Numerical Methods and Simulation Techniques for flow with pressure dependent viscosity

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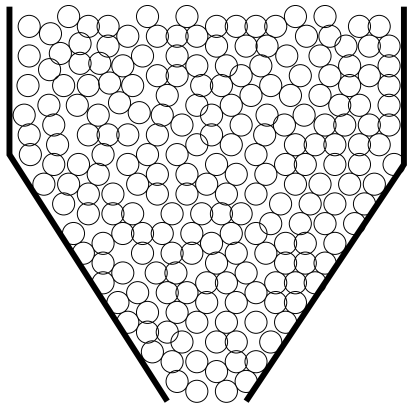
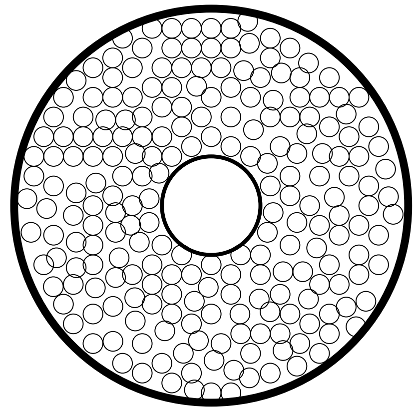
Examples of Applications

Granular Material

Pharmaceutical Industry, Food Processing, Soil mechanics ...



Granular material storage



Couette flow

Mathematical Model: Schaeffer Law

- **Flow rule of von Mises**

$$\sum_{i=1}^{3}(\sigma_i - p) \leq k^2 p^2 \tag{1}$$

where $p = \frac{1}{3}\operatorname{tr}\tau$, $k = \sqrt{2}\sin\phi$ is a constant characteristic of the materials, and σ_i are the eigenvectors of τ_{ij}.

- **Plastic Deformation**

For the material to deform plastically, equality must hold in (Eq.1) i.e.

$$\sum_{i=1}^{3}(\sigma_i - p) = k^2 p^2 \tag{2}$$

Mathematical Model: Schaeffer Law

- **Co-axiality flow rule**

$$\mathbf{D} = q(\tau - pI) \tag{3}$$

This flow rule contains also the assumption of incompressibility

$$\operatorname{div} \boldsymbol{u} = -\operatorname{tr} \mathbf{D} = q \operatorname{tr}(\tau - pI) = 0 \tag{4}$$

- **Constitutive equation:** David Schaeffer 1987

$$\tau = p(I + \sqrt{2} \sin \phi \frac{\mathbf{D}}{\|\mathbf{D}\|}) \tag{5}$$

where

$$\mathbf{D}_{ij} = -\frac{1}{2}\left(\frac{\partial \boldsymbol{u}_i}{\partial x_j} + \frac{\partial \boldsymbol{u}_j}{\partial x_i}\right)$$

Equations of motion

- Incompressible General Stokes problem

$$-\nabla\cdot[2\nu(D_{\mathrm{II}}(u),p)\mathbf{D}(\boldsymbol{u})]+\nabla p=f,\qquad \nabla\cdot\boldsymbol{u}=0 \tag{6}$$

the nonlinear viscosity $\nu(\cdot,\cdot)$ is a function of $D_{\mathrm{II}}(u)$ and p,

$$D_{\mathrm{II}}(u)=\frac{1}{2}D(u):D(u)=\frac{1}{2}\sum_{i,j}D_{i,j}(u)D_{i,j}(u).$$

- Power law defined for

$$\nu(z,p)=\nu_0 z^{\frac{r}{2}-1}$$

- Bingham law defined for

$$\nu(z,p)=\nu_0 z^{\frac{-1}{2}}$$

- Schaeffer law (including the pressure) defined for

$$\nu(z,p)=pz^{\frac{-1}{2}}$$

Nonlinear Solver

Let $\boldsymbol{u}^l$ being the initial state, the (continuous) Newton method consists of finding $\boldsymbol{u}$ such that

$$\begin{aligned}
&\int_\Omega 2\nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l)\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v})dx \\
&\qquad + \int_\Omega 2\partial_1\nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l)[\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{u})][\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})]dx \\
&\qquad\qquad + \int_\Omega 2\partial_2\nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l)[\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})]p\,dx \\
&\qquad\qquad\quad = \int_\Omega \boldsymbol{f}\boldsymbol{v} - \int_\Omega 2\nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l)\mathbf{D}(\boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})dx, \quad \forall \boldsymbol{v},
\end{aligned} \tag{7}$$

where $\partial_i\nu(\cdot,\cdot); i = 1,2$ is the partial derivative of ν related to the first and second variable respectively.

New Linear Algebraic Problem

The algorithm consists of finding $(\boldsymbol{u}, p)$ as solution of the linear system

$$\begin{cases} A(\boldsymbol{u}^l, p^l)\boldsymbol{u} + \delta_d A^*(\boldsymbol{u}^l, p^l)\boldsymbol{u} + Bp + \delta_p B^*(\boldsymbol{u}^l, p^l)p & = R_u(\boldsymbol{u}^l, p^l), \\ B^T \boldsymbol{u} & = R_p(\boldsymbol{u}^l, p^l), \end{cases} \tag{8}$$

