

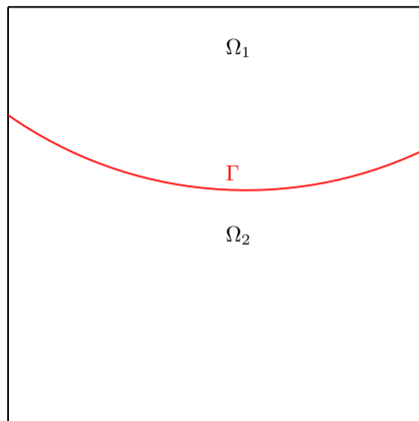
# Gradient-penalty stabilization of sharp and diffuse interface formulations in unfitted Nitsche finite element methods

Maxim Olshanskii, Jan-Phillip Bäcker, Dmitri Kuzmin

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MS07 - Advances in Numerical Methods for Problems from Fluid Mechanics

Lipschitz domain  $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2 \subset \mathbb{R}^d$

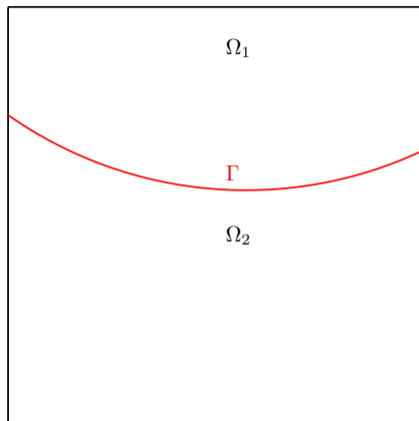


## Interface description

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Level set function  $\phi \in C(\bar{\Omega})$



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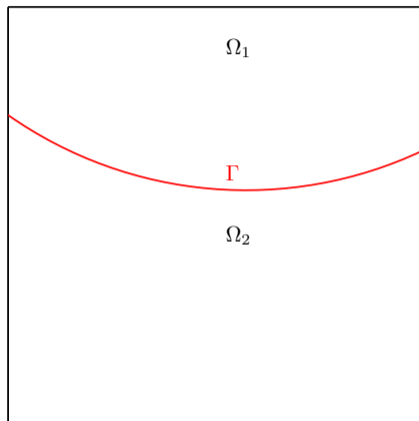
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Level set function  $\phi \in C(\bar{\Omega})$

$$\Omega_1 = \{\mathbf{x} \in \Omega : \phi(\mathbf{x}) > 0\}$$

$$\Gamma = \{\mathbf{x} \in \Omega : \phi(\mathbf{x}) = 0\}$$

$$\Omega_2 = \{\mathbf{x} \in \Omega : \phi(\mathbf{x}) < 0\}$$



## Strong form

$$\begin{aligned} -\nabla \cdot (\mu \nabla u) &= f && \text{in } \Omega, \\ u &= \bar{u} && \text{on } \partial\Omega, \\ \llbracket u \rrbracket &= 0 && \text{on } \Gamma, \\ \llbracket \mu \nabla u \rrbracket &= 0 && \text{on } \Gamma \end{aligned}$$

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$$u = \begin{cases} u_1 & \text{in } \Omega_1, \\ u_2 & \text{in } \Omega_2, \end{cases} \quad \mu = \begin{cases} \mu_1 & \text{in } \Omega_1, \\ \mu_2 & \text{in } \Omega_2, \end{cases} \quad f = \begin{cases} f_1 & \text{in } \Omega_1, \\ f_2 & \text{in } \Omega_2, \end{cases}$$

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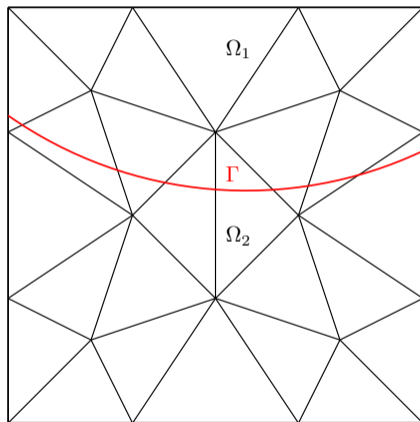
$$\llbracket u \rrbracket = u_1 - u_2 \text{ and } \llbracket \mu \nabla u \rrbracket = (\mu_1 \nabla u_1 - \mu_2 \nabla u_2) \cdot \mathbf{n}$$

