Newton Multi-grid Solver for the Solution of Incompressible Navier-Stokes Equations with Least-Squares FEM

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Incompressible Navier-Stokes eq.

Conservation of mass and momentum

$$\begin{cases} \boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p-\nu\triangle\boldsymbol{u}=f & \text{in } \Omega \\ \nabla\cdot\boldsymbol{u}=0 & \text{in } \Omega \\ \boldsymbol{u}=\boldsymbol{g}_D & \text{on } \Gamma_D \\ \boldsymbol{n}\cdot\boldsymbol{\sigma}=\boldsymbol{g}_N & \text{on } \Gamma_N \end{cases}$$

along with the zero mean pressure constraint

$$\int_\Omega p = 0$$

LSFEM in general

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Advantages

- Symmetric and positive definite systems ($\checkmark$)
- No compatibility condition (LBB) enforced ($\checkmark$)
- No stabilization technique required (?)

Disadvantage

- Lack of local mass conservation (!)
- Discontinuities of the pressure ($\times$)
- Implementation of Natural B.Cs. (!)

Other issues

- Direct application of least-squares technique to 2nd order equations must be avoided $\Rightarrow$ first-order reformulation

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First-order formulation

Define vorticity $\omega = \nabla \times \boldsymbol{u}$

Use $\nabla \times \nabla \times \boldsymbol{u} = -\triangle \boldsymbol{u} + \nabla(\nabla \cdot \boldsymbol{u})$ and $\nabla \cdot \boldsymbol{u} = 0$

$$\begin{cases} \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p - \nu \triangle \boldsymbol{u} = f \\ \nabla \cdot \boldsymbol{u} = 0 \\ \boldsymbol{u} = \boldsymbol{g}_D \\ \boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{g}_N \end{cases} \Rightarrow \begin{cases} \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p + \nu \nabla \times \omega = f & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \omega - \nabla \times \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g}_D & \text{on } \Gamma_D \\ \boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{g}_N & \text{on } \Gamma_N \end{cases}$$

First-order Velocity-Vorticity-Pressure (V-V-P) system of equations

Newton linearisation

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V-V-P formulation

$$\begin{cases} \underbrace{\boldsymbol{u} \cdot \nabla \boldsymbol{u}} + \nabla p + \nu \nabla \times \omega = f & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \omega - \nabla \times \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g}_D & \text{on } \Gamma_D \\ \boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{g}_N & \text{on } \Gamma_N. \end{cases}$$

Newton linearisation

$$\boldsymbol{u}^{n+1} \cdot \nabla \boldsymbol{u}^{n+1} \approxeq \boldsymbol{u}^{n} \cdot \nabla \boldsymbol{u}^{n+1} + \boldsymbol{u}^{n+1} \cdot \nabla \boldsymbol{u}^{n} - \boldsymbol{u}^{n} \cdot \nabla \boldsymbol{u}^{n}$$

Error tolerance

$$\frac{\left\| U^{n+1} - U^{n} \right\|_2}{\left\| U^{n+1} \right\|_2} < \epsilon , \quad U = [\boldsymbol{u}, p, \omega]^T .$$

Continuous least-squares principle

L^2-norm least-squares energy functional

$$\mathcal{J}(\boldsymbol{v}, q, \xi; f) = \|\boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla q + \nu \nabla \times \xi - f\|_0^2 \\ + \alpha \|\nabla \cdot \boldsymbol{v}\|_0^2 + \|\xi - \nabla \times \boldsymbol{v}\|_0^2 \quad \forall (\boldsymbol{v}, q, \xi) \in \boldsymbol{V}$$

and,

$$\mathcal{J}_\nu(\boldsymbol{v}, q, \xi; f) = \frac{1}{\nu} \|\boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla q + \nu \nabla \times \xi - f\|_0^2 \\ + \alpha \|\nabla \cdot \boldsymbol{v}\|_0^2 + \|\xi - \nabla \times \boldsymbol{v}\|_0^2 \quad \forall (\boldsymbol{v}, q, \xi) \in \boldsymbol{V}$$

Find $(\boldsymbol{u}, p, \omega) \in \boldsymbol{V}$ such that

$$(\boldsymbol{u}, p, \omega) = \underset{(\boldsymbol{v}, q, \xi) \in \boldsymbol{V}}{\operatorname{argmin}} \mathcal{J}(\boldsymbol{v}, q, \xi; f)$$

