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Gradient-penalty stabilization of sharp and diffuse interface formulations in unfitted Nitsche finite element methods

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Interface description

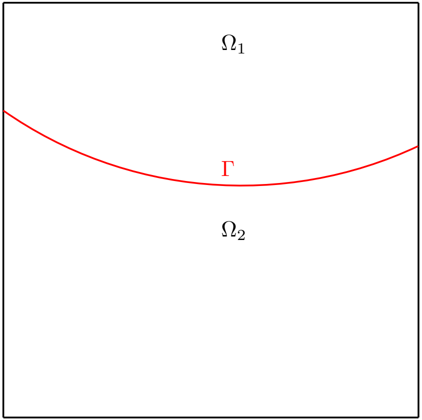
Lipschitz domain $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2 \subset \mathbb{R}^d$

Level set function $\phi \in C(\bar{\Omega})$

$\Omega_1 = \{\mathbf{x} \in \Omega : \phi(\mathbf{x}) > 0\}$

$\Gamma = \{\mathbf{x} \in \Omega : \phi(\mathbf{x}) = 0\}$

$\Omega_2 = \{\mathbf{x} \in \Omega : \phi(\mathbf{x}) < 0\}$



Elliptic interface problem

Strong form

$$
\begin{aligned}
-\nabla \cdot (\mu \nabla u) &= f \quad \text{in } \Omega, \\
u &= \bar{u} \quad \text{on } \partial\Omega, \\
[\![u]\!] &= 0 \quad \text{on } \Gamma, \\
[\![\mu \nabla u]\!] &= 0 \quad \text{on } \Gamma
\end{aligned}
$$

$$
u = \begin{cases} u_1 & \text{in } \Omega_1, \\ u_2 & \text{in } \Omega_2, \end{cases} \qquad \mu = \begin{cases} \mu_1 & \text{in } \Omega_1, \\ \mu_2 & \text{in } \Omega_2, \end{cases} \qquad f = \begin{cases} f_1 & \text{in } \Omega_1, \\ f_2 & \text{in } \Omega_2, \end{cases}
$$

$$
[\![u]\!] = u_1 - u_2 \text{ and } [\![\mu \nabla u]\!] = (\mu_1 \nabla u_1 - \mu_2 \nabla u_2) \cdot \mathbf{n}
$$

Triangulation $\mathcal{T}_h$ fitted to Ω:

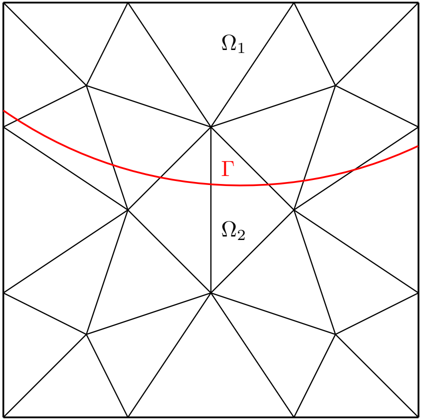
$$\bigcup_{K\in\mathcal{T}_h} K = \bar{\Omega}$$

Sub-triangulations:

$$\mathcal{T}_{h,k}(\delta) = \{K \in \mathcal{T}_h : \exists \mathbf{x} \in K : \text{dist}(\mathbf{x}, \Omega_k) \leq \delta\}$$

Mesh-dependent domains:

$$\Omega_{h,k}(\delta) = \text{int}\Big(\bigcup_{K\in\mathcal{T}_{h,k}(\delta)} K\Big)$$



Discrete weak form

$$\int_{\Omega_1} \mu_1 \nabla u_{h,1} \cdot \nabla w_{h,1} + \int_{\Omega_2} \mu_2 \nabla u_{h,2} \cdot \nabla w_{h,2} + \int_\Gamma \alpha [\![u_h]\!] [\![w_h]\!]$$

$$- \int_\Gamma [\![u_h]\!] \{\mu \nabla w_h\} - \int_\Gamma \{\mu \nabla u_h\} [\![w_h]\!]$$

$$= \int_{\Omega_1} f_1 w_{h,1} + \int_{\Omega_2} f_2 w_{h,2}$$

$$\{\mu \nabla u_h\} = (\kappa_1 \mu_1 \nabla u_{h,1} + \kappa_2 \mu_2 \nabla u_{h,2}) \cdot \mathbf{n} \qquad \text{with} \qquad \kappa_i|_{K^e} = \frac{|K^e \cap \Omega_{i,h}|}{|K^e|}$$

