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Workshop on

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Multigrid Method for Stabilized Nonconforming Finite Element for Incompressible Flow

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1 Motivation

1.1 Quasi-Newtonian fluids

A fluid is called Quasi-Newtonian if the Cauchy stress tensor is given by

$$\sigma = 2\nu(D_{II}(u))D(u) \qquad (1)$$

where

$$D_{II}(u) = \frac{1}{2}D(u) : D(u) = \frac{1}{2}\sum_{i,j} D_{i,j}(u)D_{i,j}(u),$$

and $\nu(\cdot)$ the viscosity

Models

- Power law defined for

$$\nu(z) = \nu_0 z^{\frac{r}{2}-1}$$

- Carreau law defined for

$$\nu(z) = \nu_\infty + (\nu_0 - \nu_\infty)(1 + \lambda z)^{\frac{r}{2}-1}$$

- Schaeffer model (Granular flow) defined for

$$pz^{\frac{-1}{2}}$$

- Further extension (planned)

$$\nu(D, p, T)$$

1.2 Nonlinear Newton variants

Let u^l being an initial state, Newton method reads

$$\begin{aligned} & \int_\Omega \boxed{2\nu(D_{II}(u^l))} D(u) : D(v) \\ + \quad & \int_\Omega \boxed{2\nu'(D_{II}(u^l))} [D(u^l) : D(u)][D(u^l) : D(v)] \\ = \quad & \int_\Omega fv - \int_\Omega 2\nu(D_{II}(u^l)) D(u^l) : D(v) \qquad \forall v \in V \end{aligned} \tag{2}$$

where $\nu'(\cdot)$ is the derivative of ν

This leads to the sequence of linear problem

$$A(u^l)v + \delta A^*(u^l)v = Res(u^l).$$

2 Finite Element Approximation

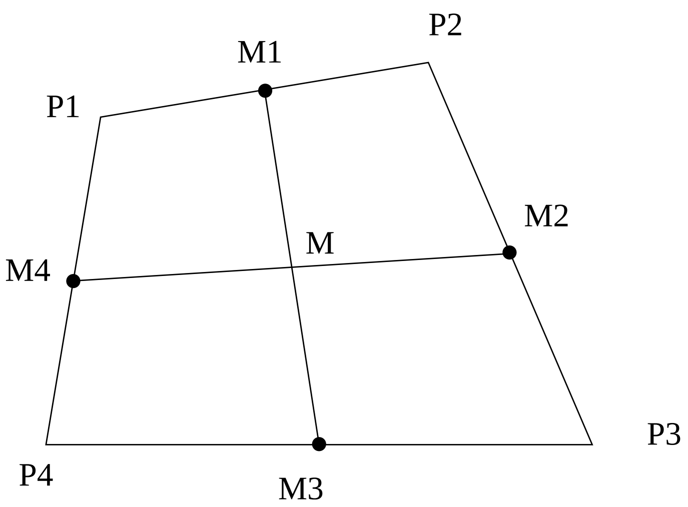


Figure 1: Rannacher-Turek element.

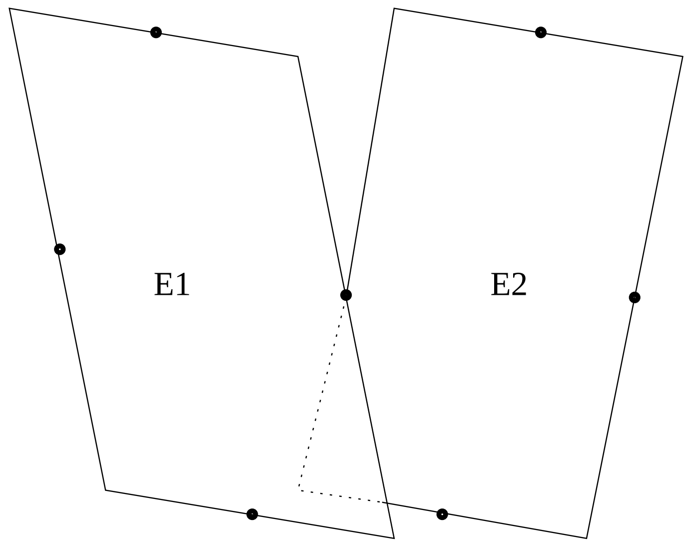


Figure 2: Continuity across the edges.