Continuous least-squares principle

Variational Problem

$$\mathcal{A}(\boldsymbol{u}, p, \omega; \boldsymbol{v}, q, \xi) = \mathcal{F}(\boldsymbol{v}, q, \xi) \quad \forall (\boldsymbol{v}, q, \xi) \in \boldsymbol{V}$$

Bilinear form

$$\begin{aligned}\mathcal{A}(\boldsymbol{u}, p, \omega; \boldsymbol{v}, q, \xi) := {} & \alpha(\nabla \cdot \boldsymbol{u}^{n+1}, \nabla \cdot \boldsymbol{v})_0 + (\omega - \nabla \times \boldsymbol{u}^{n+1}, \xi - \nabla \times \boldsymbol{v})_0 \\ & + \frac{1}{\nu}(\boldsymbol{u}^{n} \cdot \nabla \boldsymbol{u}^{n+1} + \nabla p + \nu \nabla \times \omega, \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla q - \nu \nabla \times \xi)_0\end{aligned}$$

Linear form

$$\mathcal{F}(\boldsymbol{v}, q, \xi) := \frac{1}{\nu}(f, \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla q + \nu \nabla \times \xi)_0$$

Note: W-LSFEM means scaling the equations with $\frac{1}{\nu}$

Continuous least-squares principle

Restrict to the $C_0^\infty(\Omega)$ functions to obtain operator form

$$\begin{pmatrix} -\alpha\nabla\nabla\cdot+\nabla\times\nabla\times & 0 & -\nabla\times \\ 0 & -\frac{1}{\nu}\nabla\cdot\nabla & 0 \\ -\nabla\times & 0 & I+\nabla\times\nabla\times \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \\ \omega \end{pmatrix} = RHS$$

Positive definite, symmetric coefficient matrix $\Rightarrow$ CG solver

Differentially diagonal dominant $\Rightarrow$ Efficient multigrid performance with higher order finite elements

Discrete least-squares principle

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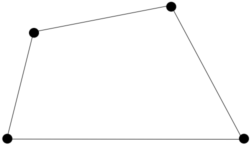
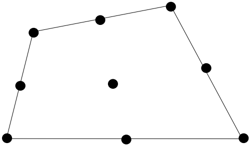
Restrict our variational problem to finite dimensional spaces $\boldsymbol{V_h}$

$$\mathcal{A}_h(\boldsymbol{u}_h, p_h, \omega_h; \boldsymbol{v}_h, q_h, \xi_h) = \mathcal{F}_h(\boldsymbol{v}_h, q_h, \xi_h) \quad \forall(\boldsymbol{v}_h, q_h, \xi_h) \in \boldsymbol{V_h}$$

- conforming finite elements, $\boldsymbol{V_h} \subset \boldsymbol{V}$
- no LBB(inf-sup) condition, equal order finite elements for all variables
- bi-linear Q_1 and bi-quadratic Q_2 finite elements





Linear solver

Available solvers : conjugate gradient (CG) and Multigrid

Goal : combining multigrid and CG to improve the efficiency of the overall solution method

Idea : geometric multigrid solver as a preconditioner for the CG (MPCG)

- direct Gauss elimination coarse-grid solver
- preconditioned CG smoothers (e.g. SSOR preconditioner)
- F-cycle multigrid
- intergrid transfer and coarse grid correction based on the underlying mesh hierarchy and the finite elements

Parameter free smoothing & higher numerical stability of CG

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Problem description

Laminar flow at $Re = 100$ in a square domain $\Omega = [0,1] \times [0,1]$

Boundary conditions

- horizontal walls: $[u, v] = [0, 0]$
- inflow: $[u, v] = [y(1-y), 0]$
- outflow: $[u, v] = [y(1-y), 0]$ or, zero stress b.c.