Non-stabilized weak form

$$q_1(u_1, u_2, w_1) = (u_2 - u_1)(\kappa_1 \mu_1 \nabla w_1) \cdot \mathbf{n} - (\kappa_1 \mu_1 \nabla u_1 + \kappa_2 \mu_2 \nabla u_2) w_1 \cdot \mathbf{n} - \alpha(u_2 - u_1) w_1$$

$$q_2(u_1, u_2, w_2) = (u_2 - u_1)(\kappa_2 \mu_2 \nabla w_2) \cdot \mathbf{n} + (\kappa_1 \mu_1 \nabla u_1 + \kappa_2 \mu_2 \nabla u_2) w_2 \cdot \mathbf{n} + \alpha(u_2 - u_1) w_2$$

$$\int_{\Omega_1} \mu_1 \nabla u_{h,1} \cdot \nabla w_{n,1} + \int_{\Omega_2} \mu_2 \nabla u_{h,2} \cdot \nabla w_{h,2}$$
$$+ \int_\Gamma q_1(u_{h,1}, u_{h,2}, w_{h,1}) + \int_\Gamma q_2(u_{h,1}, u_{h,2}, w_{h,2})$$
$$= \int_{\Omega_1} f_1 w_{h,1} + \int_{\Omega_2} f_2 w_{h,2}$$

Stabilization term

$$s_h(u,w) := \int_{\Omega_{h,1}(\delta)} \mu_1(\nabla u_1 - \mathbf{g}_{h,1}) \cdot \nabla w_1 + \int_{\Omega_{h,2}(\delta)} \mu_2(\nabla u_2 - \mathbf{g}_{h,2}) \cdot \nabla w_2$$

Lumped-mass L^2-projection

$$\mathbf{g}_{h,k} = \sum_{j=1}^{N_{h,k}} \mathbf{g}_{j,k}\varphi_j, \qquad \mathbf{g}_{j,k} = \frac{\int_{\Omega_{h,k}(\delta)} \varphi_j \nabla u_k}{\int_{\Omega_{h,k}(\delta)} \varphi_j}$$

Stabilized sharp interface form

$$
\begin{aligned}
&\int_{\Omega_{h,1}(\delta)} H(\phi)\mu_1 \nabla u_{h,1} \cdot \nabla w_{h,1} + \int_\Gamma q_1(u_{h,1}, u_{h,2}, w_{h,1}) \\
&+ \int_{\Omega_{h,2}(\delta)} (1 - H(\phi))\mu_2 \nabla u_{h,2} \cdot \nabla w_{h,2} + \int_\Gamma q_2(u_{h,1}, u_{h,2}, w_{h,2}) \\
&+ \int_{\Omega_{h,1}(\delta)} \mu_1(\nabla u_{h,1} - \mathbf{g}_{h,1}) \cdot \nabla w_{h,1} + \int_{\Omega_{h,2}(\delta)} \mu_2(\nabla u_{h,2} - \mathbf{g}_{h,2}) \cdot \nabla w_{h,2} \\
&= \int_{\Omega_1} f_1 w_{h,1} + \int_{\Omega_2} f_2 w_{h,2}
\end{aligned}
$$

Properties of PG stabilization

$$s_h(u,v) = \sum_{k=1}^{2} s_{h,k}(u_k, v_k), \quad s_{h,k}(u,v) = \int_{\Omega_{h,k}(\delta)} (\nabla u - \mathbf{g}_{h,k}) \cdot \nabla v$$

