Efficient Numerical Methods and Simulation Techniques for Granular flow

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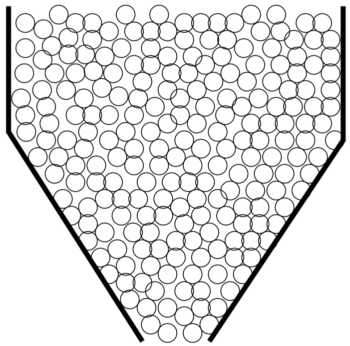
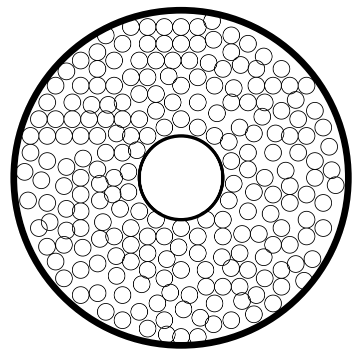
Motivation of this work

- **The flow of granular materials**
  - Example of application
    Pharmaceutical Industry, Food Processing, Soil Mechanics ...



Granular material storage

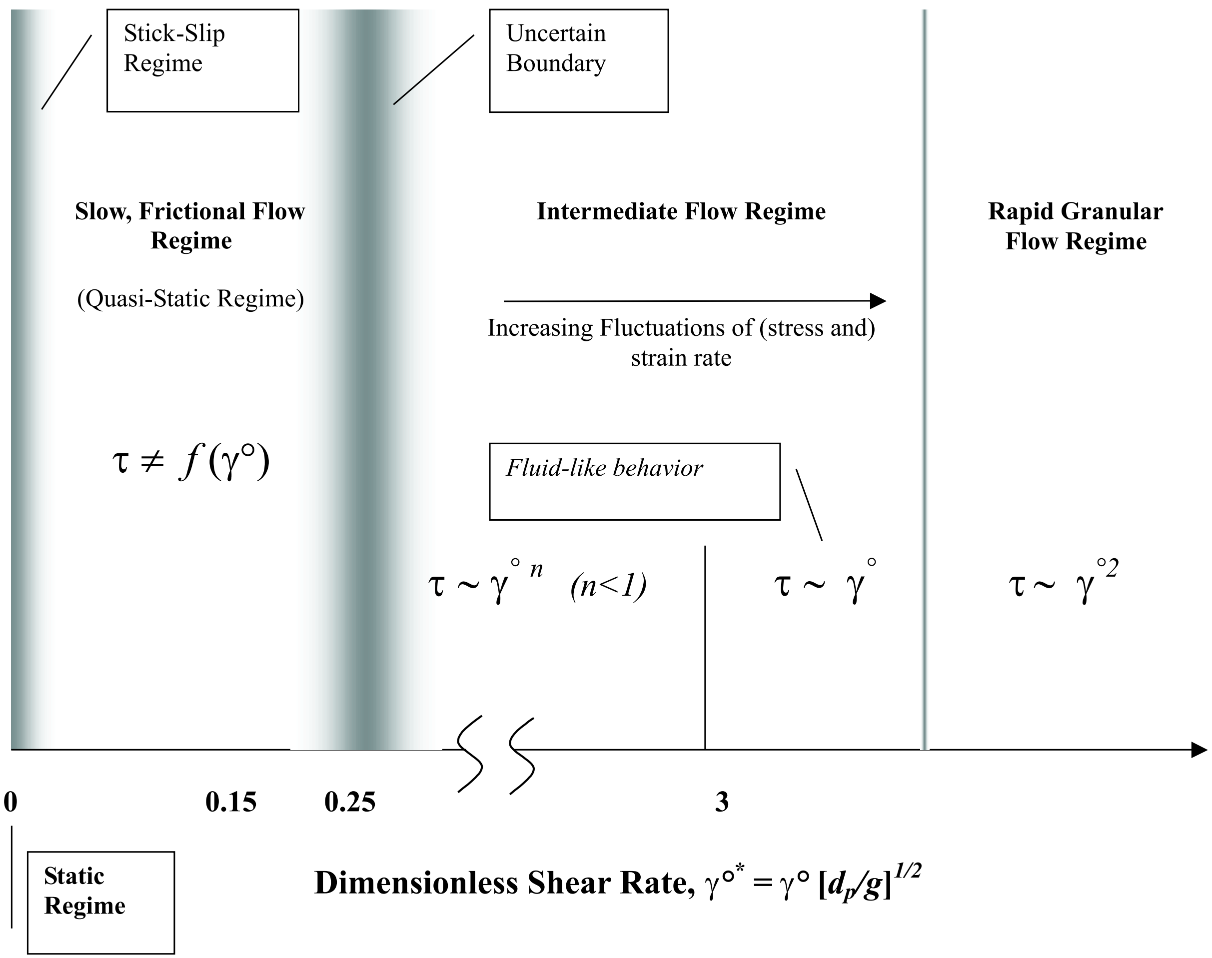


Couette flow

- **What about the viscosity !!?**
  - From engineering point of view this material do not exibit viscosity!!
  - From mathematical and numerical point of view we are able to set this type of problem in the same range of flow with generalized viscosity, since it exibits the same difficulties !?

Regimes of powder flow

Analogue to the fluid flow, the powder regimes could be represented as a function of dimensionless shear rate $\gamma^{o*} = \gamma^{o}[d_p/g]^{1/2}$ which plays the similar role as the Reynolds number Re for fluids (Tardos et al).



Regimes of powder flow

- **Quasi-static regime**
  - Any movement between two static states can be neglected
  - The static equilibrium equation can be applied.
  - No flow field can be predicted!
    This circumscribes the range of applications of this approach
- **Slow and frictional regime** (Schaeffer (1987))
  - The frictional forces between particles are predominant;
  - Inertial effect is added to the static equations
  - Consideration of the continuity, yield condition and flow rule
  - All flow fields can be computed
- **Intermediate and rapid granular regimes**
  - inter-particle friction energy,
  - collisional energy is important, too.

Equations of motion

The general equation of motion for incompressible powders

- **Conservation of mass**

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

$\frac{D*}{Dt}$ is the material derivative and $\boldsymbol{u}$ is the velocity vector

- **For an incompressible material**
the bulk density, ρ, is a constant thus

$$\nabla \cdot \boldsymbol{u} = 0$$

- **The equation of motion**

