Efficient Numerical Methods and Simulation techniques for Granular flow

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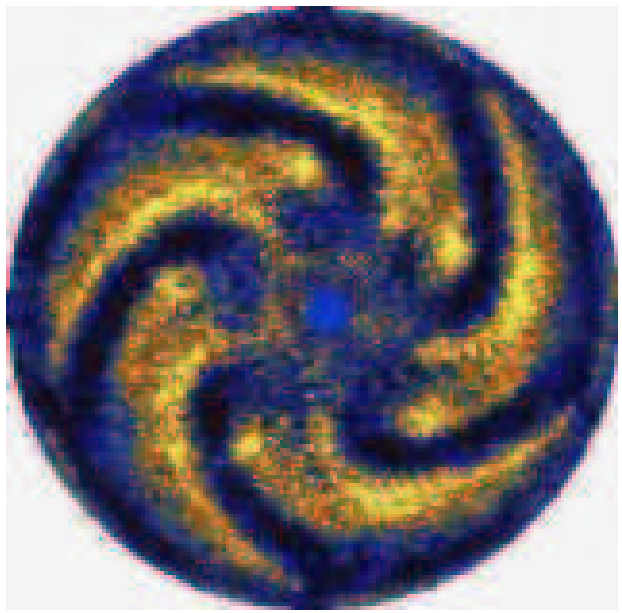
Examples of Applications

- **Paddles in a mixer or granulator**



Examples of Applications

- **Flow around inserts during emptying of bins and hoppers**

Schaeffer Law

- Incompressible Stokes problem

$$-\nabla\cdot[2\nu(D_{\mathbf{I}}(u),p)\mathbf{D}(\boldsymbol{u})]+\nabla p=f,\qquad \nabla\cdot\boldsymbol{u}=0 \tag{1}$$

the nonlinear viscosity $\nu(\cdot,\cdot)$ is a function of $D_{\mathbf{I}}(u)$ and p,

$$D_{\mathbf{I}}(u)=\frac{1}{2}D(u):D(u)=\frac{1}{2}\sum_{i,j}D_{i,j}(u)D_{i,j}(u).$$

- Power law defined for

$$\nu(z,p)=\nu_0 z^{\frac{r}{2}-1}$$

- Bingham law defined for

$$\nu(z,p)=\nu_0 z^{\frac{-1}{2}}$$

- Schaeffer law (including the pressure) defined for

$$\nu(z,p)=pz^{\frac{-1}{2}}$$

Nonlinear Solver

Let $\boldsymbol{u}^l$ being the initial state, the (continuous) Newton method consists of finding $\boldsymbol{u}$ such that

$$\begin{aligned}
&\int_\Omega 2\nu(D_{\|}(\boldsymbol{u}^l), p^l)\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v})dx \\
&\qquad + \int_\Omega 2\partial_1\nu(D_{\|}(\boldsymbol{u}^l), p^l)[\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{u})][\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})]dx \\
&\qquad\qquad + \int_\Omega 2\partial_2\nu(D_{\|}(\boldsymbol{u}^l), p^l)[\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})]p dx \\
&\qquad\qquad = \int_\Omega \boldsymbol{f}\boldsymbol{v} - \int_\Omega 2\nu(D_{\|}(\boldsymbol{u}^l), p^l)\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})dx, \quad \forall \boldsymbol{v} \in V\,, \qquad (2)
\end{aligned}$$

where $\partial_i\nu(\cdot,\cdot); i = 1, 2$ is the partial derivative of ν related to the first and second variable respectively.