Lemma

Bilinear form $s_{h,k}(u,v)$ is symmetric and

$$s_{h,k}(u,u) \simeq \|\nabla u - \mathbf{g}_{h,k}\|^2_{L^2(\Omega_{h,k}(\delta))} + h^2 \|\nabla \mathbf{g}_{h,k}\|^2_{L^2(\Omega_{h,k}(\delta))}.$$

$\Rightarrow$ semi-inner product induced by $s_{h,k}(u,v)$

Convergence results

Energy norm: $\|v\|_*^2 = \sum_{k=1}^2 \left\{ \|v_k\|^2_{H^1(\Omega_{h,k}(\delta))} + h^{-1}\|v_k\|^2_{L^2(\Gamma)} + s_{h,k}(v_k, v_k) \right\}$

Theorem

$$\|u - u_h\|_* \lesssim h \sum_{k=1}^{2} \|u\|_{H^2(\Omega_k)},$$

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^2 \sum_{k=1}^{2} \|u\|_{H^2(\Omega_k)}.$$

Diffuse interface formulation

Integration over sharp interface:

- piecewise-linear approximation to the zero level set
- numerical integration on segments of the interface

Approximation of surface integrals by volume integrals:

- Use regularized delta functions
- Extrapolate interface fluxes

Extrapolation using level sets

To be defined: constant extensions of interface fluxes

Main steps:

1. closest-point search
2. evaluation of solution and normal derivative
3. constant extrapolation

Requirements: simplicity, efficiency, accuracy

Closest point search

Interface pointer

$$\mathbf{n}_Q := -\nabla\phi_h(\mathbf{x}_Q)$$

Exact distance function ϕ

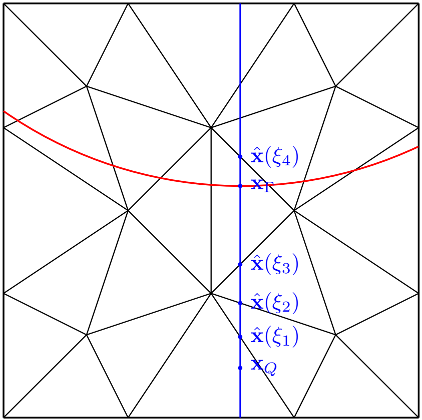
$$\mathbf{x}_\Gamma := \mathbf{x}_Q + \phi(\mathbf{x}_Q)\mathbf{n}_Q$$

Numerical approximation ϕ_h

$$\hat{\mathbf{x}}(\xi) = \mathbf{x}_Q + \xi \mathrm{sign}(\phi_h(\mathbf{x}_Q))\mathbf{n}_Q, \quad \xi \in \mathbb{R}$$

$$\phi_h(\mathbf{x}_\Gamma) = 0 \text{ at } \mathbf{x}_\Gamma = \hat{\mathbf{x}}(\xi_\Gamma)$$

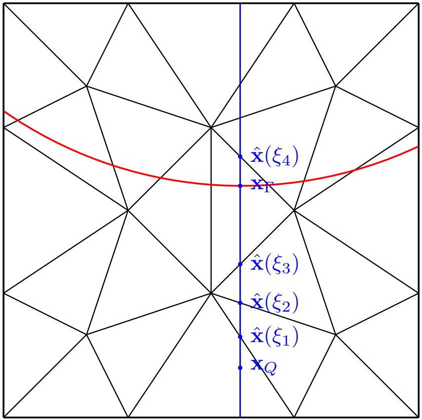
$\Rightarrow$ simple line search



Closest point search

Search algorithm

- Set $\xi_0 = 0$
- For $i > 1$: Find next intersection $\hat{\mathbf{x}}(\xi_i)$, $\xi_i > \xi_{i-1}$ of $\hat{\mathbf{x}}(\xi)$ with boundary of a mesh cell
- If $\phi(\hat{\mathbf{x}}(\xi_i))\phi(\hat{\mathbf{x}}(\xi_{i-1})) < 0$ for $i = m$ exit loop
- Solve linear/quadratic equation to find root $\xi_\Gamma \in [\xi_{m-1}, \xi_m]$ of $\phi(\hat{\mathbf{x}}(\xi))$
- Set $\mathbf{x}_\Gamma = \hat{\mathbf{x}}(\xi_\Gamma)$