$$\rho \frac{Du}{Dt} = -\nabla \cdot \mathbf{T} + \rho g$$

with, $\mathbf{T} = \mathbf{S} + p\mathbf{I}$.

Constitutive equation

The constitutive equation is devoted to correlate between the deviatoric tensor, $\mathbf{S}$, and the velocity, throught the rate of deformation $\mathbf{D} = -\frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})$, and assure the closure of equations.

- **Newtonian law**

$$\mathbf{S} = 2\nu_{\mathrm{o}}\mathbf{D}$$

- **Power law**

$$\mathbf{S} = 2\nu(D_{\|})\mathbf{D}, \quad \nu(z) = z^{\frac{r}{2}-1}, \quad r > 1$$

- **Schaeffer's law (1987):** For a powder a constitutive equation first introduced by Schaeffer (1987), which has to obey a
  - yield condition; $\|\mathbf{S}\| = \sqrt{2}p\sin\phi$, and
  - flow rule; $\mathbf{S} = \lambda\mathbf{D}$

  we use this correlation to obtain the constitutive equation

$$\mathbf{S} = \sqrt{2}p\sin\phi\frac{\mathbf{D}}{\|\mathbf{D}\|}$$

Generalized Navier-Stokes Equations

- The generalized incompressible Navier-Stokes problem

$$\rho \frac{Du}{Dt} = -\nabla p + \nabla \cdot (\nu(p, D_{\mathsf{II}})\mathbf{D}) + \rho g, \quad \nabla \cdot \boldsymbol{u} = 0$$

If we define the nonlinear ''pseudo viscosity'' $\nu(\cdot, \cdot)$ as a function of $D_{\mathsf{II}}(u) = \frac{1}{2}\mathbf{D} : \mathbf{D}$ and p, then we can show that different materials could be ranged with different viscosity law including powder;

  - Power law defined for

$$\nu(z, p) = \nu_0 z^{\frac{r}{2}-1}$$

  - Bingham law defined for

$$\nu(z, p) = \nu_0 z^{-\frac{1}{2}}$$

  - Schaeffer's law (including the pressure) defined for

$$\nu(z, p) = \sqrt{2} \sin\phi \, p z^{-\frac{1}{2}}$$

Schaeffer's law

- **Ill-posedness of the incompressible granular flow equations based on Schaeffer model:** The time dependent equations are **linearly ill-posed** according to Schaeffer; in the Navier-Stokes equations, the pressure force associated to the constraint $\operatorname{div} v = 0$ can do no work. By contrast, the pressure force in equation of granular flow can do work, and for plane waves in certain directions, it does so.