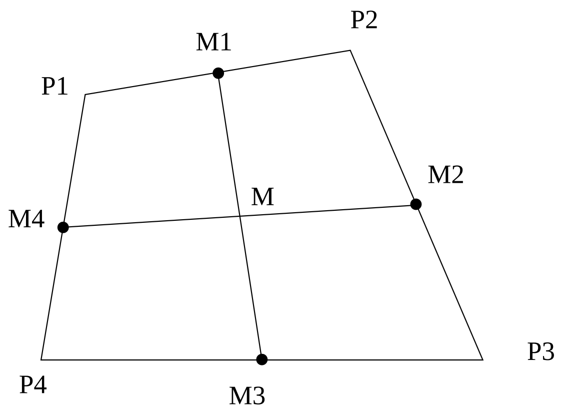
where $R_u(\cdot,\cdot)$ and $R_p(\cdot,\cdot)$ denote the corresponding nonlinear residual terms for the momentum and continuity equations, and the matrix $A^*(\boldsymbol{u}^l, p^l)$ and $B^*(\boldsymbol{u}^l, p^l)$ are defined as follows respectively

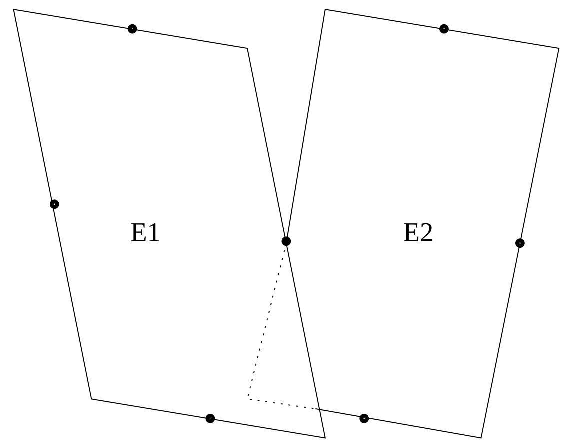
$$\langle A^*(\boldsymbol{u}^l, p^l)\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega 2\partial_1 \nu(D_{\mathrm{II}}(\boldsymbol{u}^l), p^l)[D(\boldsymbol{u}^l) : D(\boldsymbol{u})][D(\boldsymbol{u}^l) : D(\boldsymbol{v})]dx. \tag{9}$$

$$\langle B^*(\boldsymbol{u}^l, p^l)p, \boldsymbol{v}\rangle = \int_\Omega 2\partial_2 \nu(D_{\mathrm{I}}(\boldsymbol{u}^l), p^l)[D(\boldsymbol{u}^l) : D(\boldsymbol{v})]p\,dx. \tag{10}$$

Spatial discretization

- **Quadrilateral Rannacher-Turek Stokes Element**





- **Advantage:**
  - Stable and efficient for incompressible flow.
  - Compact data structures.

- **Disadvantage:** Not satisfying discrete Korn's inequality

$$\sum_{\tau \in \tau_h} \|v\|_{H^1(\tau)} \leq c(\|v\|_{0,\tau}^2 + \|D(v)\|_{0,\tau}^2)^{\frac{1}{2}} \tag{11}$$

Stabilized Rannacher-Turek Stokes Element

- **Remedy**: Stabilized R-T FEM

The stabilization consists of adding the following bilinear form

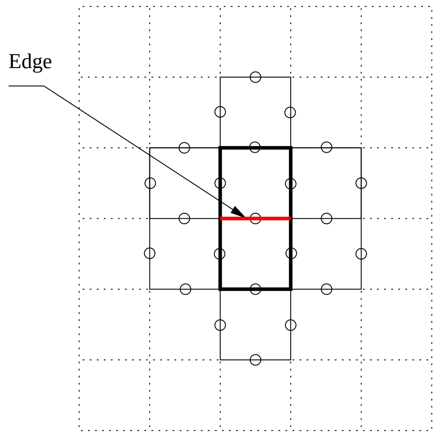
$$\sum_{E \in E_I \cup E_D} \frac{1}{|E|} \int_E [\phi_i][\phi_j] ds \tag{12}$$

for all basis function ϕ_i and ϕ_j with a weighted parameter $s = s(\nu)$ which will act as 'free' stabilization parameter. Then the corresponding matrix S is defined as:

$$< Su, v > = \sum_{E \in E_I \cup E_D} \frac{1}{|E|} \int_E [u][v] ds \tag{13}$$

Linear Multigrid solver

- Different sparsity of the matrix



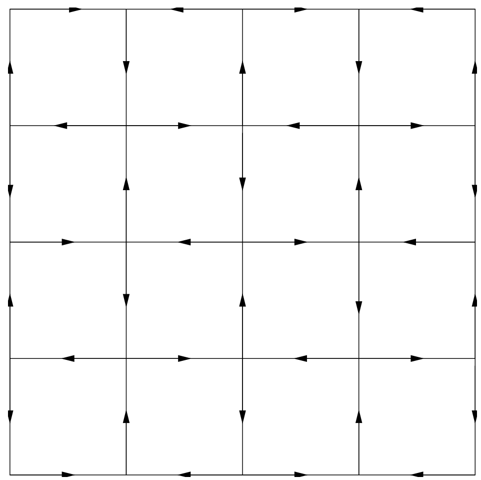
- Vanka smoother as defect correction

$$\begin{bmatrix} \boldsymbol{u}^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{l} \\ p^{l} \end{bmatrix} + \omega^l \sum_i \begin{pmatrix} F + S^*_{|\Omega_i} & \tilde{B} + \delta_p \tilde{B}^*_{|\Omega_i} \\ \tilde{B}^T_{|\Omega_i} & 0 \end{pmatrix}^{-1} \begin{bmatrix} \tilde{R}_u(\boldsymbol{u}^l, p^l) \\ \tilde{R}_p(\boldsymbol{u}^l, p^l) \end{bmatrix} \tag{14}$$

with matrix $F = \tilde{A} + \delta_d \tilde{A}^*$. For the preconditioning step only a part of the matrix, i.e. $F + S^*$, is taken

