Generalized quasi-Newtonian approach for modeling and simulating complex flows

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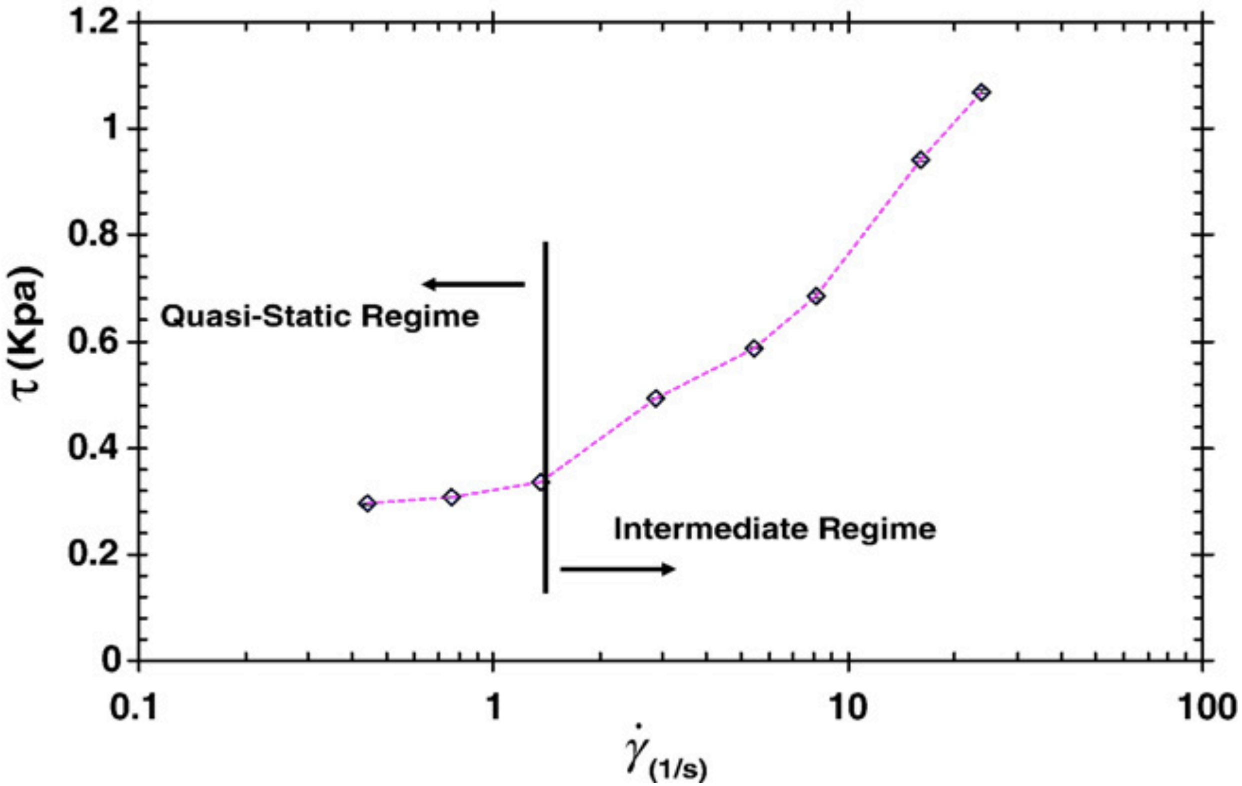
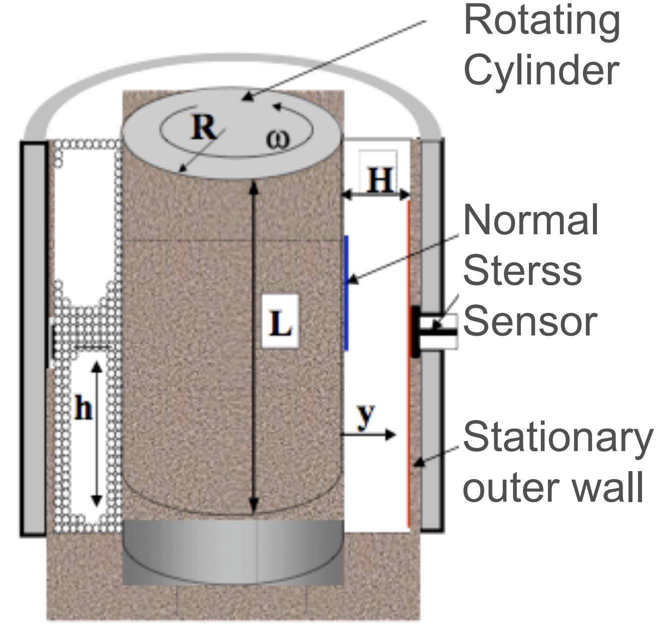
Axial flow experiment in the Couette device: spherical glass beads, 0.1 mm in diameter. (by Gabriel Tardos Group, CCNY)

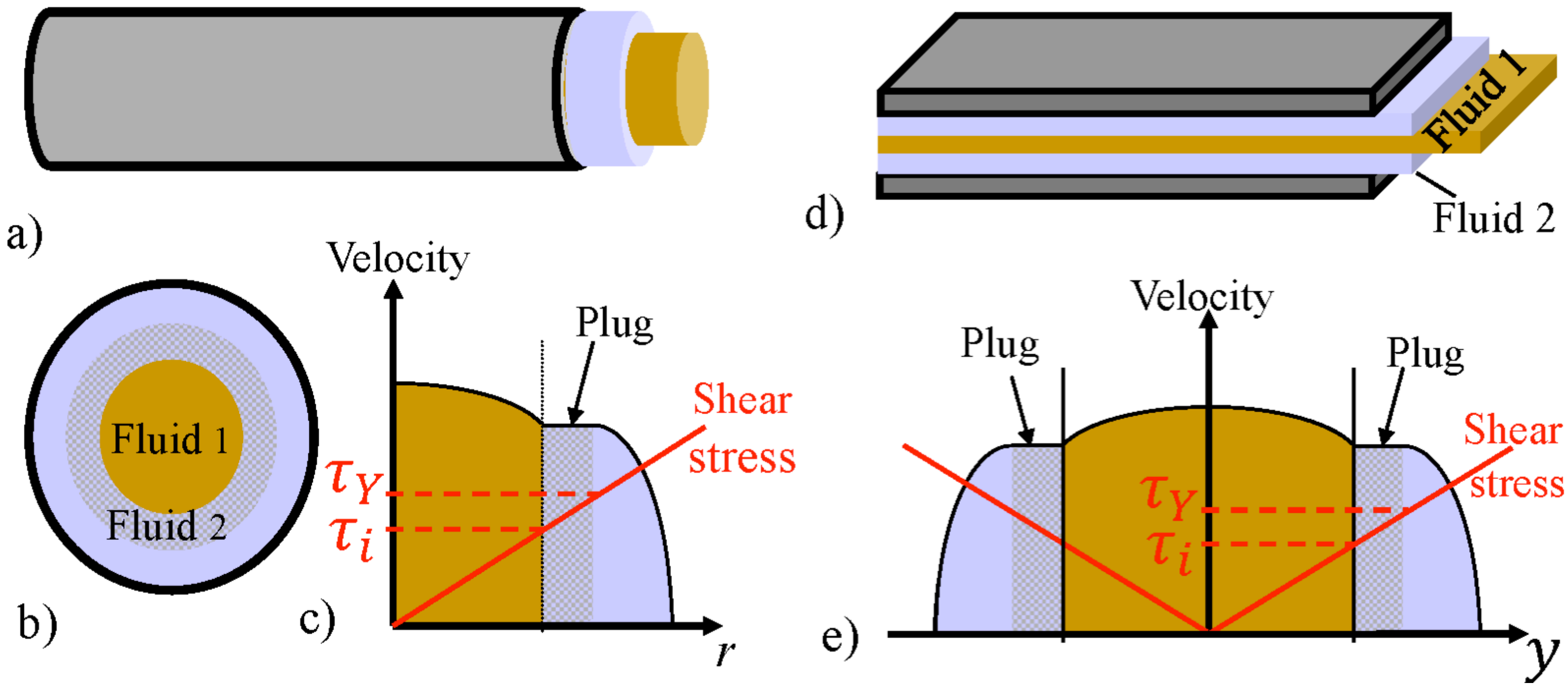


Axial flow device



- **The powder can transit from the quasi-static to the intermediate regime as the shearing rate is increased**

Shear and pressure dependent viscosity



Viscoplastic Lubricate Flow (Yield stress fluids)

- Dependent on the stress field
- Constitutive model is dependent on different flow regimes
- Non-smooth change in the constitutive relations

Model preserving the sharp changes of the constitutive equations w.r.t. flow regimes