## Space discretization

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Triangulation  $\mathcal{T}_h$  fitted to  $\Omega$ :

$$\bigcup_{K \in \mathcal{T}_h} K = \bar{\Omega}$$



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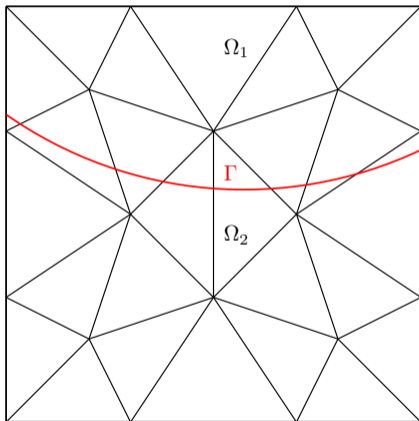
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Sub-triangulations:

$$\mathcal{T}_{h,k}(\delta) = \{K \in \mathcal{T}_h : \exists \mathbf{x} \in K : \text{dist}(\mathbf{x}, \Omega_k) \leq \delta\}$$



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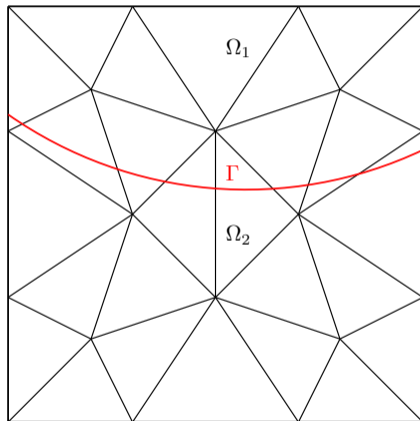
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Mesh-dependent domains:

$$\Omega_{h,k}(\delta) = \text{int}\left(\bigcup_{K \in \mathcal{T}_{h,k}(\delta)} K\right)$$



### Discrete weak form

$$\int_{\Omega_1} \mu_1 \nabla u_{h,1} \cdot \nabla w_{h,1} + \int_{\Omega_2} \mu_2 \nabla u_{h,2} \cdot \nabla w_{h,2}$$
$$= \int_{\Omega_1} f_1 w_{h,1} + \int_{\Omega_2} f_2 w_{h,2}$$

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$$\{\mu \nabla u_h\} = (\kappa_1 \mu_1 \nabla u_{h,1} + \kappa_2 \mu_2 \nabla u_{h,2}) \cdot \mathbf{n} \quad \text{with} \quad \kappa_i|_{K^e} = \frac{|K^e \cap \Omega_{i,h}|}{|K^e|}$$

$$q_1(u_1, u_2, w_1) = (u_2 - u_1)(\kappa_1 \mu_1 \nabla w_1) \cdot \mathbf{n} - (\kappa_1 \mu_1 \nabla u_1 + \kappa_2 \mu_2 \nabla u_2) w_1 \cdot \mathbf{n} - \alpha(u_2 - u_1) w_1,$$

$$q_2(u_1, u_2, w_2) = (u_2 - u_1)(\kappa_2 \mu_2 \nabla w_2) \cdot \mathbf{n} + (\kappa_1 \mu_1 \nabla u_1 + \kappa_2 \mu_2 \nabla u_2) w_2 \cdot \mathbf{n} + \alpha(u_2 - u_1) w_2$$

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### Stabilization term

$$s_h(u, w) := \int_{\Omega_{h,1}(\delta)} \mu_1(\nabla u_1 - \mathbf{g}_{h,1}) \cdot \nabla w_1 + \int_{\Omega_{h,2}(\delta)} \mu_2(\nabla u_2 - \mathbf{g}_{h,2}) \cdot \nabla w_2$$