Kernel function

- The specific kernel function takes -1 or 1 in midpoints

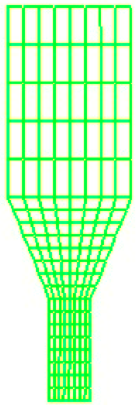


- The constants in Korn's inequality for gradient, tensor and the stabilized tensor

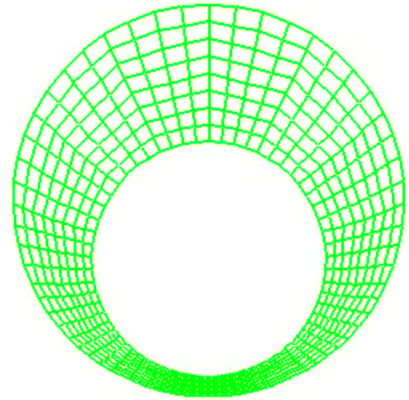
NEL	$\|u_h\|_G$	$\|u_h\|_T$	$\|u_h\|_{ST}$
256	1.3	6.9×10^{-13}	0.85
1024	1.4	1.4×10^{-13}	0.90
4096	1.4	3.0×10^{-12}	0.92
16384	1.4	6.1×10^{-12}	0.93

Domain of computation

- Mesh and geometrical domain of computation



Silo



Nonsymmetric couette

- In the follwing we restrict to the nonsymmetric couette
  - To focus on Dirichlet boundary condition
  - To have nontrivial 2d pressure behavior

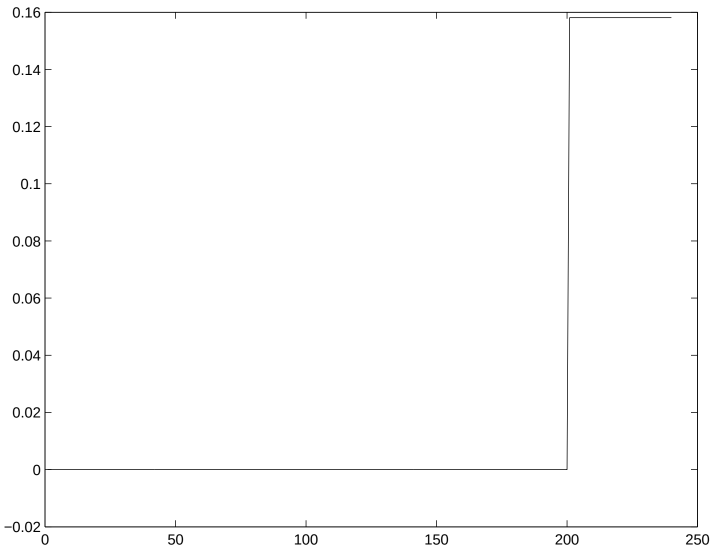
Uniqueness of the linear problem

- The matrix of the linear problem can be writen in the following form

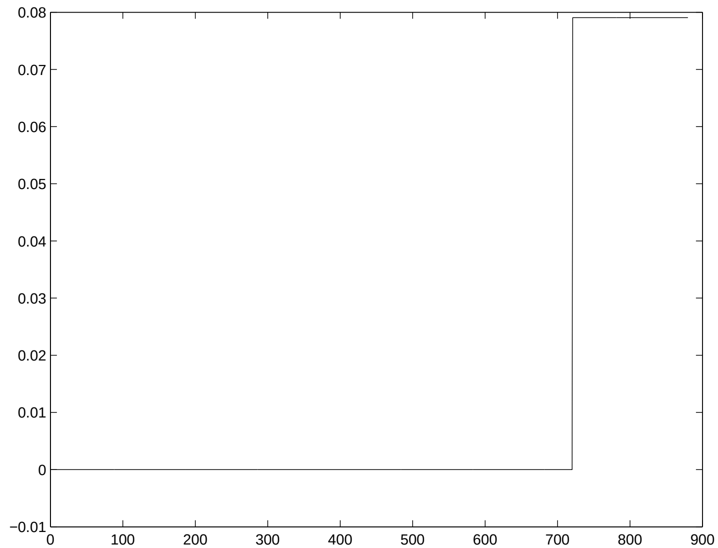
$$M_{\delta_p,\tilde{\delta}_p}(\tilde{\boldsymbol{u}},\tilde{p}) = \begin{pmatrix} A & B+\delta_p B^* \\ B^T+\tilde{\delta}_p B^{*T} & 0 \end{pmatrix} \qquad (15)$$

where $\langle B^*(\tilde{\boldsymbol{u}},\tilde{p})p,\boldsymbol{v}\rangle = \int_\Omega 2\partial_2\nu(D_{\rm II}(\tilde{\boldsymbol{u}}),\tilde{p})[D(\tilde{\boldsymbol{u}}):D(\boldsymbol{v})]p dx.$