Thixotropy concept

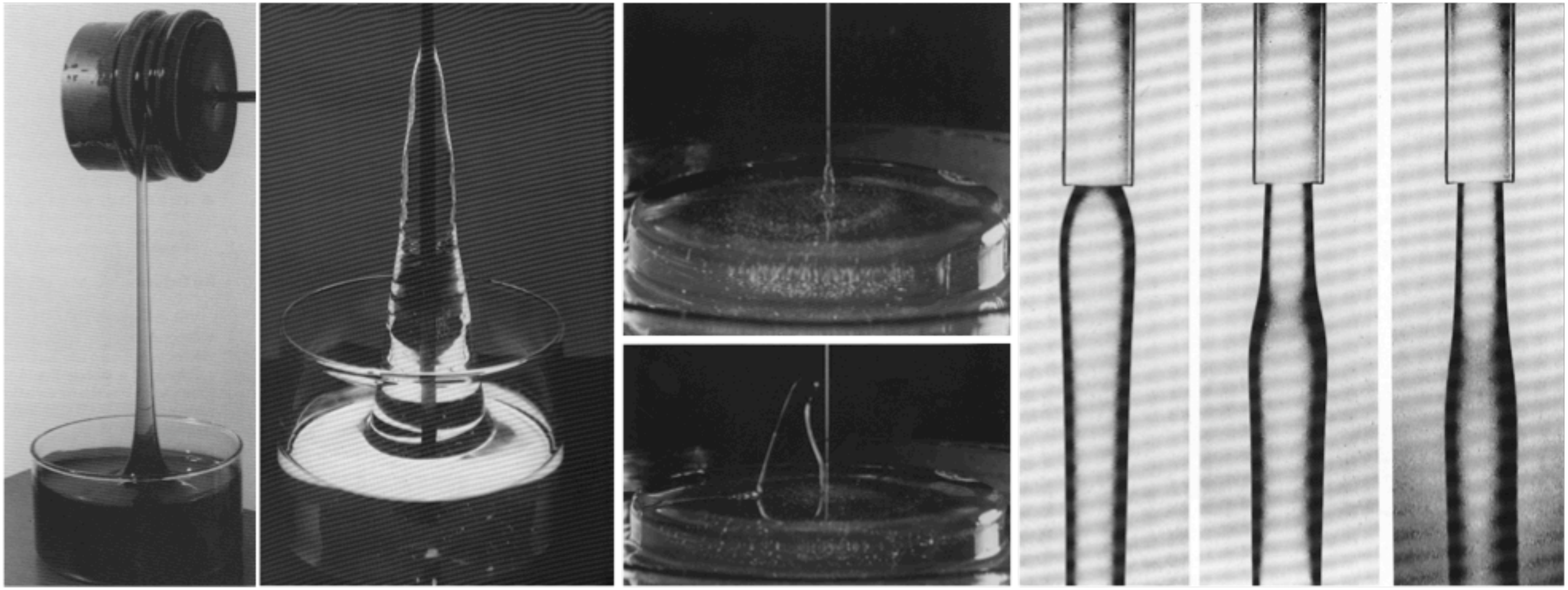
- Based on viscosity
- Flow induced by time-dependent
  decrease of viscosity
- The phenomena is reversible

- **Aging / Build-up**
  - At rest or under slow flow: fluid ages
    Increases of the viscosity in time

- **Rejuvenation / Breakdown**
  - “Faster” flow: fluid rejuvenates
    Decreases of viscosity with acceleration of the flow

Investigation of solid/liquid and liquid/solid transitions with non constant yield stress

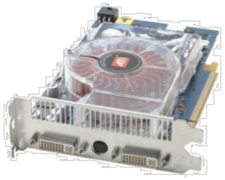
- Effects due to normal stresses
- Effects due to elongational viscosity
- The drag reduction phenomenon



Differential models

Realization in FeatFlow

HPC features:
- Moderately parallel
- GPU computing
- Open source




Hardware -oriented Numerics

Numerical features:
- Higher order FEM in space & (semi-) Implicit FD/FEM in time
- Semi-(un)structured meshes with dynamic adaptive grid deformation
- Fictitious Boundary (FBM) methods
- Newton-Multigrid-type solvers

Non-Newtonian flow module:
- generalized Newtonian model (Power-law, Carreau,...)
- viscoelastic differential model (Giesekus, FENE, Oldroyd,...)

Multiphase flow module (resolved interfaces):
- l/l – interface capturing (Level Set)
- s/l – interface tracking (FBM)
- s/l/l – combination of l/l and s/l

Engineering aspects:
- Geometrical design
- Modulation strategy
- Optimization

Here: FEM-based tools for the accurate simulation of (multiphase) flow problems, particularly with complex rheology

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- **Generalized Navier-Stokes equations**

$$\rho\left(\frac{\partial}{\partial t}+u\cdot\nabla\right)u-\nabla\cdot\sigma+\nabla p=\rho f,$$
$$\nabla\cdot u=0,$$
$$\sigma=\sigma_s+\sigma_p,$$

- **Viscous stress**

$$\sigma_s=2\eta_s(D_{\mathrm{II}},p)D(u),\qquad D_{\mathrm{II}}=\mathrm{tr}\left(D(u)^2\right).$$

- **Elastic stress**

$$\sigma_p+\mathrm{We}\frac{\delta_a\sigma_p}{\delta t}=2\eta_p D(u).$$

- **Viscous stress**

$$\sigma_s = 2\eta_s(D_{\mathrm{II}}, p) D(u), \qquad D_{\mathrm{II}} = \mathrm{tr}\left(D(u)^2\right)$$

➢ **Power law model**

$$\eta_s(z) = \eta_0 z^{r-\frac{1}{2}} \quad (\eta_0 > 0, r > 1)$$

➢ **Powder flow in the quasi-static and intermediate regimes**

$$\begin{cases} \eta_s(z,p) = \sqrt{2}p\left(\sin\phi z^{-\frac{1}{2}} + \cos\phi z^{r-\frac{1}{2}}\right) & \text{if } z \neq 0, r > 1 \\ \|\sigma_s\| \leq \sqrt{2}p\sin\phi & \text{else} \end{cases}$$

$(\phi : \text{the angle of internal friction})$

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- **Yield stress flow (Bingham Model)**

$$\begin{cases} \eta_s(z,\lambda) = \eta_0 + \tau_0 z^{-\frac{1}{2}} & \text{if } z \neq 0 \\ \|\sigma_s\| \leq \tau_0 & \text{else} \end{cases}$$

$(\tau_0 : \text{ yield stress})$

- **Thixotropic model**

$$\begin{cases} \eta_s(z,\lambda) = \eta(\lambda) + \tau(\lambda) z^{-\frac{1}{2}} & \text{if } z \neq 0 \\ \|\sigma_s\| \leq \tau(\lambda) & \text{else} \end{cases}$$

$(\lambda : \text{ structure parameter})$

  - **Structure parameter equation**

$$\frac{\partial \lambda}{\partial t} + u \cdot \nabla \lambda = a(1-\lambda) - b\lambda z^{\frac{1}{2}}$$

$(a, b \text{ are structure parameters})$

- **Elastic stress**

$$\sigma_p + \mathrm{We}\frac{\delta_a \sigma_p}{\delta t} = 2\eta_p D(u).$$

- **Upper/Lower convective derivative**

$$\frac{\delta_a \sigma}{\delta t} = \left(\frac{\partial}{\partial t} + u \cdot \triangledown\right)\sigma + g_a(\sigma, \nabla u)$$

$$g_a(\sigma, \nabla u) = \frac{1-a}{2}\left(\sigma\,\nabla u + (\nabla u)^{\mathrm{T}}\sigma\right) - \frac{1+a}{2}\left(\nabla u\,\sigma + \sigma\,(\nabla u)^{\mathrm{T}}\right) \quad (a = \pm 1)$$

- **Generalized differential constitutive model**

$$\sigma + \mathrm{We}\frac{\delta_a \sigma}{\delta t} + \mathbf{G}(\sigma, D) + \mathbf{H}(\sigma) = 2\eta_p D(u)$$

- **Oldroyd**

$$\mathbf{G} = 0, \quad \mathbf{H} = 0$$

- **Giesekus**

$$\mathbf{G} = 0, \quad \mathbf{H} = \alpha\,\mathrm{tr}(\sigma^2)$$

- **Phan-Thien and Tanner**

$$\mathbf{H} = \left[\exp\left(\alpha\,\mathrm{tr}(\sigma)\right) - 1\right]\sigma$$

- **White and Metzner**

$$\mathbf{G} = \alpha\left(2\,D : D\right)^{1/2}, \quad \mathbf{H} = 0$$

Stokes problem

- **Two-field formulation** (u, p)

$$\begin{cases} -\nabla \cdot \Big(2\eta D(u) \Big) + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = g_D & \text{on } \Gamma_D \end{cases}$$