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Lumped-mass  $L^2$ -projection

$$\mathbf{g}_{h,k} = \sum_{j=1}^{N_{h,k}} \mathbf{g}_{j,k} \varphi_j, \quad \mathbf{g}_{j,k} = \frac{\int_{\Omega_{h,k}(\delta)} \varphi_j \nabla u_k}{\int_{\Omega_{h,k}(\delta)} \varphi_j}$$

$$\begin{aligned} & \int_{\Omega_{h,1}(\delta)} H(\phi)\mu_1 \nabla u_{h,1} \cdot \nabla w_{h,1} + \int_{\Gamma} q_1(u_{h,1}, u_{h,2}, w_{h,1}) \\ + & \int_{\Omega_{h,2}(\delta)} (1 - H(\phi))\mu_2 \nabla u_{h,2} \cdot \nabla w_{h,2} + \int_{\Gamma} q_2(u_{h,1}, u_{h,2}, w_{h,2}) \\ & = \int_{\Omega_1} f_1 w_{h,1} + \int_{\Omega_2} f_2 w_{h,2} \end{aligned}$$

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$$s_h(u, v) = \sum_{k=1}^2 s_{h,k}(u_k, v_k), \quad s_{h,k}(u, v) = \int_{\Omega_{h,k}(\delta)} (\nabla u - \mathbf{g}_{h,k}) \cdot \nabla v$$

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### Lemma

Bilinear form  $s_{h,k}(u, v)$  is symmetric and

$$s_{h,k}(u, u) \simeq \|\nabla u - \mathbf{g}_{h,k}\|_{L^2(\Omega_{h,k}(\delta))}^2 + h^2 \|\nabla \mathbf{g}_{h,k}\|_{L^2(\Omega_{h,k}(\delta))}^2.$$

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$\Rightarrow$  semi-inner product induced by  $s_{h,k}(u, v)$

$$\text{Energy norm: } \|v\|_*^2 = \sum_{k=1}^2 \left\{ \|v_k\|_{H^1(\Omega_{h,k}(\delta))}^2 + h^{-1} \|v_k\|_{L^2(\Gamma)}^2 + s_{h,k}(v_k, v_k) \right\}$$

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### Theorem

$$\|u - u_h\|_* \lesssim h \sum_{k=1}^2 \|u\|_{H^2(\Omega_k)},$$

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^2 \sum_{k=1}^2 \|u\|_{H^2(\Omega_k)}.$$

## Diffuse interface formulation

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Integration over sharp interface:

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- piecewise-linear approximation to the zero level set

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## Extrapolation using level sets

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Main steps:

- 1 closest-point search
- 2 evaluation of solution and normal derivative
- 3 constant extrapolation

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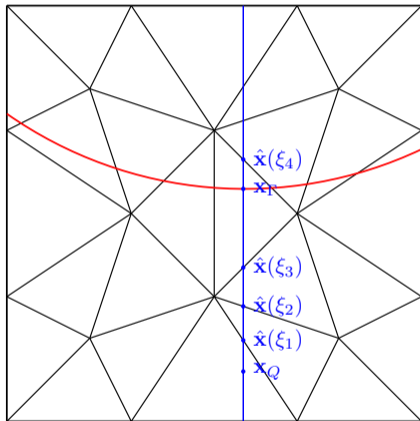
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Requirements: simplicity, efficiency, accuracy