Zero stress b.c. $\boldsymbol{\sigma} \cdot \boldsymbol{n} = [-p\mathbf{I} + \nu(\nabla \boldsymbol{u})] \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_{out}$

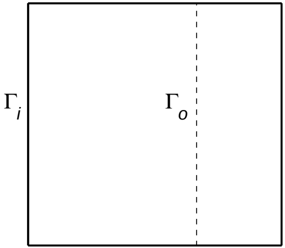
$$\mathcal{J}_{out}(\boldsymbol{v}, q, \xi; f) = \frac{1}{\nu} \|\boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla q + \nu \nabla \times \xi - f\|_0^2 + \alpha \|\nabla \cdot \boldsymbol{v}\|_0^2 + \|\xi - \nabla \times \boldsymbol{v}\|_0^2$$
$$+ \|[-q\mathbf{I} + \nu(\nabla \boldsymbol{v})] \cdot \boldsymbol{n}\|_{0,\Gamma_{out}}^2$$

Poiseuille flow

Mass conservation

Global Mass Conservation (GMC) in terms of the fractional change of mass flow rate

$$\mathrm{GMC} = \frac{\int_{\Gamma_i} \rho\,(\boldsymbol{n}\cdot\boldsymbol{u})\,d\Gamma_i - \int_{\Gamma_o} \rho\,(\boldsymbol{n}\cdot\boldsymbol{u})\,d\Gamma_o}{\int_{\Gamma_i} \rho\,(\boldsymbol{n}\cdot\boldsymbol{u})\,d\Gamma_i} \times 100$$



Poiseuille flow

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GMC values based on LSFEM, Q_1 elements

Lev.	$\alpha = 1.0$		$\alpha = 100$	
	$x = 0.3$	$x = 0.8$	$x = 0.3$	$x = 0.8$
zero normal stress				
6	0.447549	0.722513	0.101985	0.104672
7	0.138748	0.238623	0.025626	0.026484
8	0.044002	0.082468	<u>0.006470</u>	<u>0.006803</u>
Dirichlet velocity				
6	0.283028	0.250209	0.099898	0.099585
7	0.082833	0.074349	0.025029	0.024955
8	0.023621	0.022299	<u>0.006275</u>	<u>0.006262</u>

mass conservation is satisfied and improved with scaling!

Poiseuille flow

GMC values based on W-LSFEM, Q_1 elements

Lev.	$\alpha = 1.0$		$\alpha = 100$	
	$x = 0.3$	$x = 0.8$	$x = 0.3$	$x = 0.8$
zero normal stress				
6	0.722934	0.140562	0.105095	0.113889
7	0.215425	0.433627	0.026494	0.029075
8	0.057796	0.119313	<u>0.006644</u>	<u>0.007322</u>
Dirichlet velocity				
6	0.331548	0.348048	0.100521	0.100554
7	0.091683	0.098458	0.025170	0.025223
8	0.023839	0.025854	<u>0.006291</u>	<u>0.006309</u>

mass conservation is satisfied and improved with scaling!

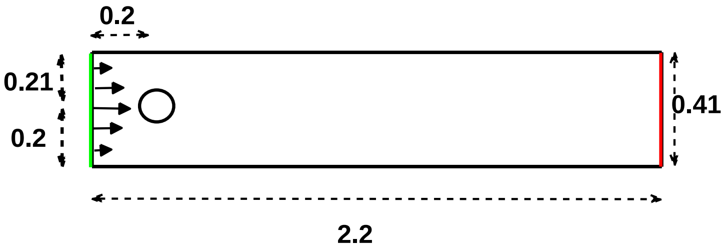
Flow around circular cylinder

Problem description

Laminar flow at $Re = 20$

Boundary conditions

- horizontal walls and the cylinder: $[u, v] = [0, 0]$
- inflow: $[u, v] = \left[\frac{1.2y(0.41-y)}{0.41^2}, 0\right]$
- outflow: zero stress b.c.



Flow around circular cylinder

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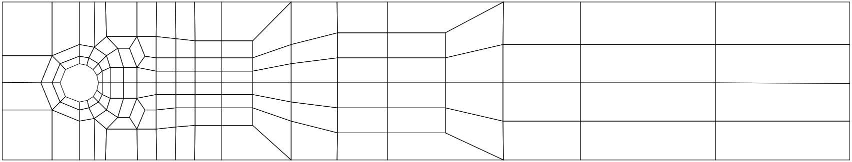
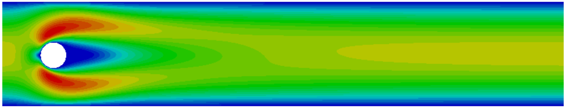
Mesh information