- **Three-field formulation** (σ, u, p)

$$\begin{cases} \sigma - 2\eta D(u) = 0 & \text{in } \Omega \\ -\nabla \cdot \Big(2\eta(1-\alpha) D(u) + \alpha\sigma \Big) + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = g_D & \text{on } \Gamma_D \end{cases}$$

- **Two-field formulation** (u, p)
  - **Set** $\mathbb{V} := \left[H_0^1(\Omega)\right]^2, \mathbb{Q} := L_0^2(\Omega)$
  - **Find** $(u, p) \in \mathbb{V} \times \mathbb{Q}$ **s.t.**

$$\left\langle \mathcal{K}(u,p), (v,q) \right\rangle = \left\langle \mathcal{L}, (v,q) \right\rangle, \quad \forall (v,q) \in \mathbb{V} \times \mathbb{Q}$$

$$\mathcal{K} = \begin{pmatrix} \mathcal{A}_u & \mathcal{B}^{\mathrm{T}} \\ \mathcal{B} & 0 \end{pmatrix}$$

  - **Compatibily constraints**

$$\sup_{v \in \mathbb{V}} \frac{\langle \mathcal{B}v, q \rangle}{\|v\|_{\mathbb{V}}} \geq \beta \|q\|_{\mathbb{Q}/Ker\mathcal{B}^{\mathrm{T}}}, \quad \forall q \in \mathbb{Q}$$

- **Three-field formulation** (σ, u, p)

  - **Set** $\mathbb{T} := \left(L^2(\Omega)\right)^4_{\text{sym}}, \mathbb{V} := \left[H^1_0(\Omega)\right]^2, \mathbb{Q} := L^2_0(\Omega)$

  - **Find** $(\sigma, u, p) \in \mathbb{T} \times \mathbb{V} \times \mathbb{Q}$ **s.t.**

$$\left\langle \mathcal{K}(\sigma, u, p), (\tau, v, q) \right\rangle = \left\langle \mathcal{L}, (\tau, v, q) \right\rangle, \quad \forall (\tau, v, q) \in \mathbb{T} \times \mathbb{V} \times \mathbb{Q}$$

$$\mathcal{K} = \begin{pmatrix} \mathcal{A}_\sigma & \mathcal{C} & 0 \\ \mathcal{C}^{\mathrm{T}} & \mathcal{A}_u & \mathcal{B}^{\mathrm{T}} \\ 0 & \mathcal{B} & 0 \end{pmatrix}$$

  - **Compatibily constraints**

$$\sup_{v \in \mathbb{V}} \frac{\langle \mathcal{B} v, q \rangle}{\|v\|_{\mathbb{V}}} \geq \beta \|q\|_{\mathbb{Q}/Ker\mathcal{B}^{\mathrm{T}}}, \quad \forall q \in \mathbb{Q}$$

$$\sup_{v \in \mathbb{V}} \frac{\langle \mathcal{C} v, \tau \rangle}{\|v\|_{\mathbb{V}}} \geq \gamma \|\tau\|_{\mathbb{T}/Ker\mathcal{C}^{\mathrm{T}}}, \quad \forall \tau \in \mathbb{T}$$

- **Conforming approximations**

$$\mathbb{T}_h \subset \mathbb{T}, \quad \mathbb{V}_h \subset \mathbb{V} = \tilde{\mathbb{V}}, \quad \mathbb{Q}_h \subset \mathbb{Q}$$
$$\mathcal{A}_{\sigma h} = \mathcal{A}_\sigma, \, \mathcal{A}_{u h} = \mathcal{A}_u, \, \mathcal{B}_h = \mathcal{B}, \, \mathcal{C}_h = \mathcal{C}$$

- **Non-conforming approximation**

$$\mathbb{T}_h \subset \mathbb{T}, \quad \mathbb{V}_h \not\subset \mathbb{V} \,\&\, \mathbb{V}_h \subset \tilde{\mathbb{V}}, \quad \mathbb{Q}_h \subset \mathbb{Q}$$
$$\mathcal{A}_{\sigma h} = \mathcal{A}_\sigma, \, \mathcal{A}_{u h} \neq \mathcal{A}_u, \, \mathcal{B}_h \neq \mathcal{B}, \, \mathcal{C}_h \neq \mathcal{C}$$

- **Discrete inf-sup condition**

$$\sup_{v_h \in \mathbb{V}_h} \frac{\left\langle \mathcal{B}_h v_h, q_h \right\rangle}{\left\| v_h \right\|_{\tilde{\mathbb{V}}}} \geq \beta_h \left\| q_h \right\|_{\mathbb{Q}/Ker\mathcal{B}_h^{\mathrm{T}}}, \quad \forall q_h \in \mathbb{Q}_h$$
$$\sup_{v_h \in \mathbb{V}_h} \frac{\left\langle \mathcal{C}_h v_h, \tau_h \right\rangle}{\left\| v_h \right\|_{\tilde{\mathbb{V}}}} \geq \gamma_h \left\| \tau_h \right\|_{\mathbb{T}/Ker\mathcal{C}_h^{\mathrm{T}}}, \quad \forall \tau_h \in \mathbb{T}_h$$

The family of non-/conforming FEM $Q_r/P_{r-1}^{\text{disc}},\ r \geq 2$ and the family of nonconforming FEM $\tilde{Q}_r/P_{r-1}^{\text{disc}},\ r \geq 2$ for (u,p)

- Inf-sup stable
- Arbitrary order with optimal convergence order
- Discontinuous pressure
  - Good for the solver
  - Element-wise mass conservation

The family of conforming FEM $Q_r/Q_r/P_{r-1}^{\text{disc}},\ r \geq 2$ for (σ,u,p) with stabilization

$$J_u(u_h,v_h) = \gamma_u \sum_{e\in\mathcal{E}_h} 2\eta\alpha h \int_e [\nabla u_h]:[\nabla v_h]\, d\Omega$$

- Both Inf-sup conditions are satisfied
- Highly consistent and symmetric stabilization, penelazing any spurious current, enhancing the preconditioner, which improve accuracy and efficiency
- None tensorial FEM approximation for the tensorial field
  - Robust solver w.r.t. the monolitic approach
  - Efficient solver w.r.t. multigird solver

- **Standard geometric multigrid solver**
- **Full Q_r and P_{r-1}^{disc} restriction and prolongation**
- **Local Multilevel Pressure Schur Complement via Vanka-like smoother**

$$\begin{pmatrix}\sigma^{l+1}\\ u^{l+1}\\ p^{l+1}\end{pmatrix} = \begin{pmatrix}\sigma^{l}\\ u^{l}\\ p^{l}\end{pmatrix} + \omega^l \sum_{T\in\mathcal{T}_h}\left(\left(\mathcal{K}_h+\mathcal{J}_u\right)_{|T}\right)^{-1}\begin{pmatrix}\mathcal{R}_{\sigma^l}\\ \mathcal{R}_{u^l}\\ \mathcal{R}_{p^l}\end{pmatrix}_{|T}$$

Coupled Monolithic Multigrid Solver !

Flow around cylinder Benchmark tests (by Aaqib Afaq)

➢ Two field formulation versus three field formulation

Level	Three-field			Two-field[1]		
	Lift	Drag	NL/LL	Lift	Drag	NL/LL
1	0.009498	5.5550	7/2	0.009498	5.5550	9/2
2	0.010601	5.5722	7/2	0.010601	5.5722	9/2
3	0.010616	5.5776	7/2	0.010616	5.5776	9/1
4	0.010618	5.5791	7/1	0.010618	5.5791	8/1

[1] Damanik. H "FEM Simulation of Non-isothermal Viscoelastic fluids", PhD Thesis

➢ Consistency of the stabilization for three field formulation

Level	α	No stabilization			With stabilization		
		Lift	Drag	NL/LL	Lift	Drag	NL/LL
2	0	0.010601	5.5722	7/2	0.010702	5.5674	7/2
3	0	0.010616	5.5776	7/2	0.010619	5.5757	7/2
4	0	0.010618	5.5791	7/1	0.010617	5.5782	7/2
5	0	0.010619	5.5794	6/2	0.010618	5.5790	6/3

Flow around cylinder Benchmark tests (by Aaqib Afaq)

- **Robustness and efficiency of the stabilization for three field formulation without any viscous contribution**

Level	α	No stabilization			With stabilization		
		Lift	Drag	NL/LL	Lift	Drag	NL/LL
2	0	0.010601	5.5722	7/2	0.010702	5.5674	7/2
3	0	0.010616	5.5776	7/2	0.010619	5.5757	7/2
4	0	0.010618	5.5791	7/1	0.010617	5.5782	7/2
5	0	0.010619	5.5794	6/2	0.010618	5.5790	6/3
2	1	———	——	—	0.010588	5.5520	7/2
3	1	———	——	—	0.010600	5.5698	7/2
4	1	———	——	—	0.010612	5.5756	7/2
5	1	———	——	—	0.010617	5.5778	7/3

Accurate, robust and efficient monolitic-multigrid Stokes solver in two-field and three-field formulations !