## Interface pointer

$$\mathbf{n}_Q := -\nabla\phi_h(\mathbf{x}_Q)$$



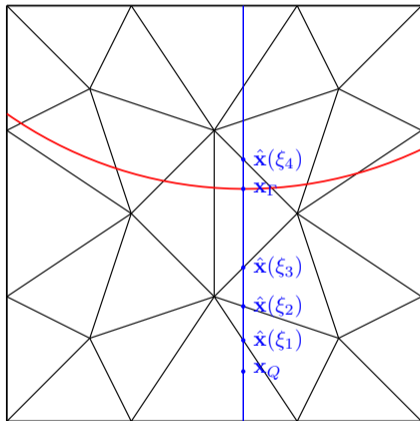
## Closest point search

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Exact distance function  $\phi$

$$\mathbf{x}_\Gamma := \mathbf{x}_Q + \phi(\mathbf{x}_Q)\mathbf{n}_Q$$



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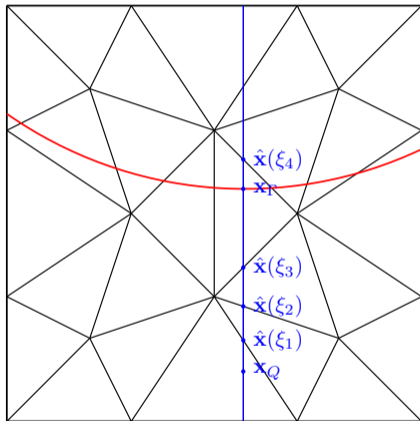
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Numerical approximation  $\phi_h$

$$\hat{\mathbf{x}}(\xi) = \mathbf{x}_Q + \xi \text{sign}(\phi_h(\mathbf{x}_Q))\mathbf{n}_Q, \quad \xi \in \mathbb{R}$$

$$\phi_h(\mathbf{x}_\Gamma) = 0 \text{ at } \mathbf{x}_\Gamma = \hat{\mathbf{x}}(\xi_\Gamma)$$



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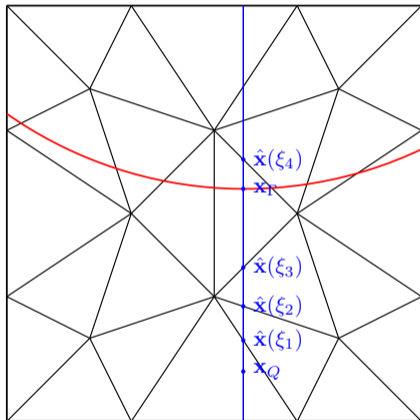
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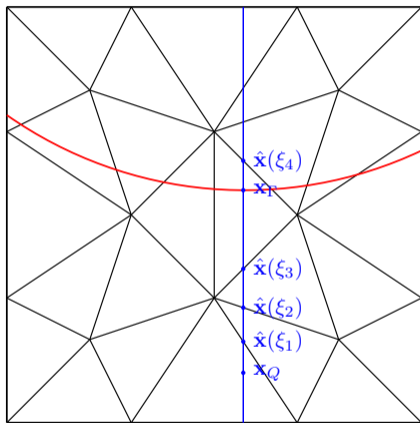
$\Rightarrow$  simple line search