Lev.	NE	NDoF	
		Q_1	Q_2
1	130	624	2288
2	520	2288	8736
3	2080	8736	34112
4	8320	34112	134784
5	33280	134784	535808

Flow around circular cylinder

Flow visualization



velocity magnitude

pressure

vorticity

Flow around circular cylinder

GMC values for Q_1 elements

	Lev.	x-coordinate of the cross-section			
		0.05	2.2	0.05	2.2
		$\alpha = 1.0$		$\alpha = 100$	
LSFEM	3	7.602904	47.600894	1.496185	5.769657
	4	3.145125	19.579627	0.473927	1.618546
	5	1.110650	7.160065	0.128078	0.481929
		$\alpha = 1.0$		$\alpha = 100$	
W-LSFEM	3	11.068663	58.389813	1.836556	7.385963
	4	5.236971	30.295098	0.648462	2.472131
	5	1.920968	12.032364	0.187898	0.777445

scaling the continuity equation ⇒ better mass conservation!

Flow around circular cylinder

GMC values for Q_2 elements

	Lev.	x-coordinate of the cross-section			
		0.05	2.2	0.05	2.2
		$\alpha = 1.0$		$\alpha = 100$	
LSFEM	3	0.217743	0.747127	0.030647	0.111722
	4	0.032186	0.107947	0.009083	0.031029
	5	0.004133	0.013817	0.001416	0.004536
		$\alpha = 1.0$		$\alpha = 100$	
W-LSFEM	3	0.424058	1.499480	0.090863	0.313636
	4	0.047289	0.163867	0.012530	0.041305
	5	0.004929	0.016752	0.001472	0.004738

bi-quadratic finite elements ⇒ no mass loss!

Quantitative analysis

$C_D = \frac{2F_D}{\rho U_m^2 D}$ and $C_L = \frac{2F_L}{\rho U_m^2 D}$ and $\triangle p$ for Q_1 elements

W-LSFEM

	Lev.	α = 1	α = 100
$C_D = 5.57953523384$	4	4.2446633	4.4068603
	5	5.0579843	5.0770015
$C_L = 0.010618948146$	4	0.0169573	0.107789
	5	0.0142112	0.0578331
$\triangle p = 0.11752016697$	4	0.0834404	0.0769076
	5	0.1042858	0.0976744

mass conservation enhancement $\Rightarrow$ less accurate results!

Quantitative analysis

$C_D = \frac{2F_D}{\rho U_m^2 D}$ and $C_L = \frac{2F_L}{\rho U_m^2 D}$ and $\triangle p$ for Q_2 elements

W-LSFEM

	Lev.	α = 1	α = 100
$C_D = 5.57953523384$	4	5.5612881	5.4392066
	5	5.5771424	5.5551275
$C_L = 0.010618948146$	4	0.0103164	0.0115738
	5	0.0105818	0.0106819
$\triangle p = 0.11752016697$	4	0.1170801	0.1151433
	5	0.1174629	0.1172354

bi-quadratic elements $\Rightarrow$ accurate results, less sensitivity to α

MPCG solver behavior

nonlinear iterations (NLI) and linear solver iterations(LI); NLI/LI

LSFEM

Lev.	Q_1 elements			Q_2 elements		
	α					
	1.0	10	100	1.0	10	100
3	8/2	8/2	8/3	10/3	10/6	10/10
4	8/2	8/3	8/4	10/3	10/4	10/9
5	8/2	8/2	8/4	10/3	10/3	10/7

Robust and grid-independent solver!

MPCG solver performance **degraded** with mass conservation enhancement!

MPCG solver behavior

nonlinear iterations (NLI) and linear solver iterations(LI); NLI/LI

W-LSFEM

Lev.	Q_1 elements			Q_2 elements		
	α					
	1.0	10	100	1.0	10	100
3	8/3	8/3	8/3	7/3	7/4	7/6
4	8/3	8/3	8/5	7/3	7/3	7/7
5	8/3	8/3	8/5	7/3	7/6	7/8

Robust and grid-independent solver!

MPCG solver performance **degraded** with mass conservation enhancement!