- The null space of the matrix $M_{0,0}$; dim(null($M_{0,0}$))=1



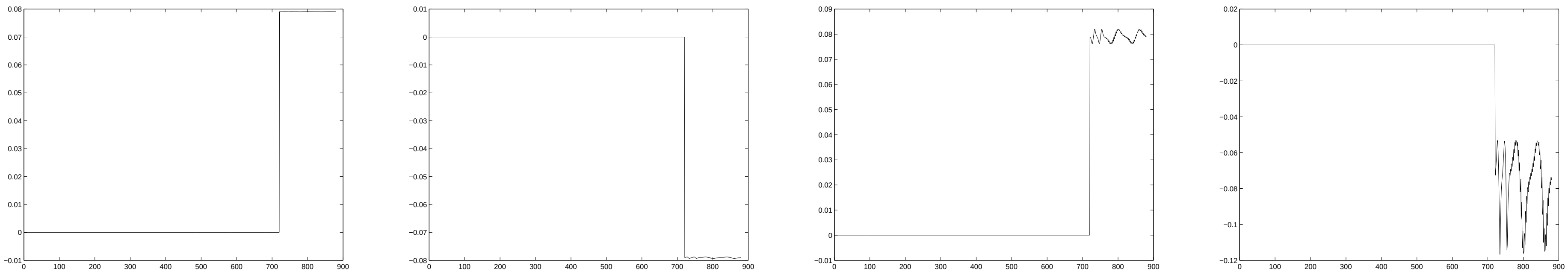
Level 1



Level 2

Uniqueness of the linear problem

- The null space of the matrix $M_{1,0}$ for increasing β with the model $\nu(\boldsymbol{u}, p) = exp(\beta p)$; dim(null($M_{1,0}$))=1



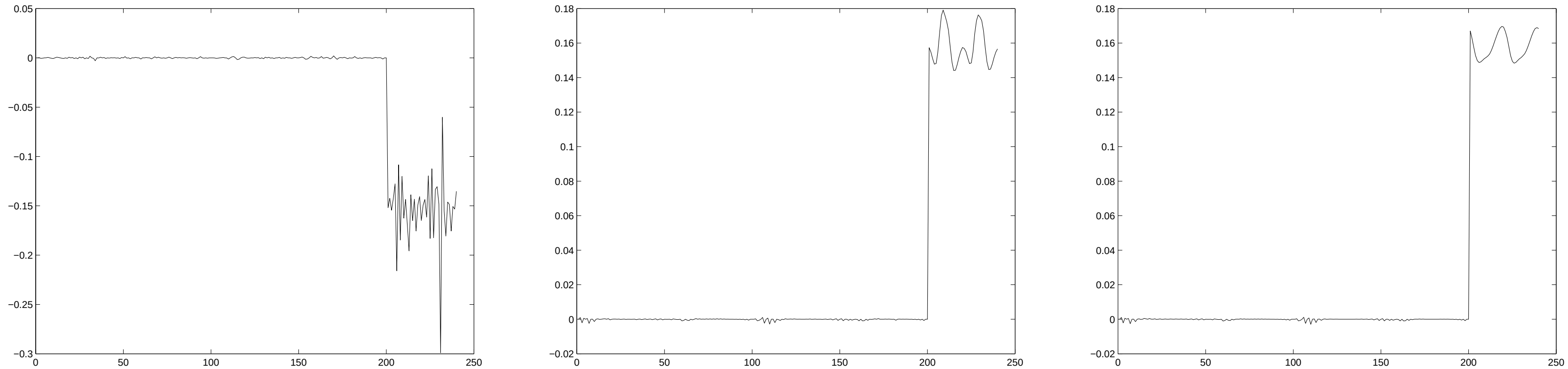
$\beta = 10^{-4}$ $\beta = 10^{-3}$ $\beta = 10^{-2}$ $\beta = 10^{-1}$

- The matrix $M_{1,1}$ has full rank,

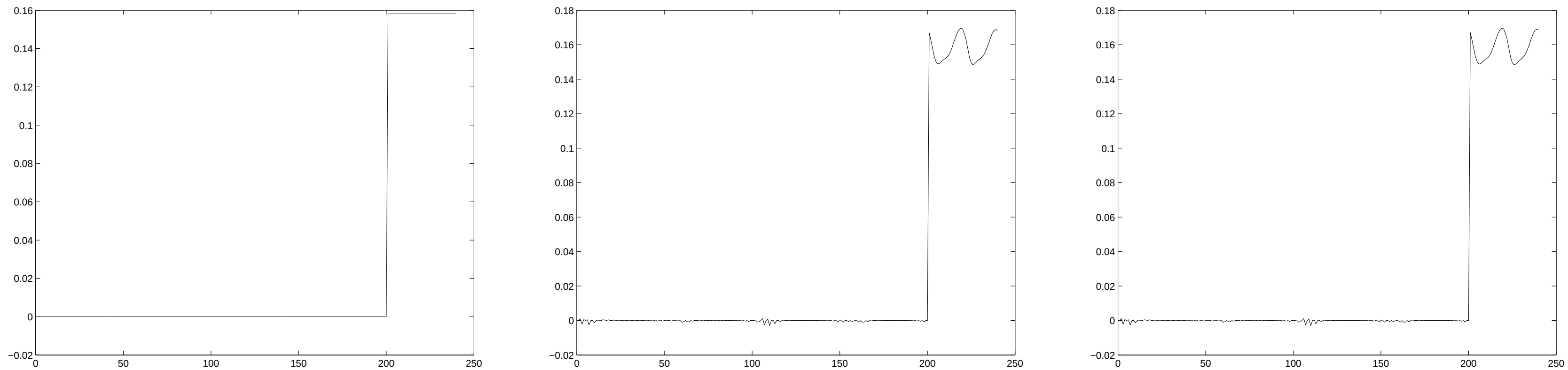
dim(null($M_{1,1}$))=0

Null space of the linear problem

- The null space of the matrix $M_{1,0}(\tilde{\boldsymbol{u}}, \tilde{p})$ for the model $\nu(\boldsymbol{u}, p) = exp(\beta p)$ during the convergence process.
  - starting by a stochastic velocity and pressure $(\tilde{\boldsymbol{u}}, \tilde{p})$