# Closest point search

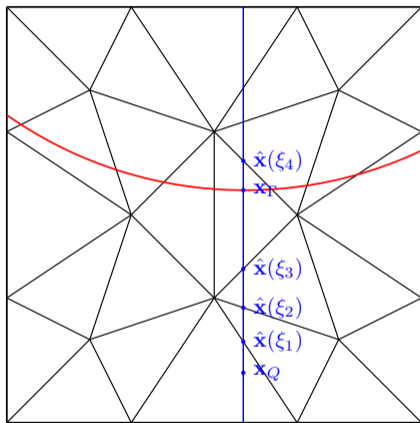
## Search algorithm

- Set  $\xi_0 = 0$



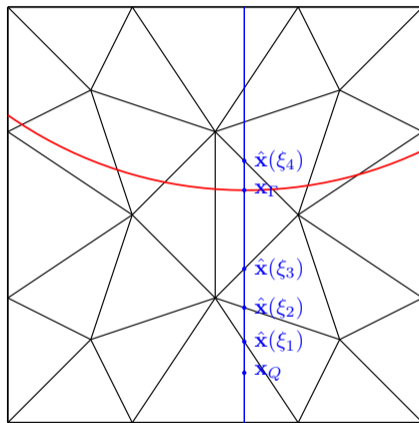
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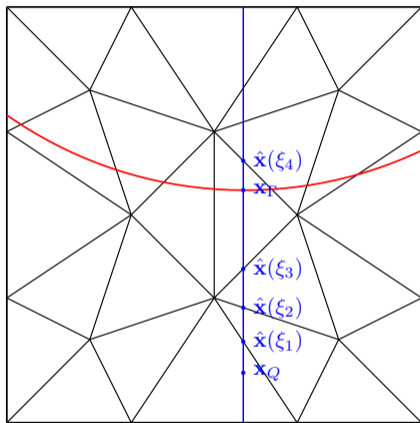
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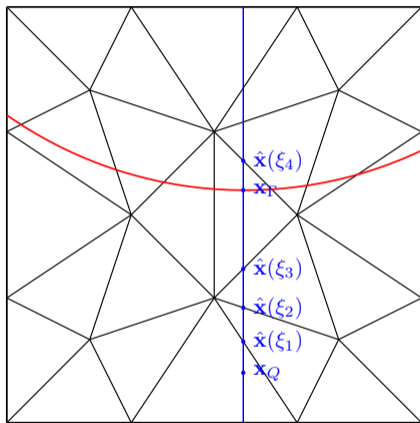
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- Set  $\mathbf{x}_\Gamma = \hat{\mathbf{x}}(\xi_\Gamma)$



### Extension operator

$$\mathcal{E}_{\text{cp}}v(\mathbf{x}) = v(\mathbf{x}_{\Gamma}(\mathbf{x}))$$

Consistency with jump conditions:

$$\mathcal{E}_{\text{cp}}q_1(u_1, u_2, w_1) + \mathcal{E}_{\text{cp}}q_2(u_1, u_2, w_2) = 0 \quad \text{at } \mathbf{x} \in \Omega$$

if

$$[[u]] = [[\mu \nabla u]] = [[w]] = 0 \quad \text{at } \mathbf{x}_{\Gamma}(\mathbf{x}) \in \Gamma$$

$$\begin{aligned} & \int_{\Omega_{h,1}(\delta)} H(\phi) \mu_1 \nabla u_{h,1} \cdot \nabla w_{h,1} + \int_{\Gamma} q_1(u_{h,1}, u_{h,2}, w_{h,1}) \\ & + \int_{\Omega_{h,2}(\delta)} (1 - H(\phi)) \mu_2 \nabla u_{h,2} \cdot \nabla w_{h,2} + \int_{\Gamma} q_2(u_{h,1}, u_{h,2}, w_{h,2}) \\ & + \int_{\Omega_{h,1}(\delta)} \mu_1 (\nabla u_{h,1} - \mathbf{g}_{h,1}) \cdot \nabla w_{h,1} + \int_{\Omega_{h,2}(\delta)} \mu_2 (\nabla u_{h,2} - \mathbf{g}_{h,2}) \cdot \nabla w_{h,2} \\ & = \int_{\Omega_1} f_1 w_{h,1} + \int_{\Omega_2} f_2 w_{h,2} \end{aligned}$$

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## Numerical examples: quasi-1D

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$$\text{Domain: } \Omega = (0, 1)^2 \quad \text{Interface: } \Gamma = \{x = 0.51\}$$

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$$u_1(x, y) = u_2(x, y) = (x - 0.01)(1.01 - x)$$

Smooth test problem:

$$\text{Domain: } \Omega = (0, 1)^2 \quad \text{Interface: } \Gamma = \{x = 0.51\}$$

$$\mu_1 = 1\text{e-08} \quad \mu_2 = 1 \quad f_1 = 2\text{e-08} \quad f_2 = 2$$

$$u_1(x, y) = u_2(x, y) = (x - 0.01)(1.01 - x)$$

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Non-smooth test problem:

$$\text{Domain: } \Omega = (0, 1)^2 \quad \text{Interface: } \Gamma = \{x = 0.51\}$$

$$\mu_1 = 0.5 \quad \mu_2 = 3 \quad f_1 = 1 \quad f_2 = 1$$

Smooth test problem:

$$\text{Domain: } \Omega = (0, 1)^2 \quad \text{Interface: } \Gamma = \{x = 0.51\}$$

$$\mu_1 = 1\text{e-08} \quad \mu_2 = 1 \quad f_1 = 2\text{e-08} \quad f_2 = 2$$

$$u_1(x, y) = u_2(x, y) = (x - 0.01)(1.01 - x)$$

Non-smooth test problem:

$$\text{Domain: } \Omega = (0, 1)^2 \quad \text{Interface: } \Gamma = \{x = 0.51\}$$

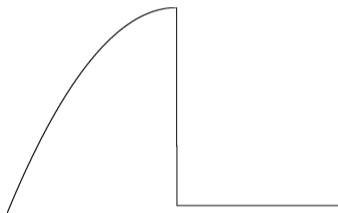
$$\mu_1 = 0.5 \quad \mu_2 = 3 \quad f_1 = 1 \quad f_2 = 1$$

$$u_1(x, y) = \frac{9}{14}(x - 0.01) - (x - 0.01)^2 \quad u_2(x, y) = \frac{5}{84} + \frac{9}{84}(x - 0.01) - \frac{(x - 0.01)^2}{6}$$

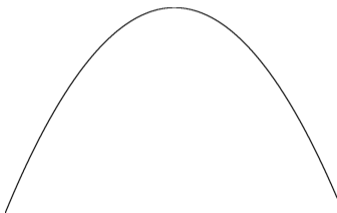
## Smooth solution: sharp interface, $\delta = 0$

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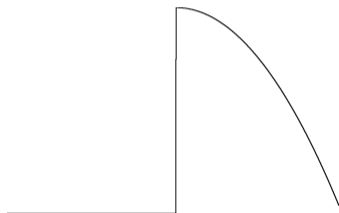
(a) extended  $u_1$



(b)  $u$



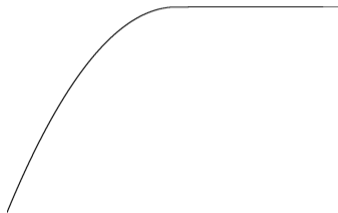
(c) extended  $u_2$



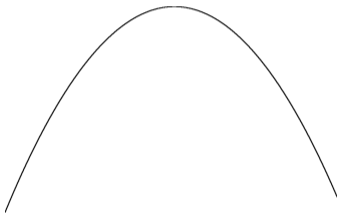
Smooth solution: diffuse interface,  $\delta = \text{diam}\bar{\Omega}$

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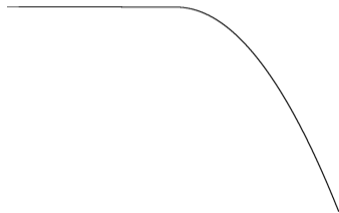
(a) extended  $u_1$



(b)  $u$



(c) extended  $u_2$



## Smooth solution: grid convergence

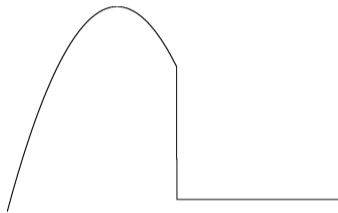
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$h^{-1}$	sharp $\delta = 0$	EOC	sharp $\delta = 6h$	EOC	diffuse $\delta = 6h$	EOC	diffuse $\delta = d_\Omega$	EOC
128	4.02e-05		4.02e-05		4.02e-05		4.02e-05	
256	1.01e-05	1.99	1.01e-05	1.99	1.01e-05	1.99	1.01e-05	1.99
512	2.54e-06	1.99	2.54e-06	1.99	2.54e-06	1.99	2.54e-06	1.99
1024	6.35e-07	2.00	6.35e-07	2.00	6.35e-07	2.00	6.35e-07	2.00
2048	1.59e-07	2.00	1.59e-07	2.00	1.59e-07	2.00	1.59e-07	2.00
4096	3.96e-08	2.01	3.99e-08	1.99	3.98e-08	2.00	3.99e-08	1.99