New Linear Algebraic Problem

The algorithm consists of finding $(\boldsymbol{u}, p)$ as solution of the linear system

$$\begin{cases} A(\boldsymbol{u}^l, p^l)\boldsymbol{u} + \delta_d A^*(\boldsymbol{u}^l, p^l)\boldsymbol{u} + Bp + \delta_p B^*(\boldsymbol{u}^l, p^l)p & = R_u(\boldsymbol{u}^l, p^l), \\ B^T\boldsymbol{u} & = R_p(\boldsymbol{u}^l, p^l), \end{cases} \qquad (3)$$

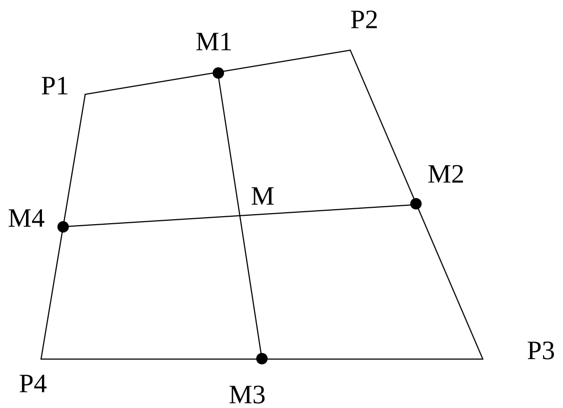
where $R_u(\cdot,\cdot)$ and $R_p(\cdot,\cdot)$ denote the corresponding nonlinear residual terms for the momentum and continuity equations, and the matrix $A^*(\boldsymbol{u}^l, p^l)$ and $B^*(\boldsymbol{u}^l, p^l)$ are defined as follows respectively

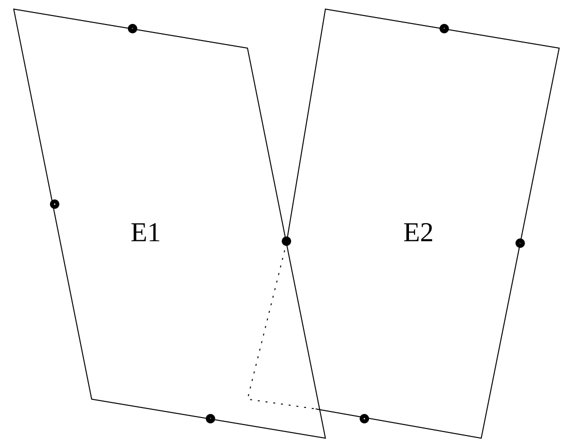
$$\langle A^*(\boldsymbol{u}^l, p^l)\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega 2\partial_1 \nu(D_{\mathbf{II}}(\boldsymbol{u}^l), p^l)[D(\boldsymbol{u}^l) : D(\boldsymbol{u})][D(\boldsymbol{u}^l) : D(\boldsymbol{v})]dx. \qquad (4)$$

$$\langle B^*(\boldsymbol{u}^l, p^l)p, \boldsymbol{v}\rangle = \int_\Omega 2\partial_2 \nu(D_{\mathbf{I}}(\boldsymbol{u}^l), p^l)[D(\boldsymbol{u}^l) : D(\boldsymbol{v})]p\,dx. \qquad (5)$$

Spatial discretization

- **Quadrilateral Rannacher-Turek Stokes Element**





- **Advantage:**
  - Stable and efficient for incompressible flow.
  - Compact data structures.
- **Disadvantage:**"Not" satisfying discrete Korn's inequality

$$\sum_{\tau \in \tau_h} \|v\|_{H^1(\tau)} \leq c(\|v\|_{0,\tau}^2 + \|D(v)\|_{0,\tau}^2)^{\frac{1}{2}} \tag{6}$$

Stabilized Rannacher-Turek Stokes Element

- **Remedy: Stabilized R-T FEM**

The stabilization consists of adding the following bilinear form

$$\sum_{E \in E_I \cup E_D} \frac{1}{|E|} \int_E [\phi_i][\phi_j] ds \tag{7}$$

for all basis function ϕ_i and ϕ_j with a weighted parameter $s = s(\nu)$ which will act as ‘free’ stabilization parameter. Then the corresponding matrix S is defined as:

$$< Su, v >= \sum_{E \in E_I \cup E_D} \frac{1}{|E|} \int_E [u][v] ds \tag{8}$$

Linear Multigrid solver

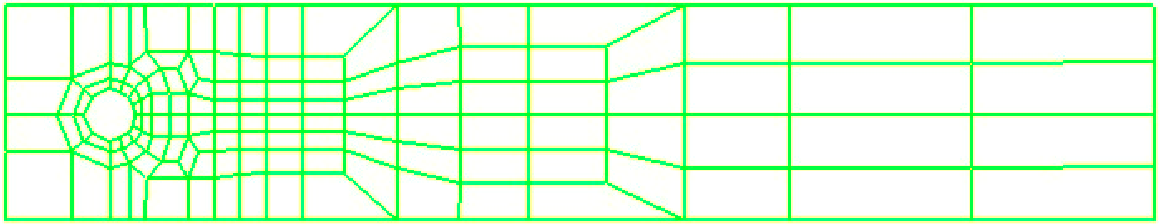
- Vanka smoother as defect correction

$$\begin{bmatrix} \boldsymbol{u}^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{l} \\ p^{l} \end{bmatrix} + \omega^l \sum_i \begin{pmatrix} F + S^*_{|\Omega_i} & \tilde{B} + \delta_p \tilde{B}^*_{|\Omega_i} \\ \tilde{B}^T_{|\Omega_i} & 0 \end{pmatrix}^{-1} \begin{bmatrix} \tilde{R}_u(\boldsymbol{u}^l, p^l) \\ \tilde{R}_p(\boldsymbol{u}^l, p^l) \end{bmatrix} \tag{9}$$

with matrix $F = \tilde{A} + \delta_d \tilde{A}^*$. For the preconditioning step only a part of the matrix, i.e. $F + S^*$, is taken

Comparison with high order scheme

- **Coarse mesh and geometrical details for ‘Flow around a cylinder’**



Mesh information			$\tilde{Q}_1/Q_0$	Q_2/P_1
Level	Elements	Vertices	Total unknowns	Total unknowns
1	156	130	702	1533
2	572	520	2686	5927
3	2184	2080	10608	23295
4	8528	8320	42016	92351
5	33696	33280	167232	367743
6	133952	133120	667264	–

Comparison with high order scheme

$$\nu(D_{\mathbf{I}}, p) = (\epsilon + D_{\|}(u))^{-\alpha/2}$$

Level	Elements	Drag	Lift	$\triangle\ p$	NNL/AVL
$\alpha = 0.9$					
2	$\tilde{Q}_1/Q_0$	819.49	2.5201	14.22	$13/2$
	Q_2/P_1	920.59	2.1805	16.76	$14/27$
3	$\tilde{Q}_1/Q_0$	917.89	3.1958	15.54	$14/2$
	Q_2/P_1	941.51	3.3310	16.02	$15/70$
4	$\tilde{Q}_1/Q_0$	916.02	3.7381	15.74	$12/2$
	Q_2/P_1	953.94	3.9217	15.82	$16/208$
5	$\tilde{Q}_1/Q_0$	935.13	3.9954	15.82	$15/3$
	Q_2/P_1	957.64	4.0587	15.87	$33/368$
6	$\tilde{Q}_1/Q_0$	946.22	4.0592	15.85	$13/5$

Efficiency of the nonlinear solver

- Shear dependent viscosity flow

$\nu(D_{II}, p) = (\epsilon + D_{II}(u))^{-\alpha/2}$				
$\tilde{Q}_1/Q_0$				
α	0.9			
$\epsilon = 10^{-2}$	Newton		Fixpoint	
Level	NNL/AVL	CPU	NNL/AVL	CPU
2	$13/2$	60	$74/2$	228
3	$14/2$	215	$114/2$	1359
4	$12/2$	740	$125/2$	5871
5	$15/3$	3957	$119/2$	21735
6	$13/5$	25926	$109/2$	80438

Efficiency of the nonlinear solver

- Shear dependent viscosity flow

$\nu(D_{II}, p) = (\epsilon + D_{II}(u))^{-\alpha/2}$				
$\epsilon = 10^{-2}$	Newton		Fixpoint	
α	NNL/AVL	CPU	NNL/AVL	CPU
0.1	$7/2$	7	$10/2$	9
0.5	$7/2$	8	$27/2$	32
0.9	$9/3$	16	$113/3$	190
1.0	$12/4$	52	$394/2$	889