- **Flow from a hoper:** The theory over-predicts the measured values by a factor of about four.

- **Further improvement:** A better treatment is by assuming that the powder is compressible!

New general equations for a powder

- **General equation of motion for a powder**

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \nabla \cdot \left[\frac{q(p,\rho)}{\left\| \mathbf{D} - \frac{1}{n}\nabla\cdot\boldsymbol{u}I \right\|} \left(\mathbf{D} - \frac{1}{n}\nabla\cdot\boldsymbol{u}I \right) \right] + \rho g, \text{ with}$$

- **Continuity equation**

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0, \text{ and}$$

- **Normality condition**

$$\nabla\cdot\boldsymbol{u} = \frac{\partial q(p,\rho)}{\partial p} \left\| \mathbf{D} - \frac{1}{n}\nabla\cdot\boldsymbol{u}I \right\|$$

- **the yield condition $q(p,\rho)$ is given by:**

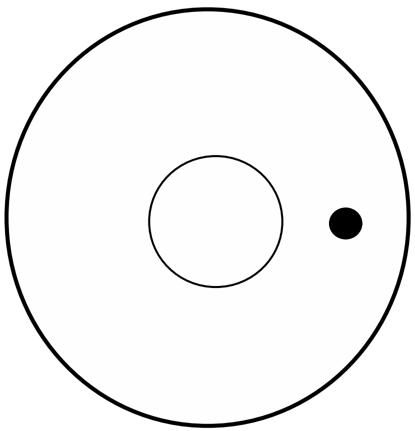
Powder properties	Non-cohesive	Cohesive
Incompressible	$p\sin\phi$ (Scheaffer model)	$p\sin\phi + c\cos\phi$
Compressible	$p\sin\phi\left[2 - \frac{p}{\rho^{\frac{1}{\beta}}}\right]$	$p\sin\phi\rho^{\frac{1}{\beta}} - C\frac{(p-\rho^{\frac{1}{\beta}})^2}{\rho^{\frac{1}{\beta}}}$

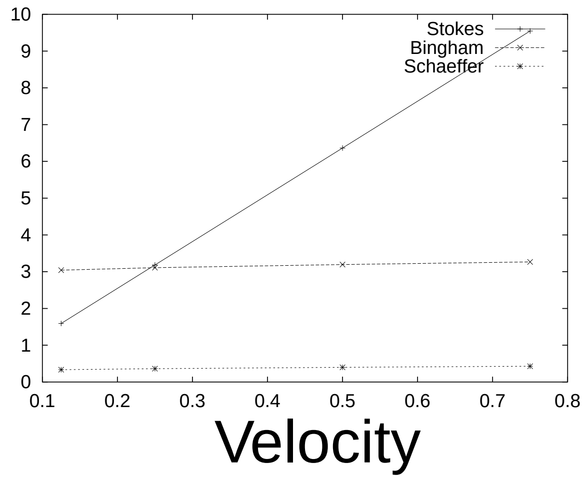
State of the art of the numerical simulations

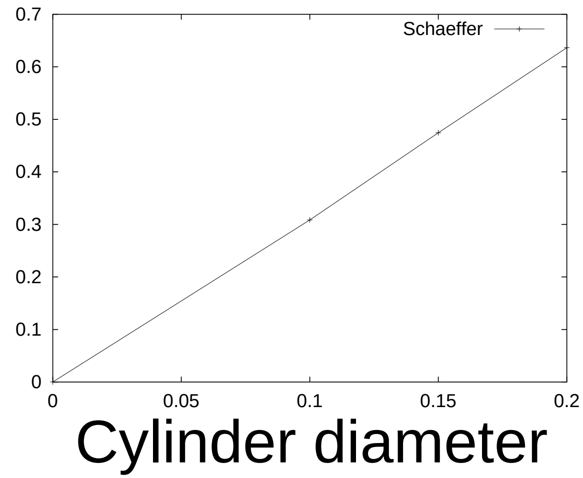
- **Spatial discretization**
  - nonconforming finite elements with edge oriented d.o.f's for velocity
  - piecewise constant finite elements for pressure
- **Nonlinear iteration:** Newton method
- **Linear solver:** multigrid for velocity and the pressure
- **Coupled solver CC2d:** stationary and nonstationary problem
  - Defect correction method as outer iteration
  - The linear coupled subproblems are solved in one iteration step
- **Projection solver PP2d:** nonstationary problem
  - Decoupling step for the velocity $\boldsymbol{u}$ and the pressure p
  - perform only one iteration for pressure each time step