Extension operator

$$\mathcal{E}_{\mathrm{cp}} v(\mathbf{x}) = v(\mathbf{x}_\Gamma(\mathbf{x}))$$

Consistency with jump conditions:

$$\mathcal{E}_{\mathrm{cp}} q_1(u_1, u_2, w_1) + \mathcal{E}_{\mathrm{cp}} q_2(u_1, u_2, w_2) = 0 \qquad \text{at } \mathbf{x} \in \Omega$$

if

$$[\![u]\!] = [\![\mu \nabla u]\!] = [\![w]\!] = 0 \qquad \text{at } \mathbf{x}_\Gamma(\mathbf{x}) \in \Gamma$$

Diffuse interface form

$$
\begin{aligned}
&\int_{\Omega_{h,1}(\delta)} H(\phi)\mu_1 \nabla u_{h,1} \cdot \nabla w_{h,1} + \int_{\Omega} \mathcal{E}_{\mathrm{cp}} q_1(u_{h,1}, u_{h,2}, w_{h,1}) \delta_\epsilon(\phi) |\nabla \phi| \\
+ &\int_{\Omega_{h,2}(\delta)} (1 - H(\phi))\mu_2 \nabla u_{h,2} \cdot \nabla w_{h,2} + \int_{\Omega} \mathcal{E}_{\mathrm{cp}} q_2(u_{h,1}, u_{h,2}, w_{h,2}) \delta_\epsilon(\phi) |\nabla \phi| \\
+ &\int_{\Omega_{h,1}(\delta)} \mu_1 (\nabla u_{h,1} - \mathbf{g}_{h,1}) \cdot \nabla w_{h,1} + \int_{\Omega_{h,2}(\delta)} \mu_2 (\nabla u_{h,2} - \mathbf{g}_{h,2}) \cdot \nabla w_{h,2} \\
&= \int_{\Omega_1} f_1 w_{h,1} + \int_{\Omega_2} f_2 w_{h,2}
\end{aligned}
$$

Numerical examples: quasi-1D

Smooth test problem:

Domain: $\Omega = (0,1)^2$ Interface: $\Gamma = \{x = 0.51\}$

$\mu_1 = $ 1e-08 $\mu_2 = 1$ $f_1 = $ 2e-08 $f_2 = 2$

$$u_1(x,y) = u_2(x,y) = (x - 0.01)(1.01 - x)$$

Non-smooth test problem:

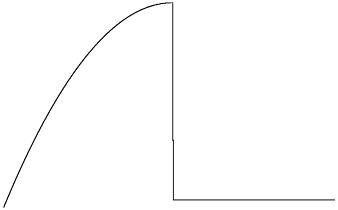
Domain: $\Omega = (0,1)^2$ Interface: $\Gamma = \{x = 0.51\}$

$\mu_1 = 0.5$ $\mu_2 = 3$ $f_1 = 1$ $f_2 = 1$

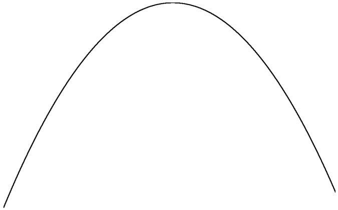
$$u_1(x,y) = \frac{9}{14}(x - 0.01) - (x - 0.01)^2 \qquad u_2(x,y) = \frac{5}{84} + \frac{9}{84}(x - 0.01) - \frac{(x - 0.01)^2}{6}$$