- Advantage:
  - Stable and efficient for incompressible flow.
  - Compact data structures.

- Disadvantage:
  - Not satisfying discrete Korn's inequality

$$\|D(u)\|_{L_2(\Omega)} \geq \text{Const}\|u\|_{H^1(\Omega)}.$$

- Remedy (cf. Larson and Hansbo work)
  Introduce **the jump** in a discrete formulation (Nitsche's method) as a stabilization parameter of **order h^{-1}**,

$$[E_h]_{i,j} = h^{-1} \sum_{\text{Edges}} \int_{\text{Edge}} [\phi_i][\phi_j] ds \qquad (3)$$

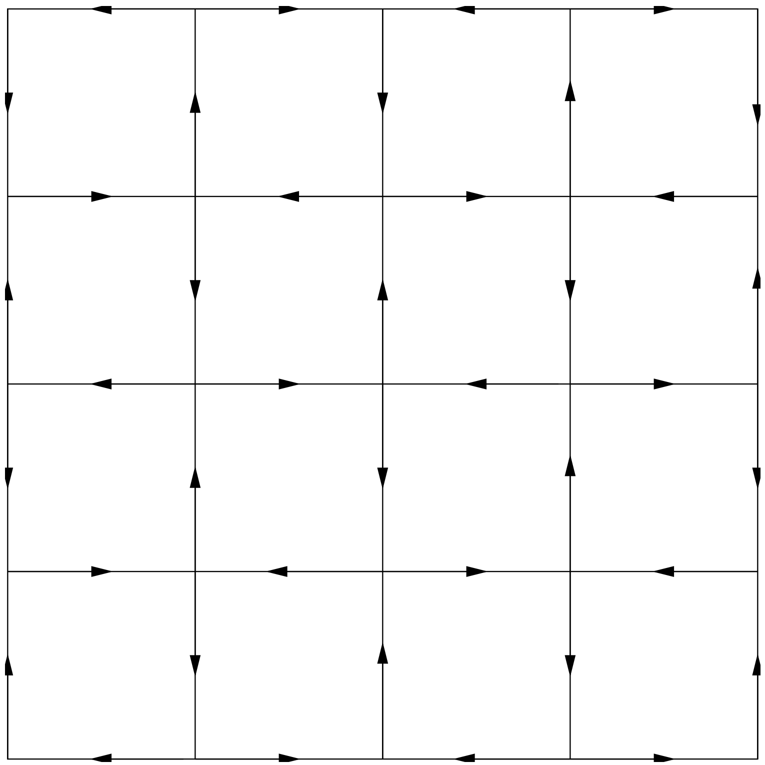
The resulting system in a matrix form reads:

$$\begin{cases} A_h(u^l)u + \delta A_h^*(u^l)u + sE_hu + B_hp = Res_h(u^l), \\ B_h^T u = 0. \end{cases} \tag{4}$$

- Linearization
  - Fixed point linearization is supplied for $\delta = 0$.
  - Full Newton linearization is supplied for $\delta = 1$.
- Result (see Hansbo et. al)
  - Stable system.
  - Optimal accuracy.
- Problem
  - The solver?!

3 Korn's Inequality

Kernel Function



The kernel function takes -1 or 1 in midpoints

Korn's Inequality

Level	NEL	$\|u_h\|_A$	$\|u_h\|_T$	$\|u_h\|_{ST}$
2	4	0.91	$0.63E-15$	0.30
3	16	1.1	$0.20E-14$	0.60
4	64	1.3	$0.41E-14$	0.76
5	256	1.3	$0.69E-14$	0.85
6	1024	1.4	$0.14E-14$	0.90
7	4096	1.4	$0.30E-13$	0.92
8	16384	1.4	$0.61E-13$	0.93