Multiphase flow problem

Incompressible N-S Equation)

$$\rho(\Gamma)\left(\frac{\partial}{\partial u} + u \cdot \nabla\right) u - \nabla \cdot \sigma_s + \nabla p = 0$$

- **Viscous stress**

$$\sigma_s = 2\eta_s(\Gamma) D(u)$$

- **Interface boundary conditions**

$$[u]_{|\Gamma} = 0$$

$$-[p\mathrm{I} + \sigma_s]_{|\Gamma} \cdot n = \sigma \kappa n$$

Multiphase flow problem

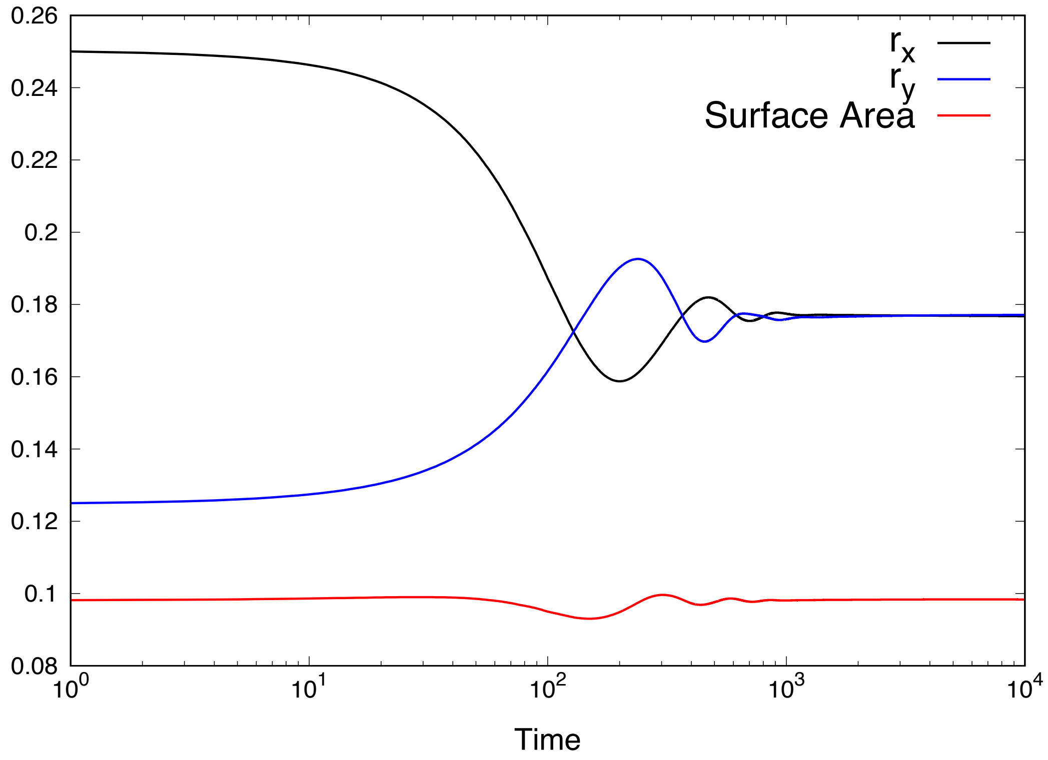
- **New extra stress for multiphase flow**

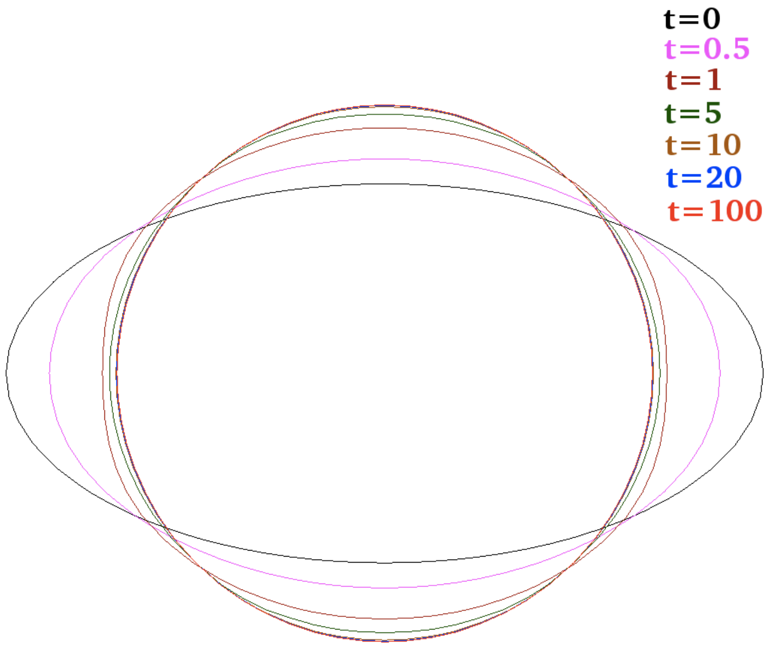
$$\sigma_m = -\sigma \left(\frac{\nabla\psi \otimes \nabla\psi}{|\nabla\psi|} \right)$$

- **Full set of equations for multiphase flow**

$$\begin{cases} \rho(\psi)\left(\dfrac{\partial u}{\partial t} + u \cdot \nabla u\right) - \nabla \cdot \tau + \nabla p = 0 \\ \nabla \cdot u = 0 \\ \tau - 2\eta_s(\psi) D(u) + \sigma \left(\dfrac{\nabla\psi \otimes \nabla\psi}{|\nabla\psi|} \right) = 0 \\ \dfrac{\partial \psi}{\partial t} + u \cdot \nabla\psi + \nabla \cdot \left(\gamma_{nc} \psi (1-\psi) \dfrac{\nabla\psi}{|\nabla\psi|} \right) \\ \qquad -\nabla \cdot \left(\gamma_{nd} \left(\nabla\psi \cdot \dfrac{\nabla\psi}{|\nabla\psi|} \right) \dfrac{\nabla\psi}{|\nabla\psi|} \right) = 0 \end{cases}$$

Oscillating bubble (by Aaqib Afaq)





Robust nonlinear solver based on Newton's method following a specefic path of convergence using the residual's convergence

- **Robust w.r.t. starting guesses**
- **Dealing with Jacobian's singularities using generalized deriviatives or approximated one**
- **Full benefit from the quadratic convergence's region of classical Newton's method**

Let $\mathcal{U} = (u,p)$, (σ,u,p), or (σ,u,p,φ) and $\mathcal{R}_{\mathcal{U}}(\mathcal{U})$ be the continuous or the discrete corresponding system's residum.

- **Update of the nonlinear iteration with the correction $\delta\mathcal{U}$ i.e.**

$$\mathcal{U}^N = \mathcal{U} + \delta\mathcal{U}$$

- **The linearization of the residual provides**

$$\begin{aligned}\mathcal{R}_{\mathcal{U}}\left(\mathcal{U}^N\right) &= \mathcal{R}_{\mathcal{U}}\left(\mathcal{U} + \delta\mathcal{U}\right) \\ &= \mathcal{R}_{\mathcal{U}}\left(\mathcal{U}\right) + \mathcal{J}\left(\mathcal{U}\right)\cdot\delta\mathcal{U}\end{aligned}$$

- **The Newton's method assuming invertible Jacobian**

$$\mathcal{U}^N = \mathcal{U} - \mathcal{J}^{-1}\left(\mathcal{U}\right)\cdot\mathcal{R}_{\mathcal{U}}\left(\mathcal{U}\right)$$

Jacobian calculations

$$\mathcal{J}\left(\mathcal{U}\right) = \left(\frac{\partial \mathcal{R}_{\mathcal{U}}\left(\mathcal{U}\right)}{\partial \mathcal{U}}\right)$$

- **Exact G-Newton based on a priori study of Jacobian's properties and decompositions**

$$\mathcal{J}\left(\mathcal{U}\right) = \left(\frac{\partial \hat{\mathcal{R}}_{\mathcal{U}}\left(\mathcal{U}\right)}{\partial \mathcal{U}}\right) + \delta \left(\frac{\partial \tilde{\mathcal{R}}_{\mathcal{U}}\left(\mathcal{U}\right)}{\partial \mathcal{U}}\right)$$

- **Inexact G-Newton based on the residum's convergence**

$$\left(\frac{\partial \bar{\mathcal{R}}_{\mathcal{U}}\left(\mathcal{U}\right)}{\partial \mathcal{U}}\right)_{ij} \approx \left(\frac{\bar{\mathcal{R}}_{\mathcal{U}}\left(\mathcal{U} + \epsilon^{+} e_j\right) - \bar{\mathcal{R}}_{\mathcal{U}}\left(\mathcal{U} - \epsilon^{-} e_i\right)}{\left(\epsilon^{+} + \epsilon^{-}\right)}\right),$$

$$\bar{\mathcal{R}} = \mathcal{R},\ \hat{\mathcal{R}},\ \text{or}\ \tilde{\mathcal{R}}\ .$$

