,024	535,776
5	88,576	357,184	1,422,976	535,776	2,134,464
6	354,304	1,422,976		2,134,464	

- Benchmark quantities: $$GMC_{|\Gamma_i} = \frac{\int_{\Gamma_i} \rho(\boldsymbol{n}\cdot\boldsymbol{v})d\Gamma_i - \int_{\Gamma_o} \rho(\boldsymbol{n}\cdot\boldsymbol{v})d\Gamma_o}{\int_{\Gamma_i} \rho(\boldsymbol{n}\cdot\boldsymbol{v})d\Gamma_i} \times 100,$$

$$C_D = \frac{2F_D}{\rho U_m^2 D}, \quad C_L = \frac{2F_L}{\rho U_m^2 D}, \ \triangle p.$$

Flow around cylinder Re = 20

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- V-V-P formulation

	Level	C_D	C_L	$\triangle p$	$GMC\|_{x=2.2}$
Q_1, $\alpha = 1$	4	4.8914483	0.0043424	0.1009819	12.828753
	5	5.3687854	0.0086759	0.1124486	3.871370
	6	5.5234496	0.0101034	0.1161682	1.025463
Q_2, $\alpha = 1$	3	5.5668223	0.0104928	0.1172298	0.114906
	4	5.5779343	0.0106055	0.1174841	0.010779
	5	5.5792792	0.0106169	0.1175144	0.001096
Q_1, $\alpha = 100$	4	5.2716595	0.0245279	0.1044718	1.053974
	5	5.4769244	0.0132686	0.1128306	0.283655
	6	5.5500584	0.0109745	0.0116144	0.073273
Q_2, $\alpha = 100$	3	5.4744825	0.0094397	0.1157215	0.035024
	4	5.5620013	0.0104345	0.1172938	0.004049
	5	5.5762393	0.0105925	0.1174909	0.000473

ref.: $C_D = 5.57953523384$, $C_L = 0.010618948146$, $\triangle p = 0.11752016697$

- **mass conservation enhancement** $\Rightarrow$ **less accurate results!**
- **higher order element** $\Rightarrow$ **accurate results, less sensitivity to** α

Flow around cylinder Re $= 20$

- S-V-P formulation

	Level	C_D	C_L	$\triangle p$	$GMC_{\vert x=2.2}$
Q_1	4	5.1716353	0.0210522	0.0103135	1.114501
	5	5.4440131	0.0142939	0.1117922	0.299773
	6	5.5415463	0.0117584	0.1152451	0.077866
Q_2	3	5.5588883	0.0101360	0.1165546	0.022791
	4	5.5769755	0.0105355	0.1173265	0.003022
	5	5.5792424	0.0106064	0.1174766	0.000556

ref.: $C_D = 5.57953523384$, $C_L = 0.010618948146$, $\triangle p = 0.11752016697$

- **higher order element** $\Rightarrow$ **accurate results**
- **S-V-P formulation** $\Rightarrow$ **better mass conservation!**

Flow around cylinder Re $= 20$

- MPCG solver for the V-V-P and S-V-P formulation

Level	V-V-P						S-V-P	
	Q_1			Q_2			Q_1	Q_2
α	1.0	10	100	1.0	10	100	1.0	1.0
3	8/4	8/4	8/6	6/5	6/6	6/10	7/19	6/12
4	8/4	8/4	8/6	6/5	6/6	6/9	7/17	6/12
5	7/4	7/4	8/6	6/5	6/6	6/9	7/17	6/12

Robust and grid-independent solver!

MPCG solver performance **degraded** with mass enhancement!

Summary

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Numerical investigations of Least squares FEM for Navier-Stokes equations w.r.t

- different first order systems of equations V-V-P and S-V-P
- different choice of FE spaces
- Newton-Multigrid precondioner for CG
- mass conservation

Outlook

Investigations of Least squares FEM w.r.t

- "do-nothing" outflow boundary condition; $\frac{1}{2}\left\|\boldsymbol{\sigma} \cdot \boldsymbol{n}\right\|_{0,\Gamma_{out}}$
- multiphase flow using S-V-P formulation to analyze the jump in the stress i.e. due to shear stress and/or normal stress (discontinuous pressure)
- Generalized Newtonian fluids; $\boldsymbol{\sigma} = \nu(\dot{\gamma})\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right) - p\mathbf{I}$