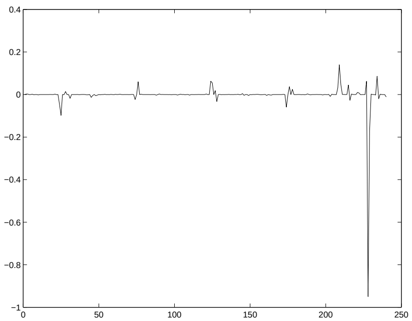
  - starting by zero



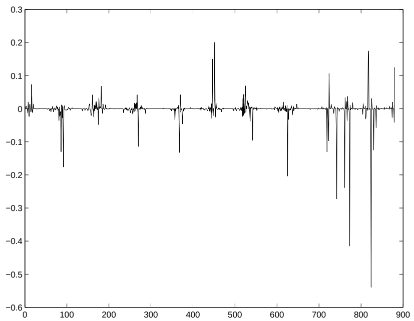
Null space of $M_{1,0}(\tilde{\boldsymbol{u}}, \tilde{p})$

- The null space of the matrix $M_{1,0}(\tilde{\boldsymbol{u}}, \tilde{p})$ for the model $\nu(\boldsymbol{u}, p) = exp(\beta p)$ for different level.
  - For the stochastic $(\tilde{\boldsymbol{u}}, \tilde{p})$

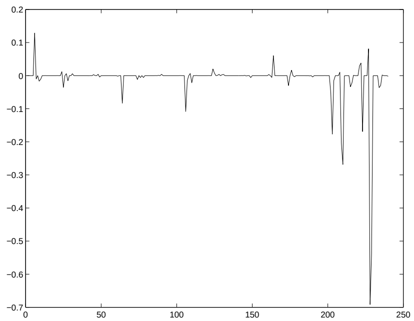
$\beta = 0.5$ $\beta = 1.0$



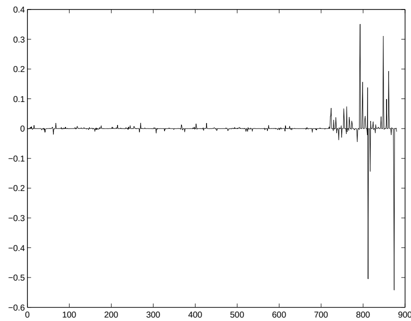
Level 1



Level 2



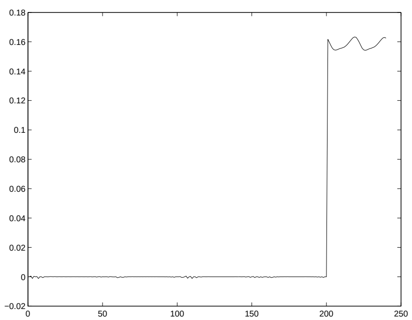
Level 1



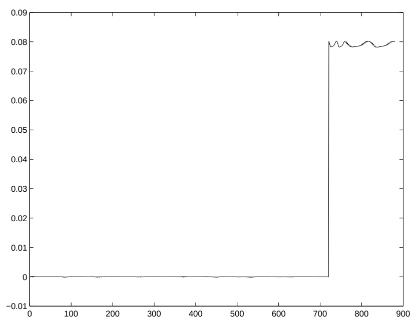
Level 2

  - For the convergence solution $(\tilde{\boldsymbol{u}}, \tilde{p})$

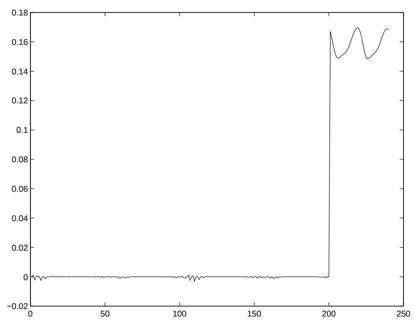
$\beta = 0.5$ $\beta = 1.0$



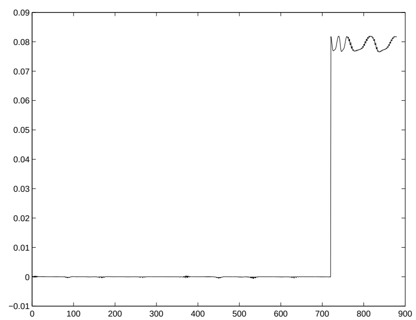
Level 1



Level 2



Level 1



Level 2

Complete closure of equations

- Fix the pressure ?!!

$$\int_\Omega P dx = C \tag{16}$$

  - The effect on the global solution

$\nu(\boldsymbol{u}, p) = exp(\beta p), \beta = 10^{-1}$				
ΔC	0	2	4	6
$\Delta \nu$	0	3.31×10^{-1}	7.31×10^{-1}	13.80×10^{-1}
$\Delta \boldsymbol{u}$	0	2.07×10^{-4}	4.72×10^{-4}	8.15×10^{-4}

  - This leads to think of the appropriate physical choice ?!