Lid-driven cavity

Problem description

Laminar flow at $1 \leq Re \leq 1000$ in a square domain $\Omega = [0,1] \times [0,1]$

Boundary conditions

- vertical and lower horizontal walls: $[u, v] = [0, 0]$
- cavity lid: $[u, v] = [-16x^2(1-x^2), 0]$ $\Rightarrow$ regularized cavity

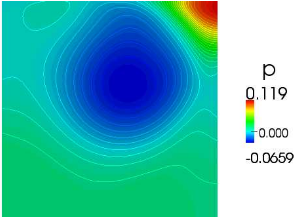
Note: a pressure datum, $p = 0$, on the mid-width of the lower cavity wall

Global quantities

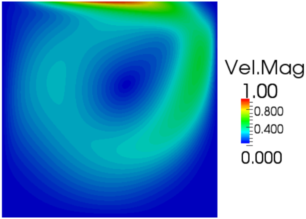
Kinetic energy: $E = \frac{1}{2} \left\| \boldsymbol{u}_h \right\|_{0,\Omega}^2$

Enstophy: $Z = \frac{1}{2} \left\| w_h \right\|_{0,\Omega}^2$

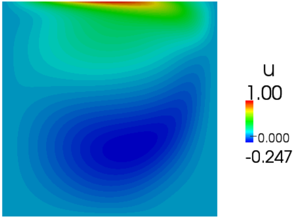
Flow visualization, $Re = 400$



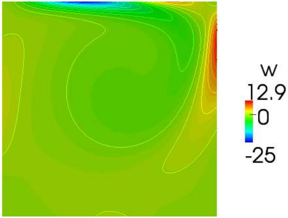
pressure



velocity magnitude



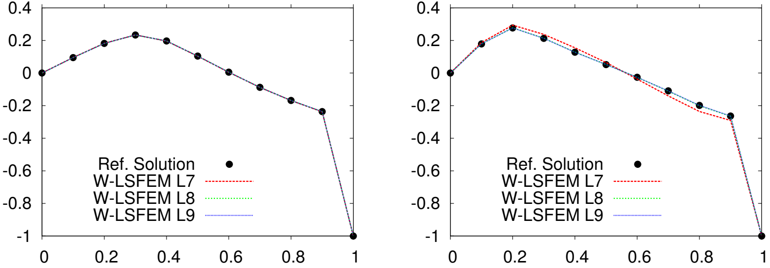
x-velocity



vorticity

x-velocity cutlines

W-LSFEM with Q_2 elements



$Re = 400$ $Re = 1000$

excellent agreement with reference solution!

Quantitative analysis

Convergence of the kinetic energy $E = \frac{1}{2}\left\|\boldsymbol{u}_h\right\|_{0,\Omega}^2$ with Q_2 elements

Lev.	W-LSFEM	Mixed FEM
$Re = 400$		
7	2.133053E-02	2.131706E-02
8	2.131581E-02	2.131547E-02
9	2.131537E-02	2.131537E-02
$Re = 1000$		
7	2.552796E-02	2.277790E-02
8	2.287704E-02	2.276761E-02
9	2.277389E-02	2.276691E-02

accurate results were obtained at "level 9" for $Re = 1000$

Quantitative analysis

Convergence of the Enstophy $Z = \frac{1}{2} \|w_h\|^2_{0,\Omega}$ with Q_2 elements

W-LSFEM

Lev.	$Re = 400$	$Re = 1000$
6	3.181475	5.300352
7	3.225070	4.806740
8	3.229481	4.827331
9	3.229801	4.830225

convergence at $Re = 1000$ was obtained with relatively finer grids!

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Summary

- Performance of the LSFEM is investigated for the solution of the V-V-P formulation using Q_1 and Q_2 elements
- Efficient (Geometric) Multigrid-Preconditioned Conjugate Gradient solver is studied
- The effects of the "continuity equation scaling" on mass conservation, accuracy and solver performance are studied
- The advantage of using higher order finite elements in the recovery of mass conservation is demonstrated
- The necessity of using higher order finite elements to achieve accurate results is demonstrated
- The natural boundary condition, outflow BC, is successfully implemented

Outlook

- Solution of the NS equations using different finite elements, e.g. div-conforming finite elements, non-conforming finite elements
- Investigation of different 1st-order formulations, e.g. velocity-stress-pressure, velocity-velocity gradient-pressure
- Use of discontinuous finite elements for the pressure
- Error analysis and convergence rate investigations
- Solution of the non-stationary NS equations