Three-field viscoplastic application

- **Viscoplatic constitutive law**

  - **Bingham constitutive law**

$$\begin{cases} \sigma = 2\eta D(u) + \tau_0 \dfrac{D(u)}{\|D(u)\|} & \text{if } \|D(u)\| \neq 0 \\ \|\sigma\| \leq \tau_0 & \text{if } \|D(u)\| = 0 \end{cases}$$

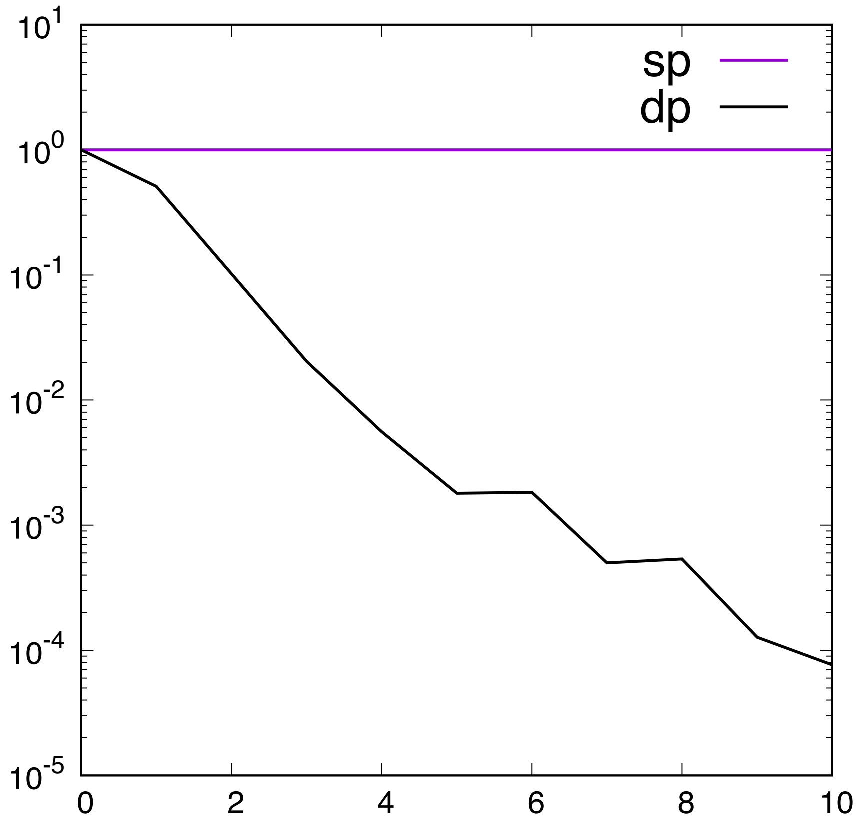
  - **New extra stress σ_{Y_0} for viscoplastic flow s.t.**

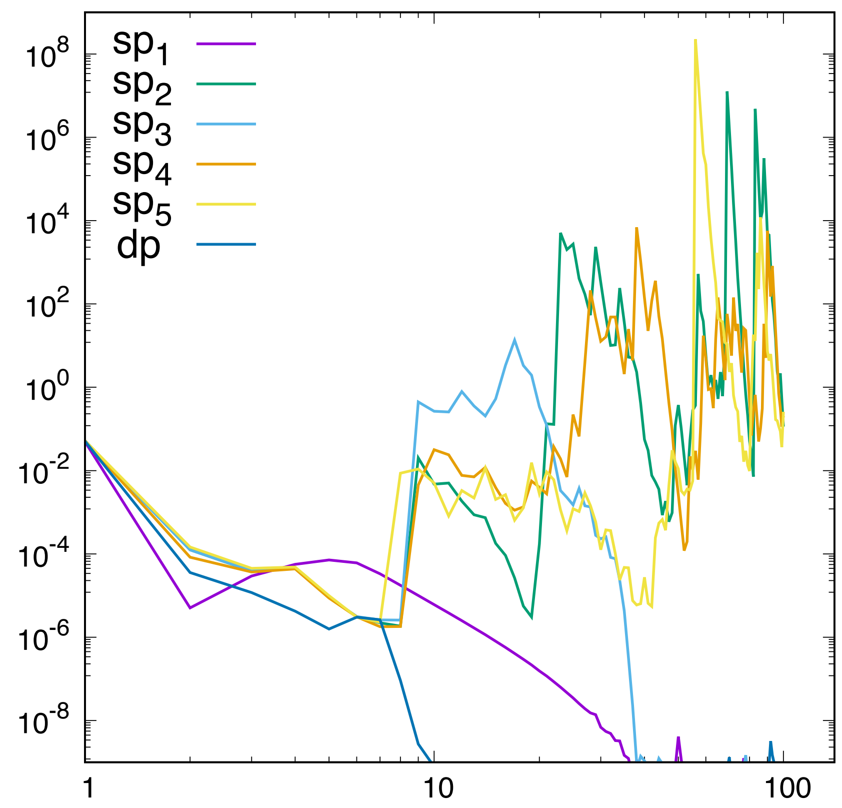
$$\|D(u)\| \sigma_{Y_0} = D(u)$$

- **Three-field viscoplastic set of equations**

$$\begin{cases} \|D(u)\| \sigma_{Y_0} - D(u) = 0 & \text{in } \Omega \\ -\nabla \cdot \left(2\eta D(u) + \tau_0 \sigma_{Y_0} \right) + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \end{cases}$$

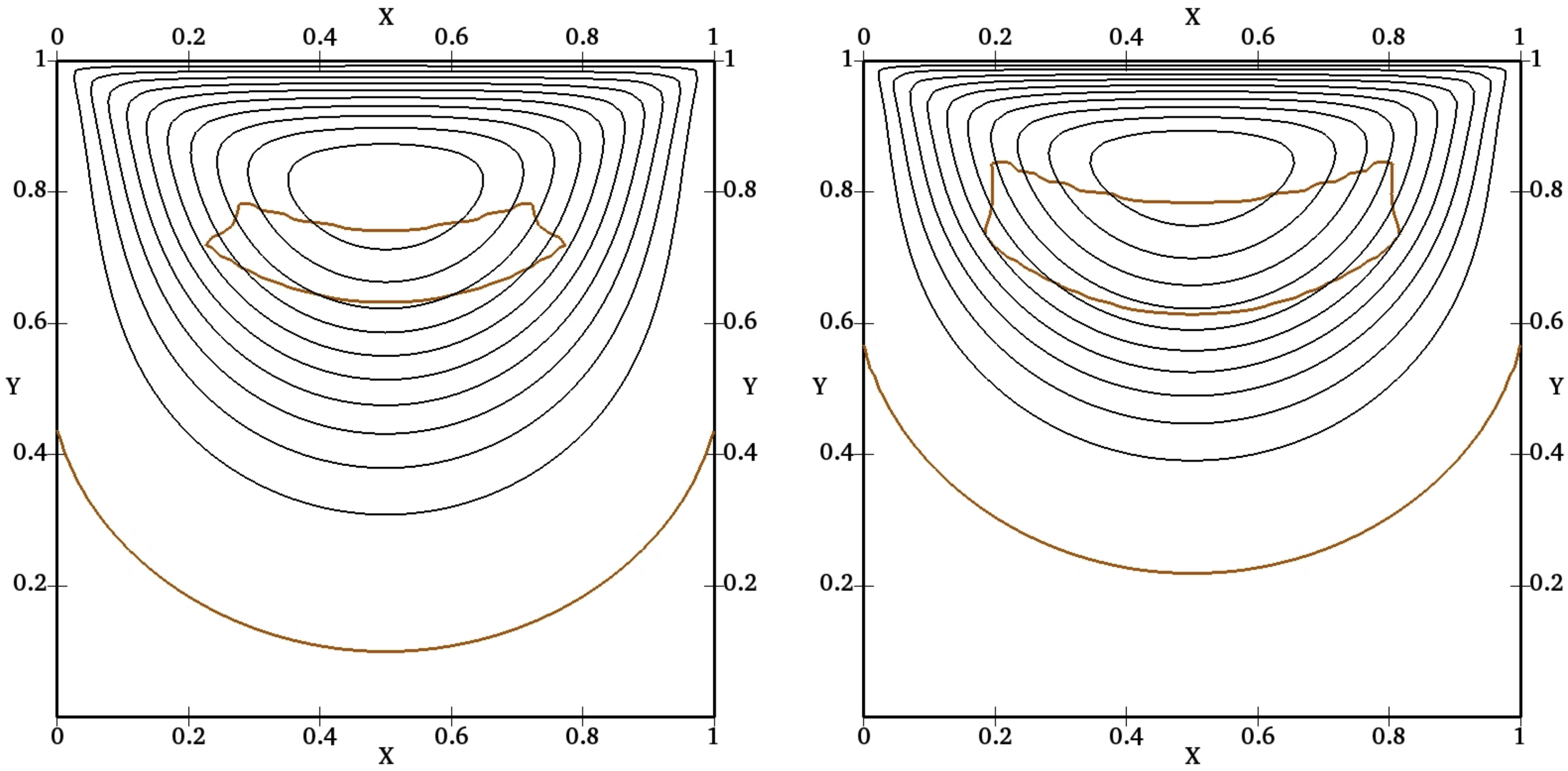
Dynamic path versus static one w.r.t. number of iterations, and the corresponding convergence of the residium (by Arooj Fatima)