- Alternative way
  Think about! the matrix$M_{1,1}$! since it has a full rank. Then, the continuity equation should be modified to get equivalent problem.

$$\langle B^{*T}(\tilde{\boldsymbol{u}}, \tilde{p})u, q\rangle = \int_\Omega 2\partial_2 \nu(D_{\mathbf{I}}(\tilde{\boldsymbol{u}}), \tilde{p})[D(\tilde{\boldsymbol{u}}) : D(\boldsymbol{u})] q dx, \qquad \forall q. \tag{17}$$

Multigrid behavior of the linear problem

- The convergence rate of F-cycle multigrid type

β	10^{-4}	10^{-3}	10^{-2}	10^{-1}
Level	$(\delta_p = 0, \tilde{\delta}_p = 0)^1$			
3	0.22	0.22	0.14	0.26
4	0.23	0.24	0.23	0.36
5	0.26	0.25	0.24	0.37
Level	$(\delta_p = 1, \tilde{\delta}_p = 0)^1$			
3	0.22	0.22	0.15	0.27
4	0.23	0.24	0.23	0.37
5	0.26	0.25	0.24	0.38
Level	$(\delta_p = 1, \tilde{\delta}_p = 1)^2$			
3	0.22	0.22	0.15	0.27
4	0.23	0.24	0.23	0.44
5	0.26	0.25	0.24	0.50

[1] with fixing the pressure [2] without any condition on the pressure

Efficiency of the nonlinear solver

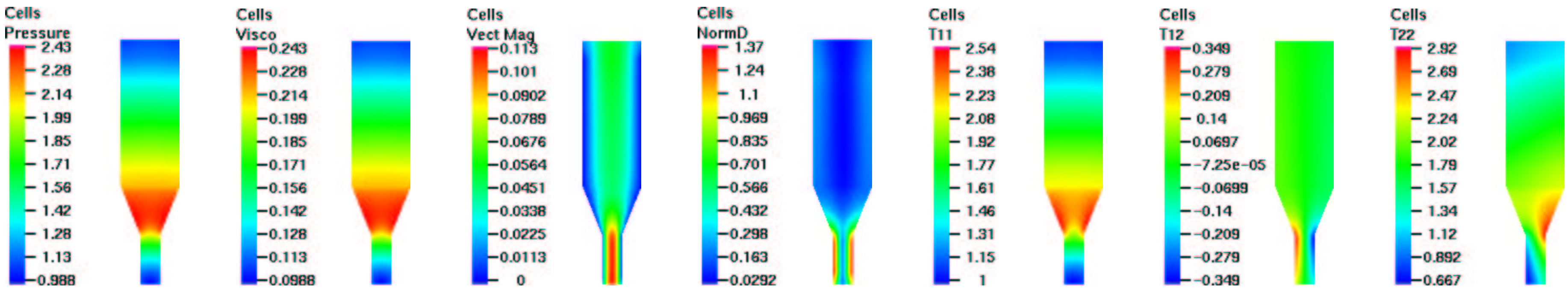
- Pressure dependent viscosity flow with the model $\nu(D_{\mathbf{II}}, p) = \exp(\beta p)$

NNL/AVL						
NSM$=3$	Fixpoint			Newton		
β	0.1	0.3	0.5	0.1	0.3	0.5
Level	Deformation formulation					
4	$6/2$	$10/2$	$19/2$	$3/2$	$3/3$	$4/2$
5	$6/2$	$12/2$	$33/2$	$3/3$	$4/2$	$4/3$
6	$5/3$	$11/3$	$65/2$	$3/3$	$3/3$	$3/3$
Level	Gradient formulation					
4	$5/2$	$8/2$	$21/2$	$3/3$	$4/2$	$4/3$
5	$6/2$	$11/2$	$34/22$	$3/3$	$4/2$	$4/3$
6	$5/3$	$9/3$	$76/2$	$3/3$	$3/3$	$5/3$