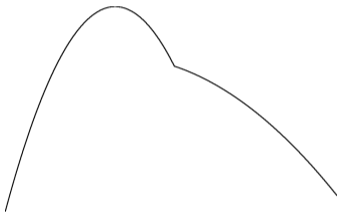
## Non-smooth solution: sharp interface, $\delta = 0$

---

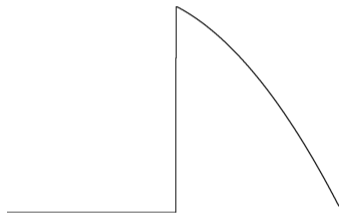
(a) extended  $u_1$



(b)  $u$



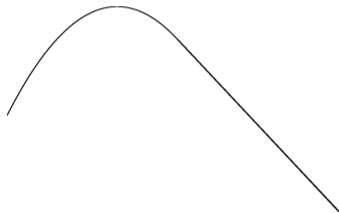
(c) extended  $u_2$



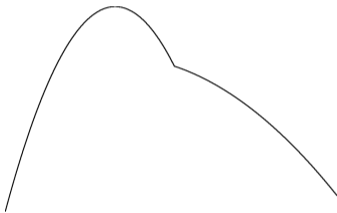
Non-smooth solution: diffuse interface,  $\delta = \text{diam}\bar{\Omega}$

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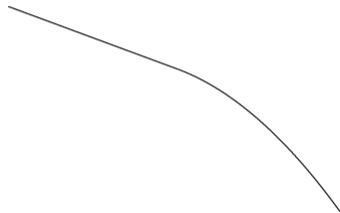
(a) extended  $u_1$



(b)  $u$



(c) extended  $u_2$



## Non-smooth solution: grid convergence

---

$h^{-1}$	sharp $\delta = 0$	EOC	sharp $\delta = 6h$	EOC	diffuse $\delta = 0$	EOC	diffuse $\delta = d_\Omega$	EOC
128	2.91e-05		2.91e-05		2.67e-05		2.67e-05	
256	7.31e-06	1.99	7.31e-06	1.99	8.29e-06	1.69	8.29e-06	1.69
512	1.83e-06	2.00	1.83e-06	2.00	1.38e-06	2.59	1.38e-06	2.59
1024	4.57e-07	2.00	4.57e-07	2.00	4.16e-07	1.73	4.16e-07	1.73
2048	1.03e-07	2.15	1.03e-07	2.15	9.79e-08	2.09	9.79e-07	2.09
4096	2.29e-08	2.17	2.29e-08	2.17	2.75e-08	1.83	2.75e-08	1.83

Domain:  $\Omega = (-1, 1)^2$

Level set function:  $\phi(x, y) = 0.75 - \sqrt{x^2 + y^2}$

$$\text{Domain: } \Omega = (-1, 1)^2$$

$$\text{Level set function: } \phi(x, y) = 0.75 - \sqrt{x^2 + y^2}$$

$$\mu_1 = 1 \quad \mu_2 = 10^3 \quad f_1 = f_2 = 4$$

$$\text{Domain: } \Omega = (-1, 1)^2$$

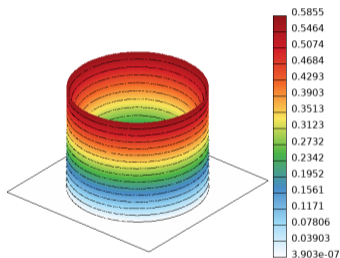
$$\text{Level set function: } \phi(x, y) = 0.75 - \sqrt{x^2 + y^2}$$