Efficiency of the nonlinear solver

- Pressure dependent viscosity flow

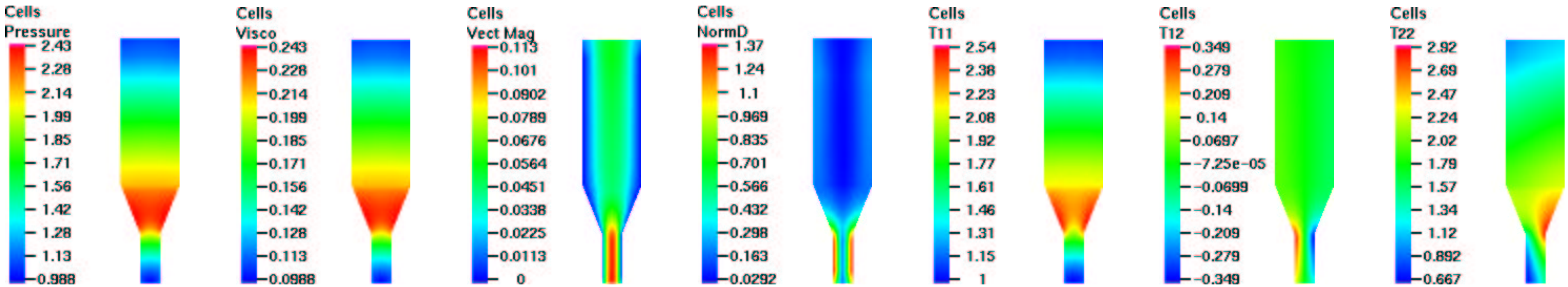
$\nu(D_{II}, p) = exp(\beta p)$	Newton		Fixpoint	
β	NNL/AVL	CPU	NNL/AVL	CPU
5×10^{-2}	$5/2$	9	$10/2$	21
1×10^{-1}	$6/2$	13	$15/2$	33
2×10^{-1}	$7/4$	33	$30/3$	86

$\nu(D_{II}, p) = \beta p$	Newton		Fixpoint	
β	NNL/AVL	CPU	NNL/AVL	CPU
5×10^{-2}	$9/6$	63	$94/4$	296
1×10^{-1}	$8/7$	65	$103/4$	426
2×10^{-1}	$7/16$	120	$124/8$	1042