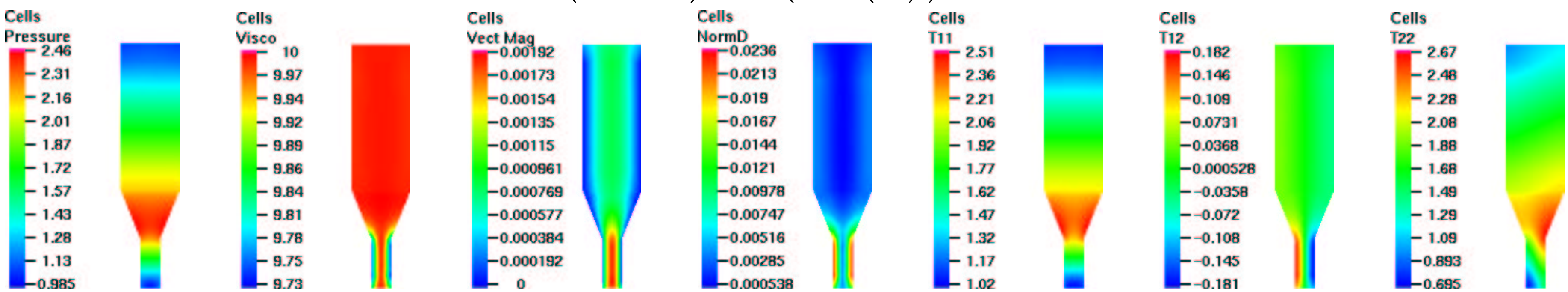
Steady state flow in Silos

- Schaeffer flow under gravity force

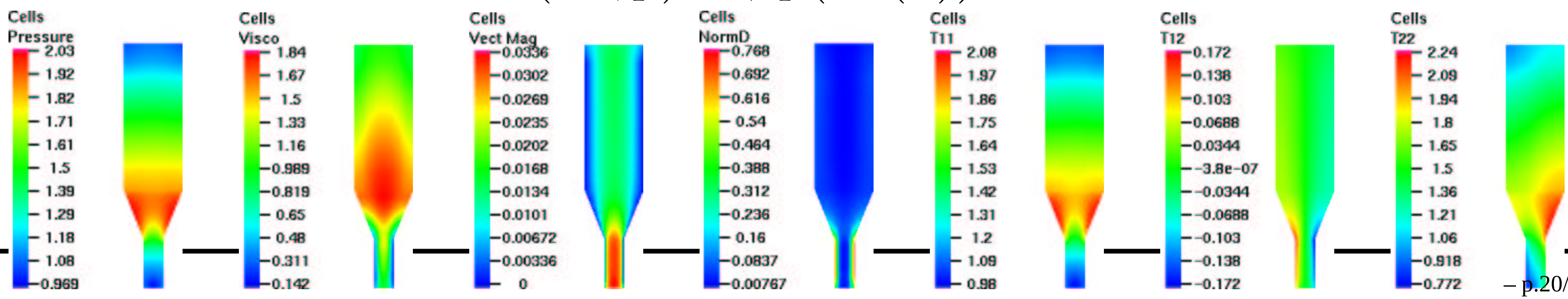
$$\nu(D_{II}, p) = \beta p$$



$$\nu(D_{II}, p) = (D_{II}(u))^{-1/2}$$

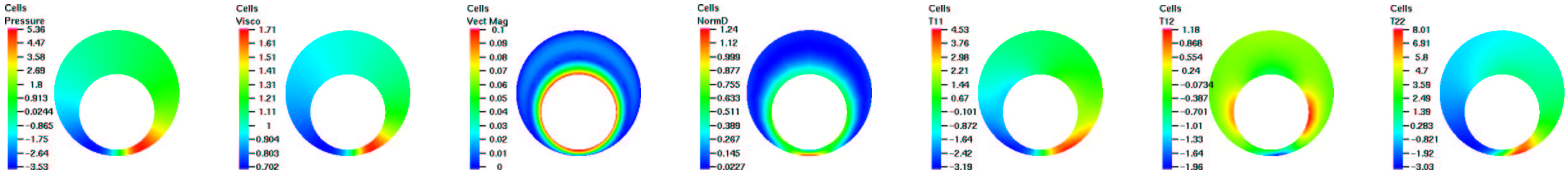


$$\nu(D_{II}, p) = \beta p (D_{II}(u))^{-1/2}$$

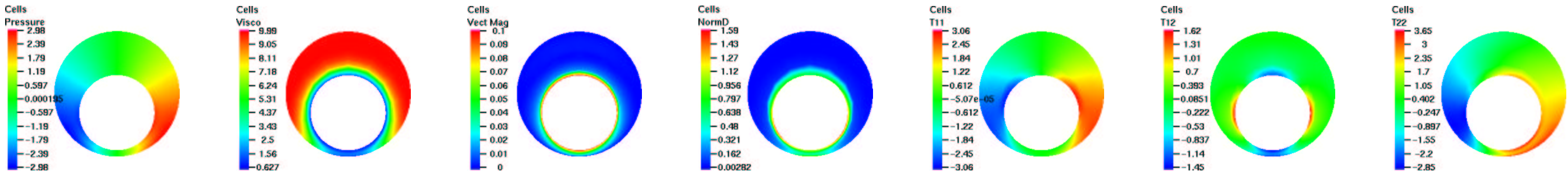


Steady state flow in Couette device

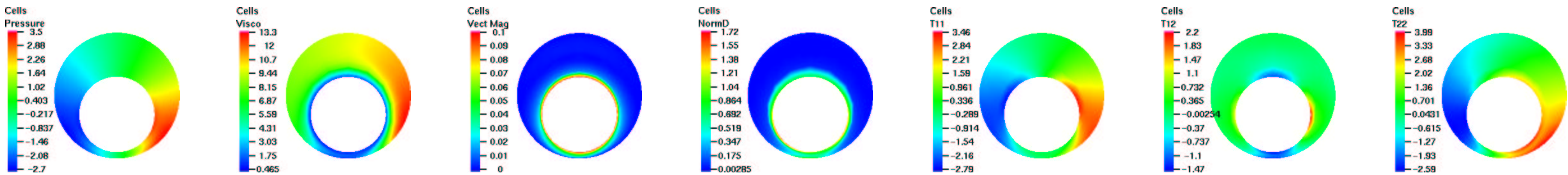
$$\nu(D_{\mathrm{II}}, p) = exp(\beta p)$$