Smooth solution: sharp interface, $\delta = 0$

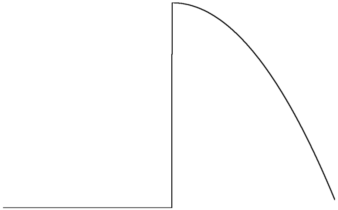
(a) extended u_1



(b) u

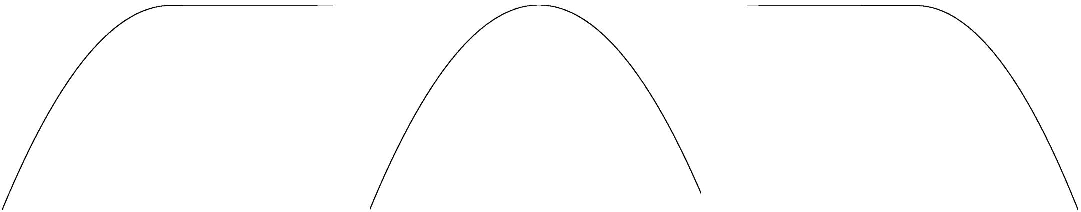


(c) extended u_2



Smooth solution: diffuse interface, $\delta = \mathrm{diam}\bar{\Omega}$

(a) extended u_1 (b) u (c) extended u_2

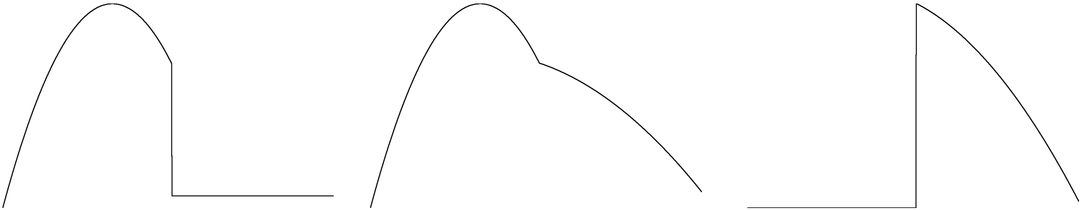


Smooth solution: grid convergence

h^{-1}	sharp $\delta = 0$	EOC	sharp $\delta = 6h$	EOC	diffuse $\delta = 6h$	EOC	diffuse $\delta = d_\Omega$	EOC
128	4.02e-05		4.02e-05		4.02e-05		4.02e-05	
256	1.01e-05	1.99	1.01e-05	1.99	1.01e-05	1.99	1.01e-05	1.99
512	2.54e-06	1.99	2.54e-06	1.99	2.54e-06	1.99	2.54e-06	1.99
1024	6.35e-07	2.00	6.35e-07	2.00	6.35e-07	2.00	6.35e-07	2.00
2048	1.59e-07	2.00	1.59e-07	2.00	1.59e-07	2.00	1.59e-07	2.00
4096	3.96e-08	2.01	3.99e-08	1.99	3.98e-08	2.00	3.99e-08	1.99

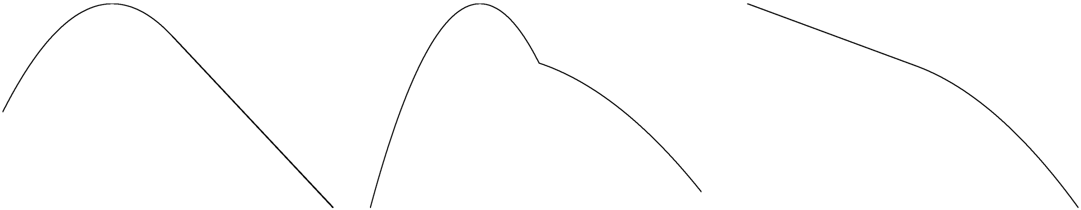
Non-smooth solution: sharp interface, $\delta = 0$

(a) extended u_1 (b) u (c) extended u_2



Non-smooth solution: diffuse interface, $\delta = \mathrm{diam}\bar{\Omega}$

(a) extended u_1 (b) u (c) extended u_2



Non-smooth solution: grid convergence

h^{-1}	sharp $\delta = 0$	EOC	sharp $\delta = 6h$	EOC	diffuse $\delta = 0$	EOC	diffuse $\delta = d_\Omega$	EOC
128	2.91e-05		2.91e-05		2.67e-05		2.67e-05	
256	7.31e-06	1.99	7.31e-06	1.99	8.29e-06	1.69	8.29e-06	1.69
512	1.83e-06	2.00	1.83e-06	2.00	1.38e-06	2.59	1.38e-06	2.59
1024	4.57e-07	2.00	4.57e-07	2.00	4.16e-07	1.73	4.16e-07	1.73
2048	1.03e-07	2.15	1.03e-07	2.15	9.79e-08	2.09	9.79e-07	2.09
4096	2.29e-08	2.17	2.29e-08	2.17	2.75e-08	1.83	2.75e-08	1.83