Lid driven cavity benchmark

Unyielded zone for two different yield stresses, $\tau_0 = 2$, and $\tau_0 = 5$ (by Arooj Fatima)



Generalized differential constitutive model

$$\sigma + \mathrm{We}\frac{\delta_a \sigma}{\delta t} + \mathbf{G}(\sigma, D) + \mathbf{H}(\sigma) = 2\eta_p D(u)$$

- **Oldroyd**

$$\mathbf{G} = 0, \quad \mathbf{H} = 0$$

- **Giesekus**

$$\mathbf{G} = 0, \quad \mathbf{H} = \alpha\,\mathrm{tr}(\sigma^2)$$

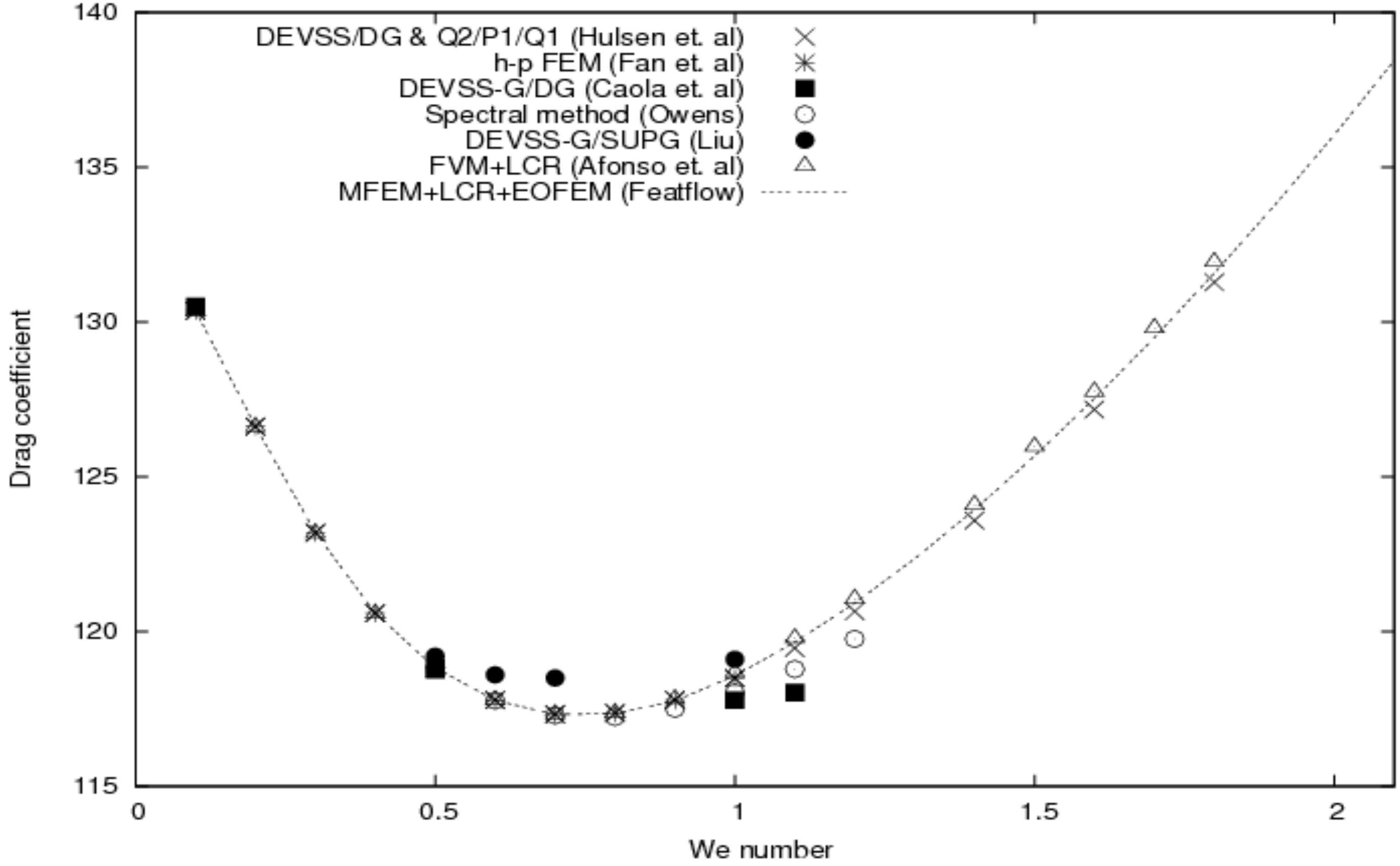
- **Phan-Thien and Tanner**

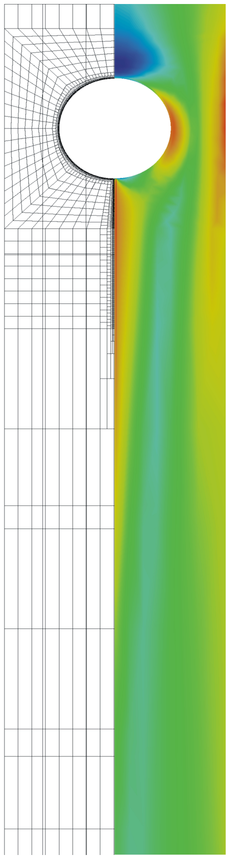
$$\mathbf{H} = \left[\exp\left(\alpha\,\mathrm{tr}(\sigma)\right) - 1\right]\sigma$$

- **White and Metzner**

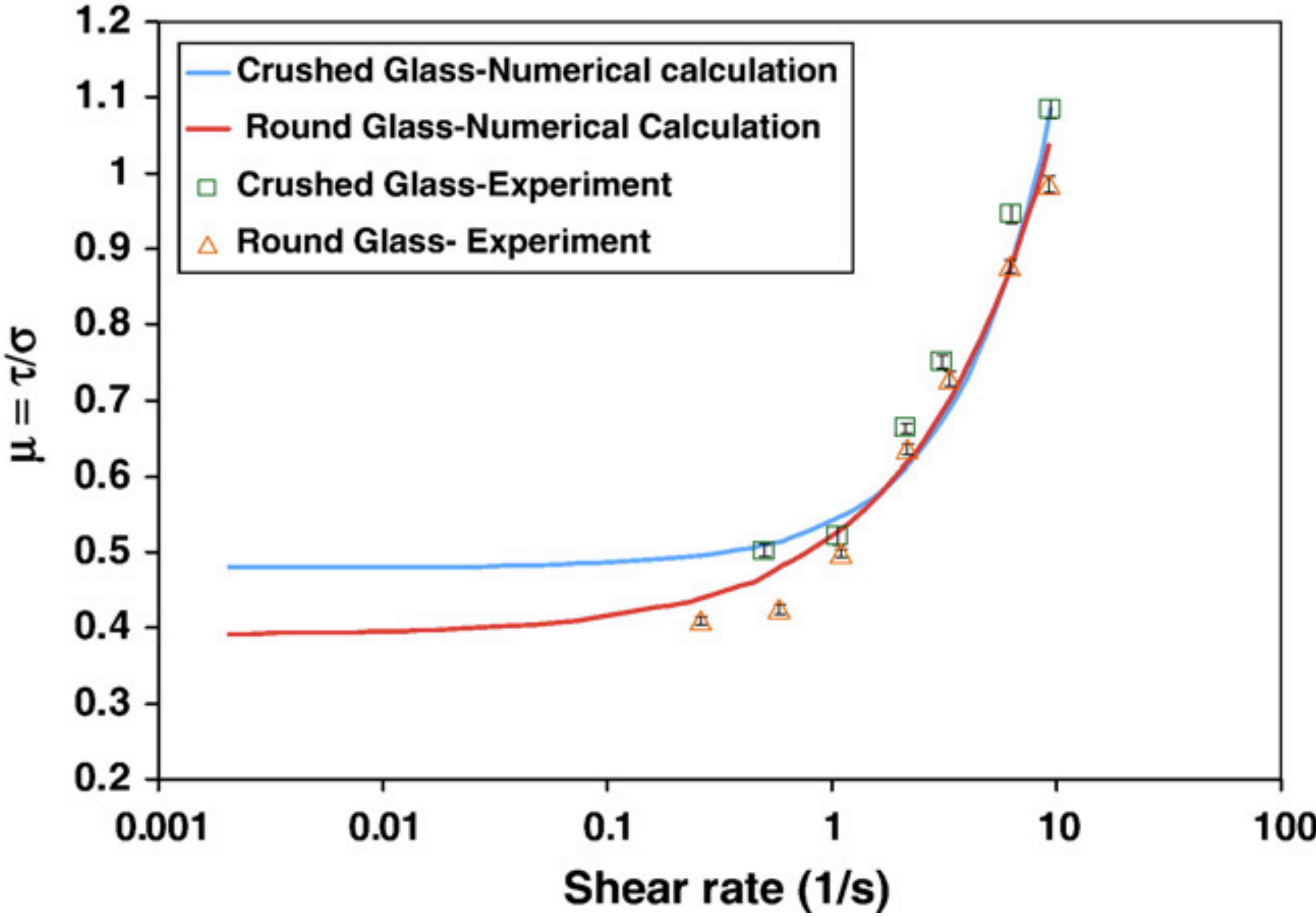
$$\mathbf{G} = \alpha\left(2\,D : D\right)^{1/2}, \quad \mathbf{H} = 0$$