where

$$u_h = \sum_i X_i \phi_i$$

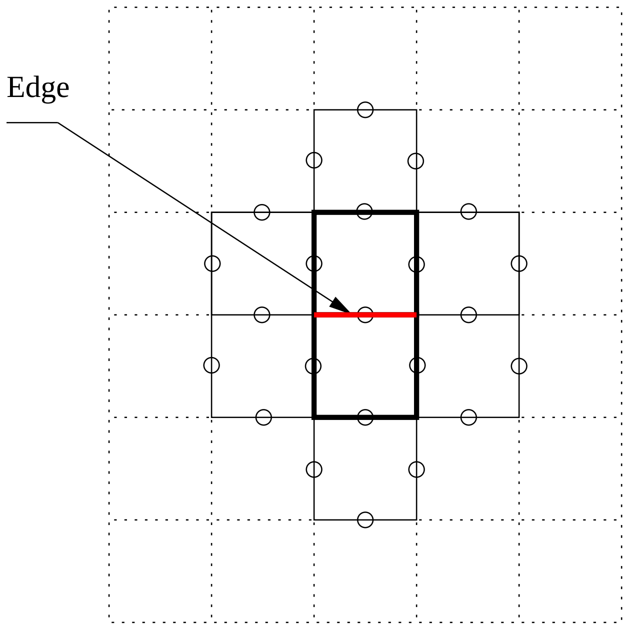
$$\|u_h\|_M = \left(\sum_{i,j} M_{ij} X_i X_j\right)^{\frac{1}{2}}$$

A, T and ST be the resulting matrices from gradient, tensor and stabilized tensor discretization respectively .

4 Linear multigrid solver

4.1 Sparsity of the matrix

Involved Elements for each Edge



- Seems to contradict the FEM storage philosophy!

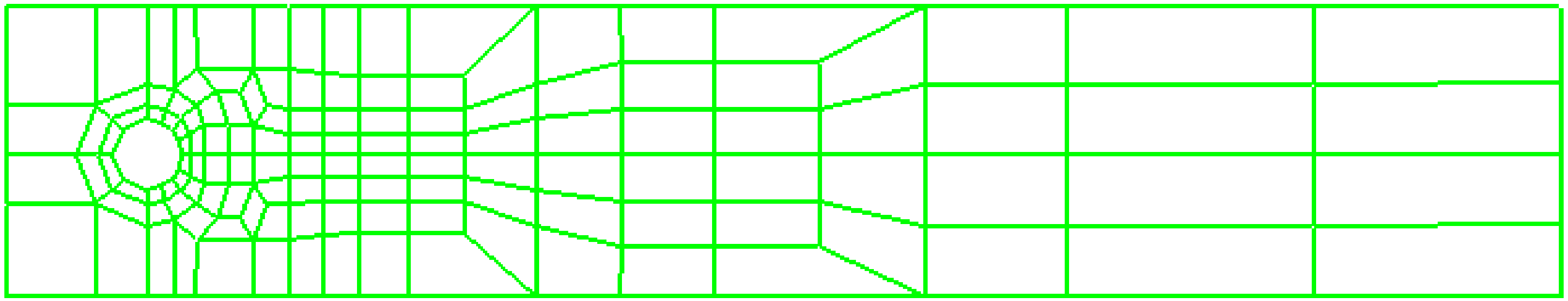
4.2 Vanka smoother as defect correction

$$\begin{bmatrix} u^n \\ p^n \end{bmatrix} = \begin{bmatrix} u^{n-1} \\ p^{n-1} \end{bmatrix} + \boxed{\sum_i \begin{pmatrix} N^*_{|\Omega_i} & B_{|\Omega_i} \\ B^T_{|\Omega_i} & 0 \end{pmatrix}^{-1}} \begin{bmatrix} Res_1(u^{n-1}, p^{n-1}) \\ Res_2(u^{n-1}, p^{n-1}) \end{bmatrix}$$

- Ω_i is a "patch" (which is only one element in our recent tests)
- $N^* = A + \delta A^*$
- Res is the full defect $(N^* + sE_h)$
- "$\sum_i$" means Jacobi, or Gauss-Seidel, or some kind of average.

5 Effect of convective term

The Coarse Mesh for Flow around cylinder Benchmark



Lift and Drag Forces

Level 5				
$1/\nu$		grad	Tensor	S. Tensor
1	lift	$0.31252D+04$	$0.31221D+04$	$0.31231D+04$
	drag	$0.30898D+02$	$0.30924D+02$	$0.30936D+02$
10	lift	$0.31258D+03$	$0.31227D+03$	$0.31237D+03$
	drag	$0.36832D+01$	$0.36851D+01$	$0.36864D+01$
100	lift	$0.32016D+02$	$0.31981D+02$	$0.31990D+02$
	drag	$0.79786D+00$	$0.79758D+00$	$0.79770D+00$
1000	lift	$0.55657D+01$	$0.55531D+01$	$0.55535D+01$
	drag	$0.10180D-01$	$0.10259D-01$	$0.10277D-01$

No differences !