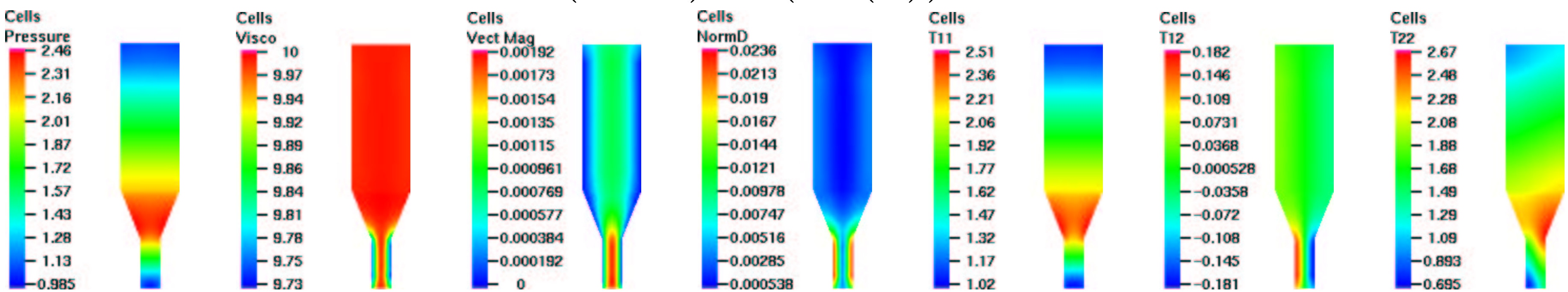
Steady state flow in Silos

- Schaeffer flow under gravity force

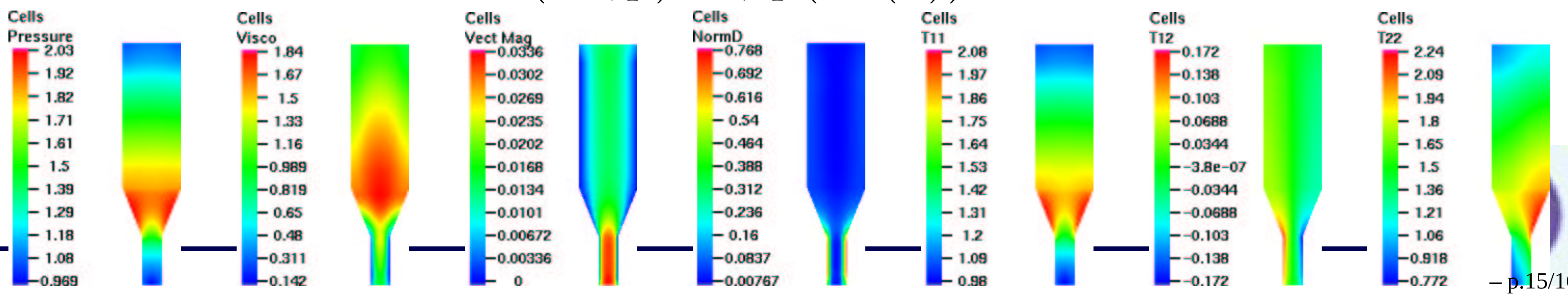
$$\nu(D_{II}, p) = \beta p$$



$$\nu(D_{II}, p) = (D_{II}(u))^{-1/2}$$

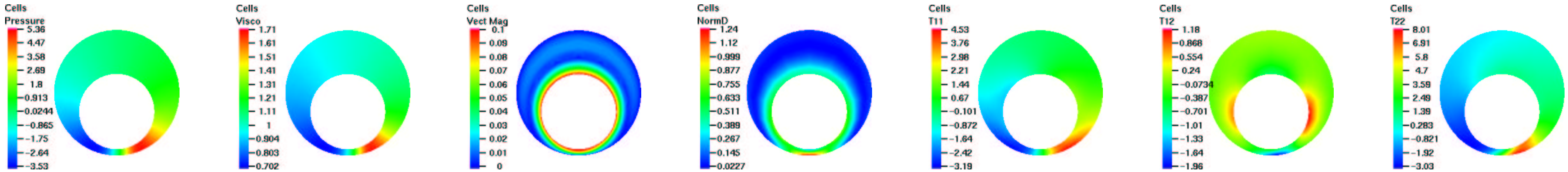


$$\nu(D_{II}, p) = \beta p (D_{II}(u))^{-1/2}$$

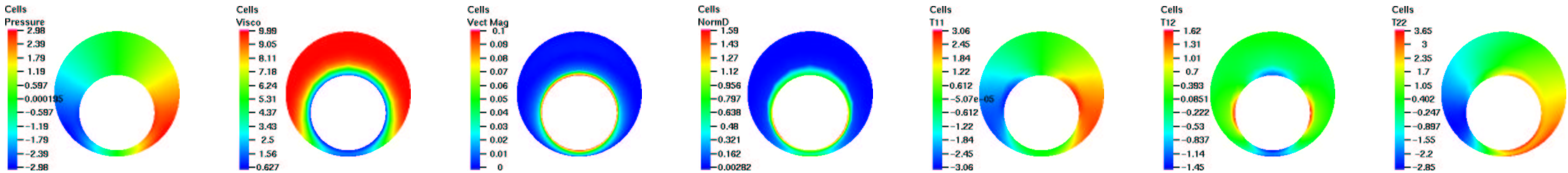


Steady state flow in Couette device

$$\nu(D_{\mathbf{I}}, p) = exp(\beta p)$$



$$\nu(D_{\mathbf{I}}, p) = (D_{\mathbf{I}}(u))^{-1/2}$$



$$\nu(D_{\mathbf{I}}, p) = exp(\beta p)(D_{\mathbf{I}}(u))^{-1/2}$$