- **Planar flow around cylinder Oldroyd-B (by Hogenrich Damanik)**





- **Experimental and numerical results for dry, frictional powder flows in the quasi-static and intermediate regimes**



The numerical method do not introduce errors !

- **Viscosity model for thixotropic flow i.e. extended viscosity defined on all domaine s.t.**

$$\begin{cases} \eta_s(\|D(u)\|,\lambda) = \eta(\lambda) + \tau(\lambda)\|D(u)\|^{-\frac{1}{2}} & \text{if } \|D(u)\| \neq 0 \\ \|\sigma_s\| \leq \tau(\lambda) & \text{else} \end{cases}$$

$(\lambda : \text{ structure parameter})$

- **Structure equation**

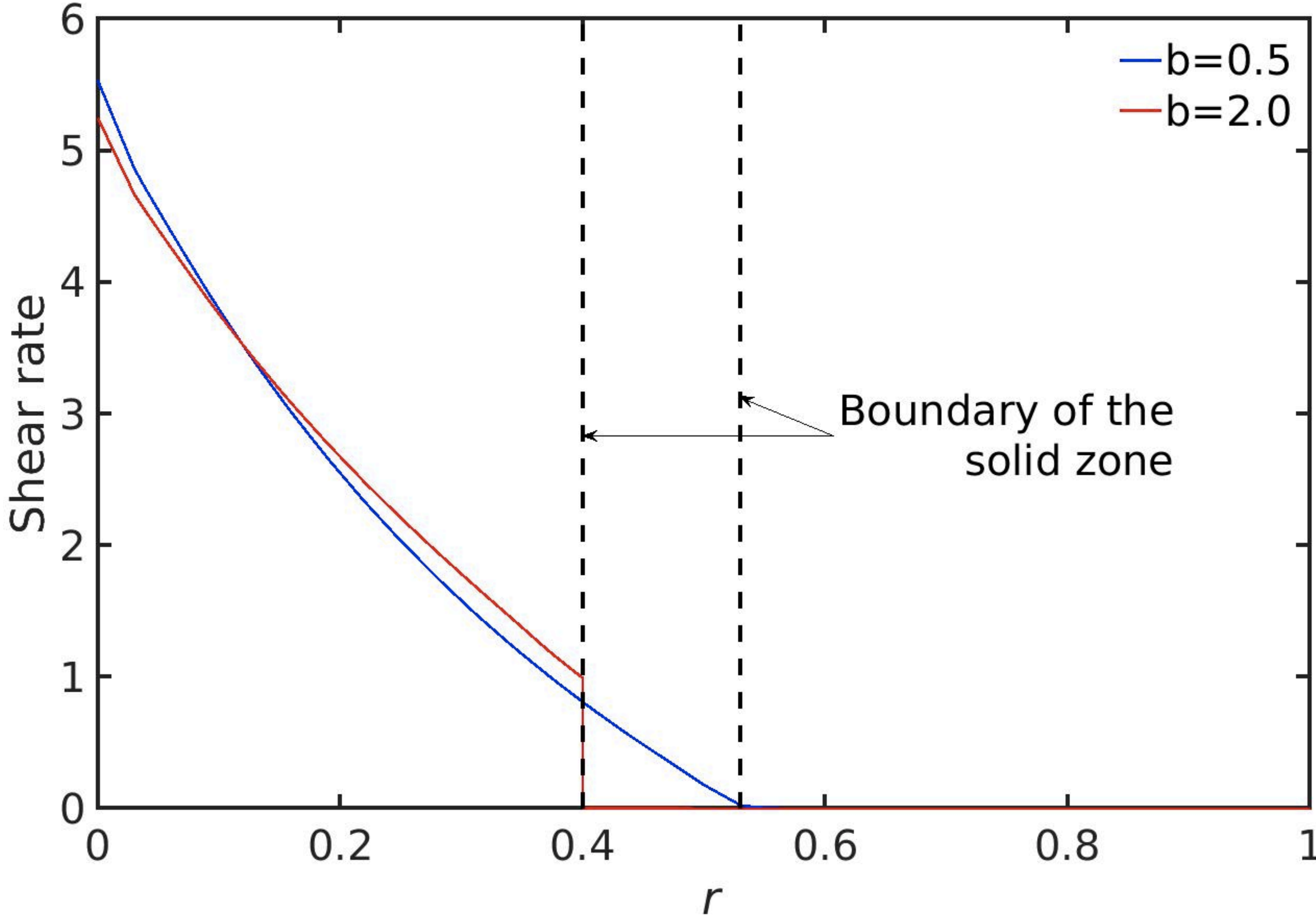
$$\frac{\partial \lambda}{\partial t} + u \cdot \nabla \lambda = a(1-\lambda) - b\lambda z^{\frac{1}{2}}$$