Effect of Convective Term

	$1/\nu$	1	10	100	$1000(Re = 20)$
Level	Formulation	NL/MG	NL/MG	NL/MG	NL/MG
4	grad	3/3	4/3	6/3	11/4
	tensor	3/15	5/17	6/13	11/4
	stabilized tensor	3/3	5/3	6/3	11/4
5	grad	3/3	4/3	6/3	11/3
	tensor	4/140	5/35	6/15	11/10
	stabilized tensor	4/3	5/3	7/3	11/3
6	grad	3/3	4/3	6/3	11/3
	tensor	4/513	4/161	7/17	11/12
	stabilized tensor	3/3	4/3	7/3	11/3

Summary

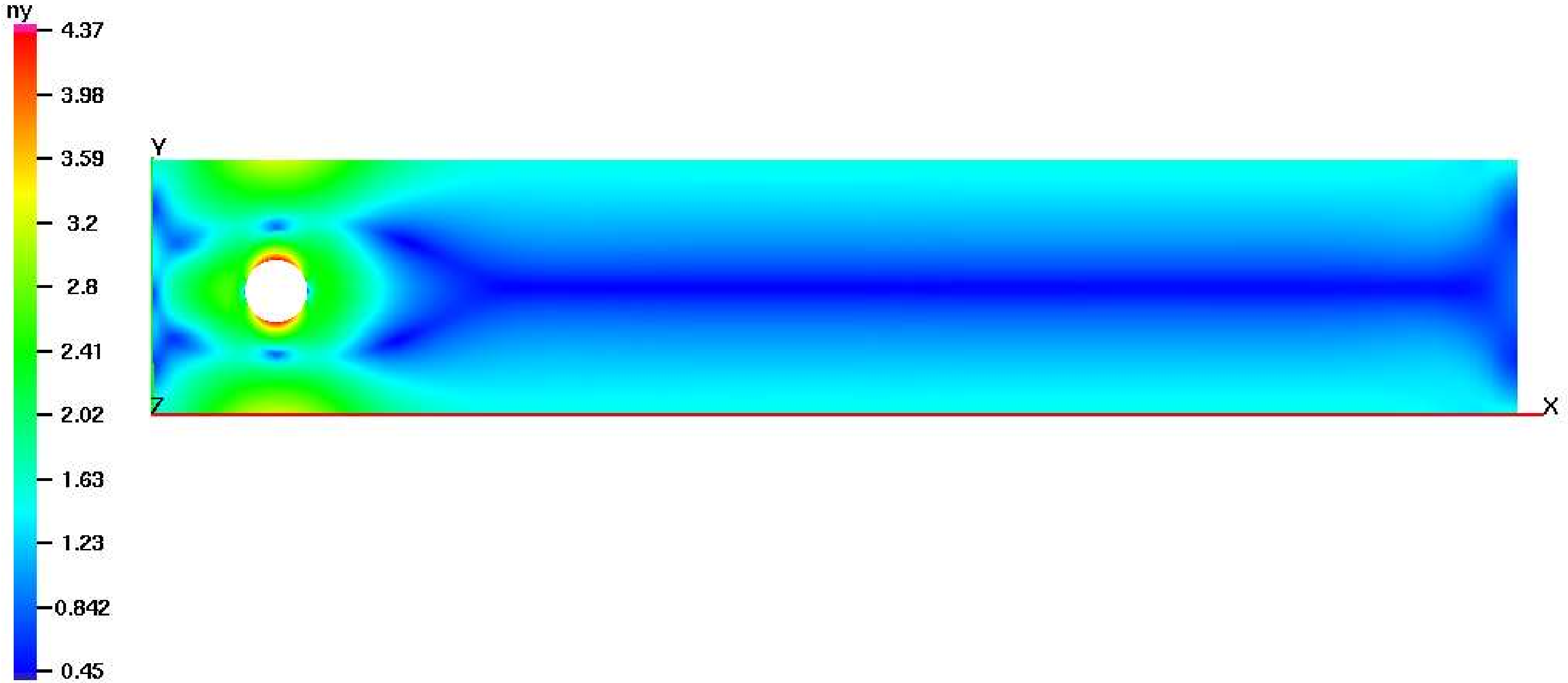
- Tensor
  - Problem **only** with multigrid for **small** *Re* number !
- Stablized tensor
  - Stable.
  - Optimal accuracy.