$$\mu_1 = 1 \quad \mu_2 = 10^3 \quad f_1 = f_2 = 4$$

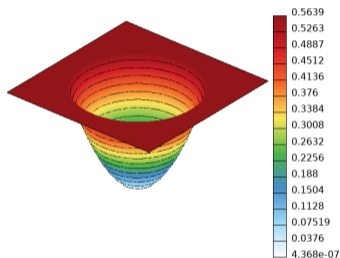
$$u_1(x, y) = x^2 - y^2$$

$$u_2(x, y) = \frac{x^2 - y^2}{1000} - \frac{0.5625}{1000} + 0.5625$$

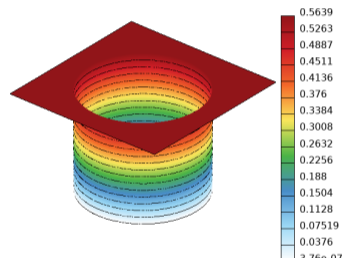
(a) extended  $u_1$



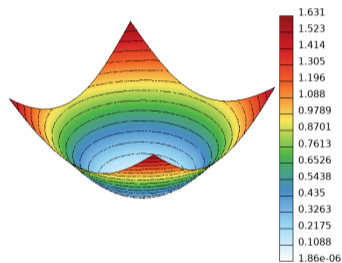
(b)  $u$



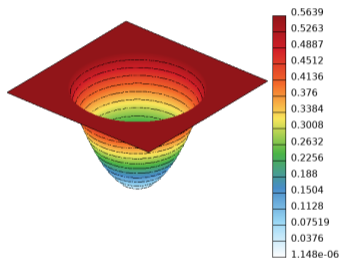
(c) extended  $u_2$



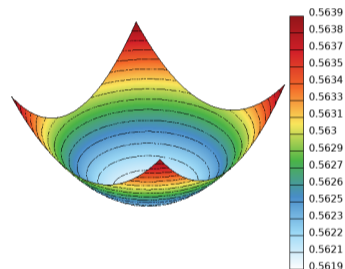
(a) extended  $u_1$



(b)  $u$



(c) extended  $u_2$



$h^{-1}$	sharp $\delta = 0$	EOC	sharp $\delta = 6h$	EOC	diffuse $\delta = 6h$	EOC	diffuse $\delta = d_\Omega$	EOC
8	6.33e-02		7.04e-02		7.03e-02		7.03e-02	
16	1.69e-02	1.91	1.97e-02	1.84	2.01e-02	1.81	2.01e-02	1.81
32	4.32e-03	1.97	5.16e-03	1.93	5.14e-03	1.97	5.16e-03	1.96
64	1.09e-03	1.99	1.25e-03	2.05	1.24e-03	2.05	1.32e-03	1.97
128	2.74e-04	1.99	2.96e-04	2.08	2.90e-04	2.10	3.30e-04	2.00
256	6.87e-05	2.00	7.16e-05	2.05	6.86e-05	2.08	8.15e-05	2.02
512	1.72e-05	2.00	1.76e-05	2.02	1.61e-05	2.09	1.97e-05	2.05
1024	4.31e-06	2.00	4.35e-06	2.02	3.60e-06	2.16	4.56e-06	2.11

## Summary





- parameter-free stabilization using projected gradients
- approximation of surface integrals by volume integrals
- fast closest-point search algorithm

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## Outlook

- extension to moving boundary problems
- application to PDE systems

-  M. Olshanskii and J.-P. Bäcker and D. Kuzmin, Gradient-penalty stabilization of sharp and diffuse interface formulations in unfitted Nitsche finite element methods. *arXiv-preprint* (2025, submitted to ESAIM:M2AN) arXiv:2501.16594.
-  D. Kuzmin and J.-P. Bäcker, An unfitted finite element method using level set functions for extrapolation into deformable diffuse interfaces. *J. Comput. Phys.* **461** (2022) 111218.
-  S. Zahedi and A.K. Tornberg, Delta function approximations in level set methods by distance function extension. *J. Comput. Phys.* **229** (2010) 2199–2219.
-  P. Hansbo and A. Hansbo, An unfitted finite element method, based on Nitsche's method, for elliptic interface problems. *Comput. Methods Appl. Mech. Engrg.* **191** (2002) 5537–5552.

**Thank you for your attention!**

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