$(a, b \text{ are structure parameters})$

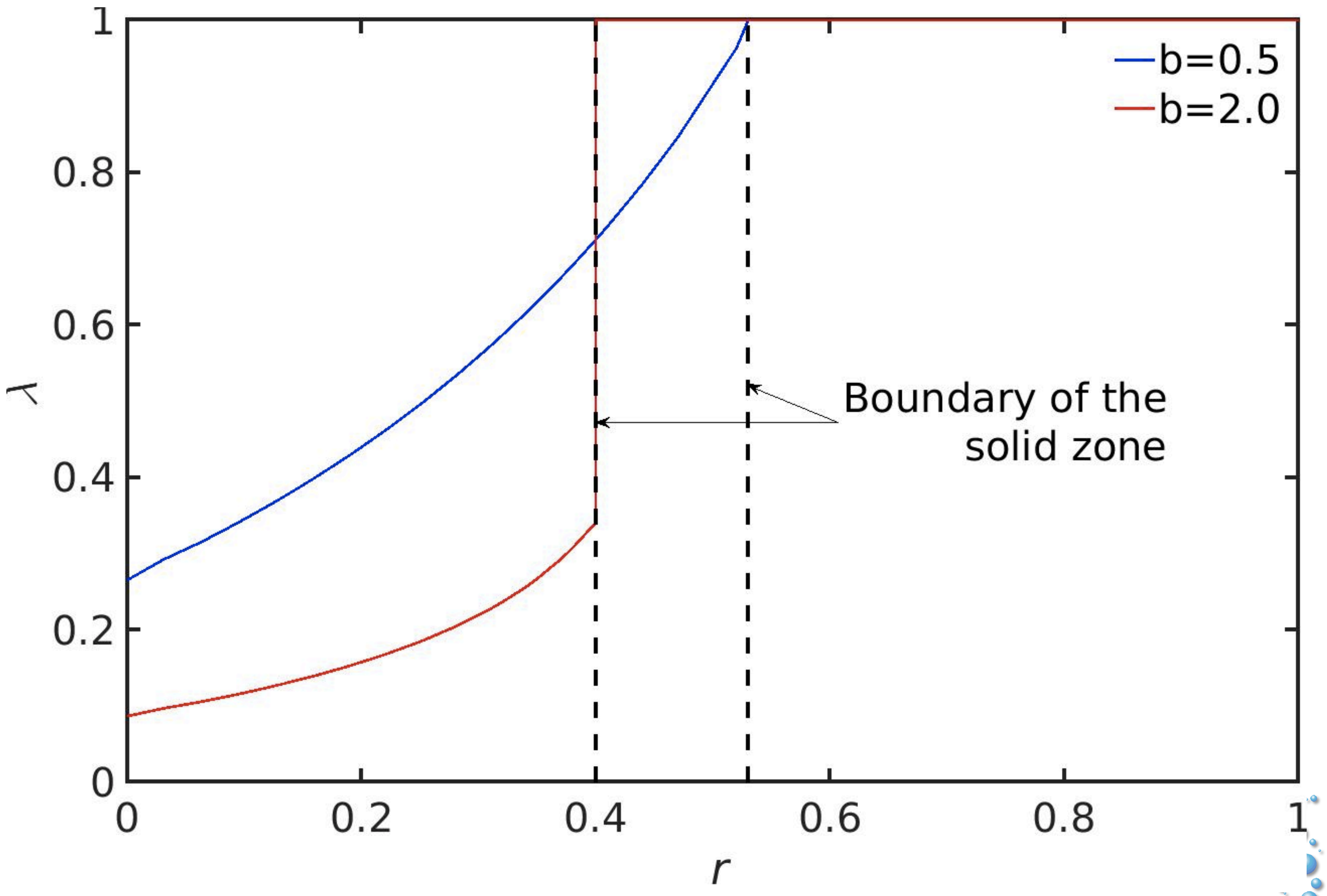
- **Full set of equations**

$$\begin{cases} \left(\frac{\partial}{\partial t} + u \cdot \nabla\right) u - \nabla \cdot \Big(2\eta_s(\|D(u)\|,\lambda)D(u)\Big) + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ \frac{\partial \lambda}{\partial t} + u \cdot \nabla \lambda - a(1-\lambda) + b\lambda\|D(u)\| = 0 & \text{in } \Omega \end{cases}$$

Shear rate in a couette w.r.t. breakdown parameter (by Naheed Begum)



Structure parameter in a couette w.r.t. breakdown (by Naheed Begum)



- ✓ **shear history effect**
- ✓ **time history effect**
- ✓ **Hysteresis**
- ✓ **stress overhoots**

A quasi-Newtonian model for thixotropic phenomena via a time and shear dependent viscosity

- **Include the non-Newtonian stress or any extra stress in diffusion operator**
- **Get rid of a tensorial field**
  - **Less constraints for the choices of FE approximation**
  - **Robust and efficient numerical algorithms**
  - **Simple evolution equations !**
- **Proof of the concept and validation**

- **Divergence form**

$$\mathcal{L}\,u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}\, u\, \frac{\partial}{\partial x_i} \right)$$

- **Weak form**

$$\mathcal{L}_{\mathrm{W}}\,u = \sum_{i,j=1}^{N} A_{ij} : \left(\nabla \cdot e_j \otimes \nabla \cdot\, e_i \right)\, u$$

Benefit of the weak form representation !

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Weak form representation 2D

$$\mathcal{L}_{\mathrm{w}}\, u = \sum_{k,l=1}^{2} \sum_{i,j=1}^{N} [A_{kl}]_{ij} : \left(\nabla \cdot e_j \otimes \nabla \cdot e_i \right) u_j^l, \quad k = 1,2$$

- **Gradient formulation**

$$A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Deformation formulation**

$$A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

Different deriviatives combinations accessibilities !

Weak form representation 2D

$$\mathcal{L}_{\mathrm{w}}\, u = \sum_{k,l=1}^{2} \sum_{i,j=1}^{N} [A_{kl}]_{ij} : \left(\nabla \cdot e_j \otimes \nabla \cdot e_i\right) u_j^l, \quad k = 1, 2$$

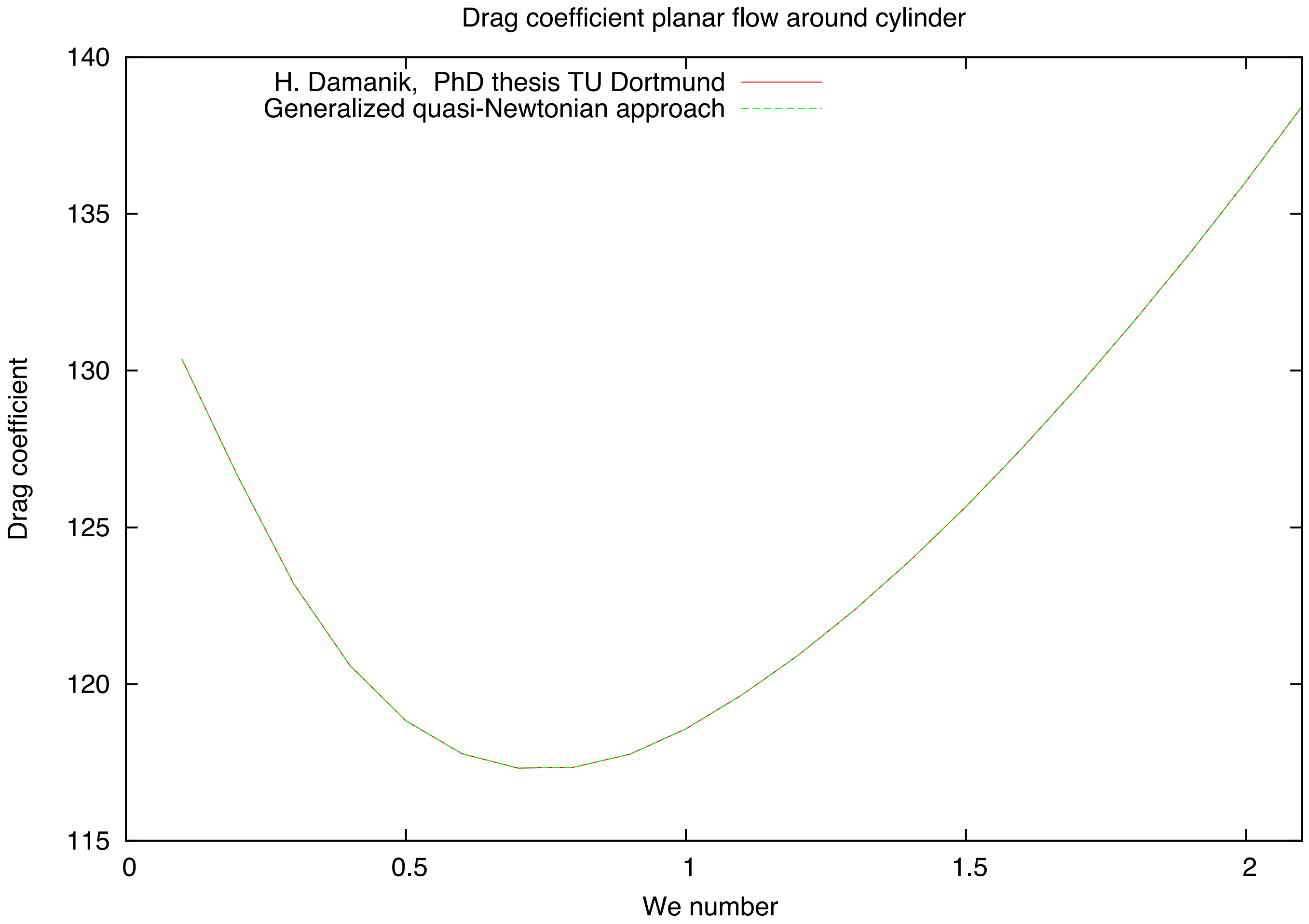
- **Generalized formulation I**

$$A_{11} = \begin{bmatrix} a_{11} & \frac{1}{2} a_{21} \\ \frac{1}{2} a_{12} & \frac{1}{4}(a_{11} + a_{22}) \end{bmatrix}, \quad A_{12} = \begin{bmatrix} \frac{1}{2} a_{12} & \frac{1}{4}(a_{11} + a_{22}) \\ 0 & \frac{1}{2} a_{12} \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} \frac{1}{2} a_{21} & 0 \\ \frac{1}{4}(a_{11} + a_{22}) & \frac{1}{2} a_{21} \end{bmatrix}, \quad A_{22} = \begin{bmatrix} \frac{1}{4}(a_{11} + a_{22}) & \frac{1}{2} a_{21} \\ \frac{1}{2} a_{12} & a_{22} \end{bmatrix}$$

More deriviatives combinations accessibilities are allowed !

- **Planar flow around cylinder Oldroyd-B (by Hogenrich Damanik)**



Genalized quasi-Newtonian approach for non-Newtonian problem i.e. Oldroyd-B !

Summary

New generalized quasi-Newtonian approach for modeling and simulating complex flows is introduced and validated.

Based on new numerical and algorithmic tools using

- ✓ **Monolithic FEM two-field and three-field Stokes solver**
- ✓ **Generalized Newton's method w.r.t. singularities with global convergent property**
- ✓ **Edge Oriented stabilization (EO-FEM)**
- ✓ **Fast Multigrid Solver with local MPSC smoother**

Extensively tested from numerical and physical perspectives via the simulations of different flow problems in different formulations to motivate the newly introduced generalized quasi-Newtonian approach.