6 Stabilization parameters

NNL/NMG for Nonlinear Viscosity

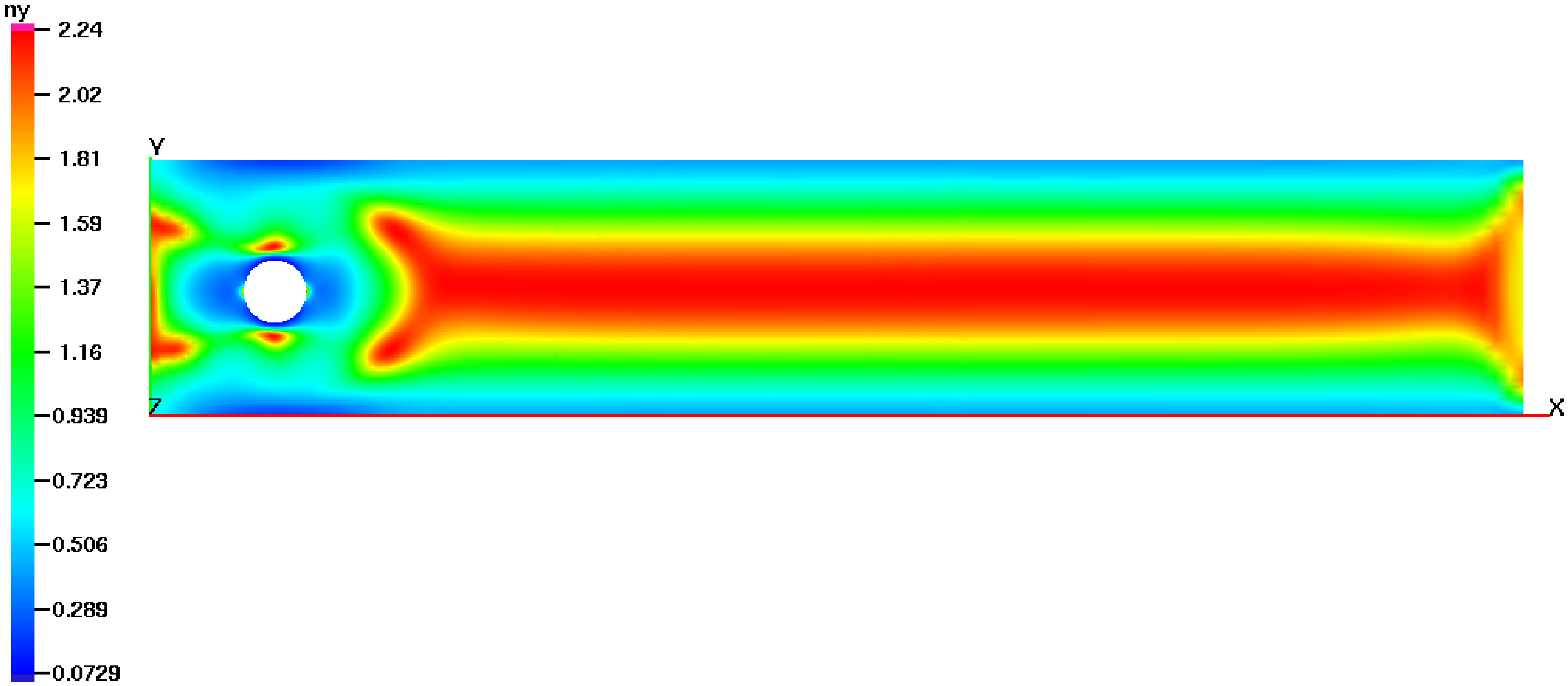
$\nu(D) = \nu_0(0.1 + \|D\|)^\alpha.$						
α	-0.7			$+0.7$		
Level	0	s_{min}	s_{local}	0	s_{min}	s_{local}
3	26/48	48/2	35/2	17/7	17/2	17/2
4	26/140	26/2	36/ 2	18/15	17/2	17/2
5	27/414	24/3	32/2	50/212	17/3	17/3
6	–	27/3	25/3	–	17/3	17/3