Numerical examples: 2D

Domain: $\Omega = (-1,1)^2$

Level set function: $\phi(x,y) = 0.75 - \sqrt{x^2+y^2}$

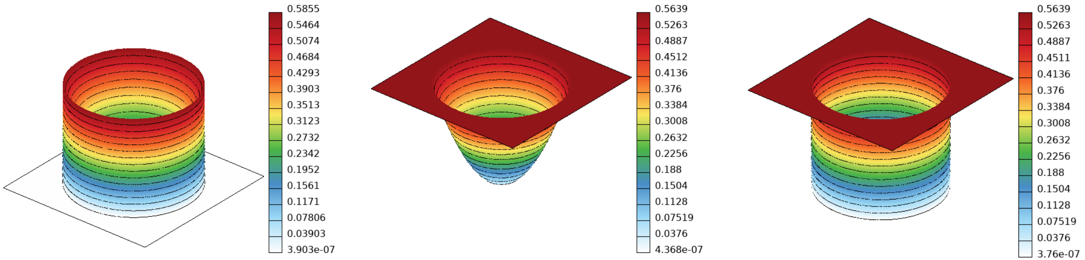
$$\mu_1 = 1 \quad \mu_2 = 10^3 \quad f_1 = f_2 = 4$$

$$u_1(x,y) = x^2 - y^2$$

$$u_2(x,y) = \frac{x^2 - y^2}{1000} - \frac{0.5625}{1000} + 0.5625$$

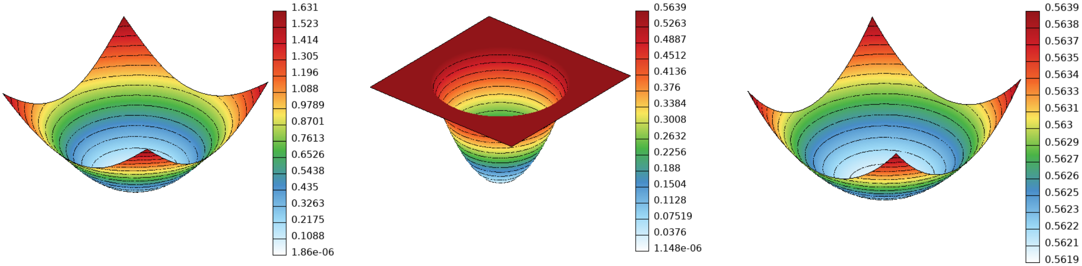
Sharp interface, $\delta = 0$

(a) extended u_1 (b) u (c) extended u_2



Diffuse interface, $\delta = \mathrm{diam}\bar{\Omega}$

(a) extended u_1 (b) u (c) extended u_2



Grid convergence

h^{-1}	sharp $\delta = 0$	EOC	sharp $\delta = 6h$	EOC	diffuse $\delta = 0$	EOC	diffuse $\delta = d_\Omega$	EOC
32	1.15e-02		1.17e-02		9.05e-04		9.74e-04	
64	2.68e-03	2.10	2.77e-03	2.08	2.09e-04	2.11	2.44e-04	2.00
128	6.59e-04	1.96	6.78e-04	2.03	5.06e-05	2.05	6.11e-05	2.00
256	1.64e-04	2.07	1.69e-04	2.00	1.23e-05	2.04	1.51e-05	2.02
512	4.10e-05	2.00	4.24e-05	1.99	2.96e-06	2.05	3.59e-06	2.07
1024	1.04e-05	1.98	1.07e-05	1.99	6.88e-07	2.11	8.00e-07	2.17

Conclusions

Summary

- parameter-free stabilization using projected gradients
- approximation of surface integrals by volume integrals
- fast closest-point search algorithm

Outlook

- extension to moving boundary problems
- application to PDE systems

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Thank you for your attention!

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