$$\nu(D_{\mathrm{II}}, p) = (D_{\mathrm{II}}(u))^{-1/2}$$

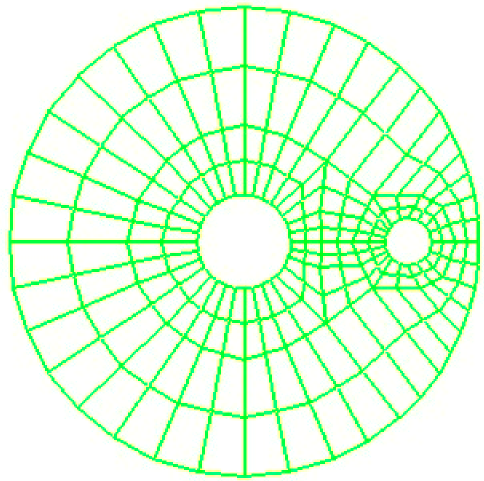


$$\nu(D_{\mathrm{II}}, p) = exp(\beta p)(D_{\mathrm{II}}(u))^{-1/2}$$



Proposed Benchemark Calculation

- Coarse mesh and geometrical details for ‘Couette Flow around a cylinder’



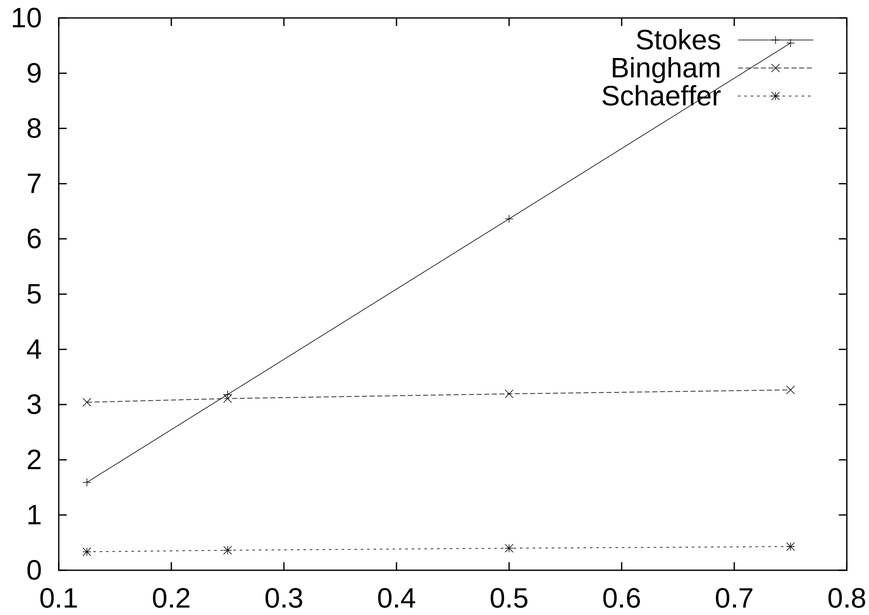
Mesh information			Q_1/Q_0
Level	Element	Vertices	Total unknowns
1	50	72	542
2	200	245	1984
3	800	891	7568
4	3200	3383	29536
5	25968	13167	116672

Couette flow: Slow Drag force

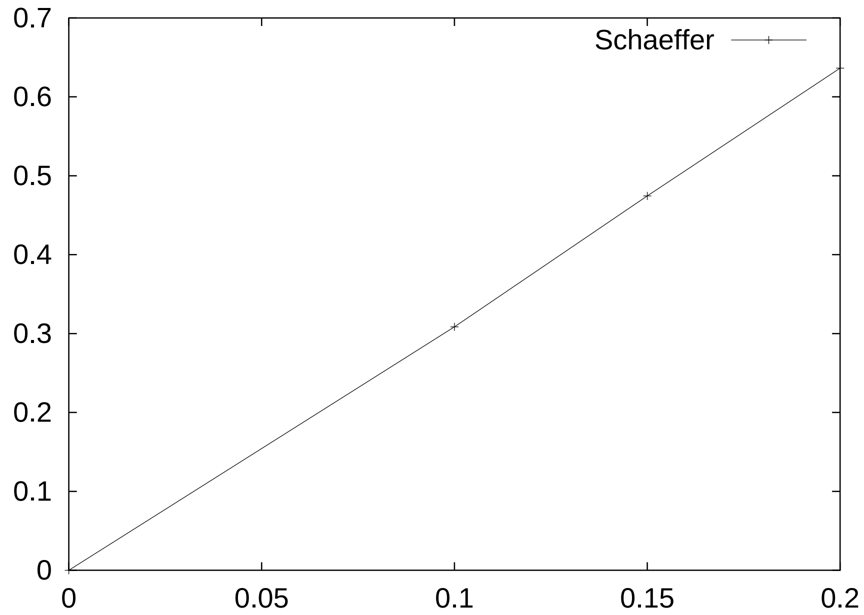
- The **velocity** and **cylinder diameter** dependence of the measured drag force in a couette flow around a cylinder

The simulations with the condition $\int_\Omega P dx = 1$, to avoid negative pressure for the Schaeffer model

With Grain Velocity



With Cylinder Diameter



Summary and outlook

- The Newton method shows its power if not its neceassity for this type of nonlinear prolem
- The linear multigrid convergence rate is
  - independent of both gradient and deformation tensor formulation: ***The stabilization for nonconforming FEM works perfectly***
  - dependent with pressure function for both fixpoint and Newton methods: ***More investigation should focus on the linear algebraic problem, beside to the existance of the nonlinear solution***
  - increasing for Newton, related to the accuracy nedded to reach the nonlinear convergence: ***This can be treated by some kind of paching***
- The physical meaning of any choice for the pressure to get the closure of the equations with Dirichlet boundary condition !?