The viscosity for Shear Thickening



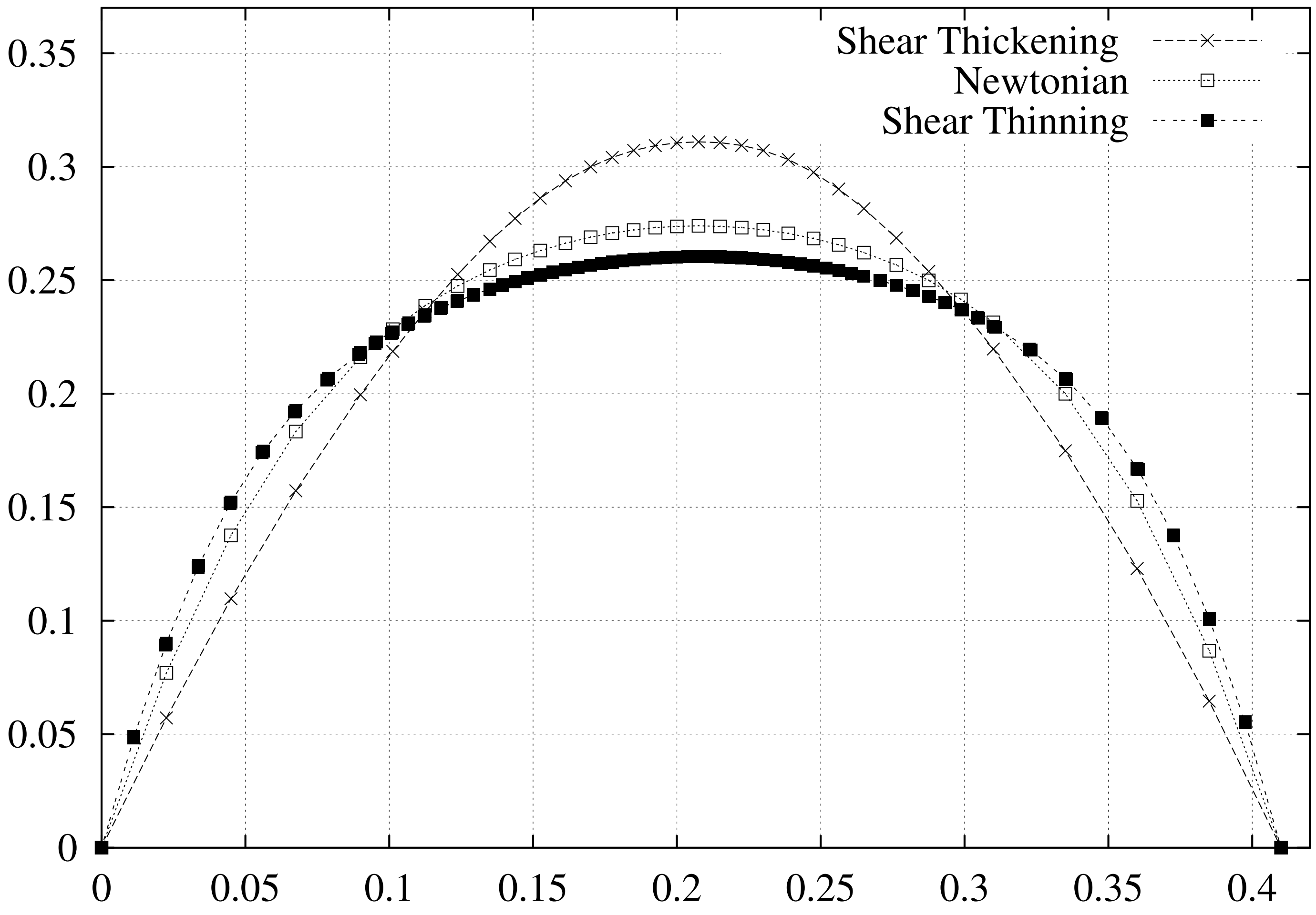
Factor of 10!

The viscosity for Shear Thinning



Factor of 30!

The velocity Profiles



7 Current work

Many challenges still remain to be overcome:

- Develop best strategy to apply the stabilization for the quasi-Newtonian models or plasticity model

$$\nu = \nu(D(u)) \longrightarrow s(.)?$$

- Develop best strategy to apply the stabilization in conjunction with the Newton linearization technique.
- Improve multigrid solver (adaptive patching).