Drag force in a granular medium

- The dependence of **drag force** with grain **velocity** in a couette flow around a cylinder for different material and for different cylinder diameter





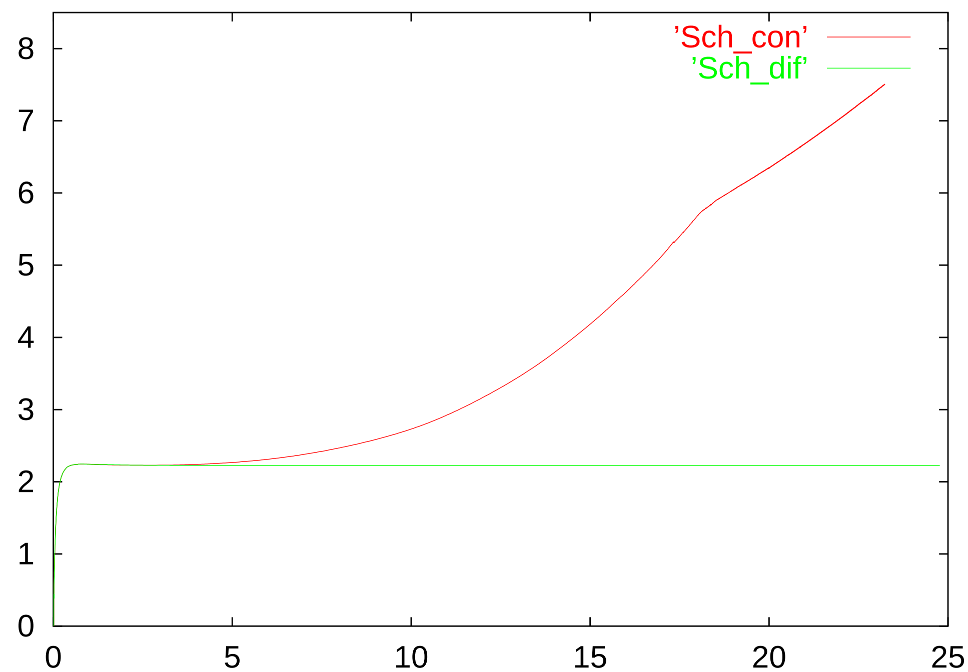


- The **drag force** for schaeffer and Bingham flow acting on cylinder is **independent** of the grain **velocity**, contrary to the stokes flow

”When mechanical ploughs replaced draught animals, it was observed that ploughing at greater speeds does not require greater forces!”

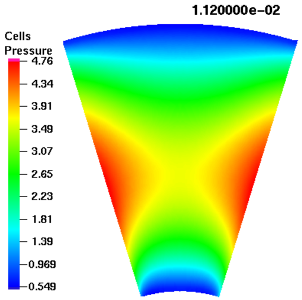
Drag force in a granular medium

The dependence of the force with velocity grain for Schaeffer model with and without convection

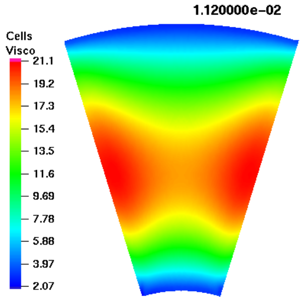


- inertia effect must be taken in consideration from certain speed

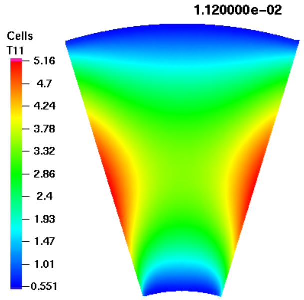
Granular medium in a hopper



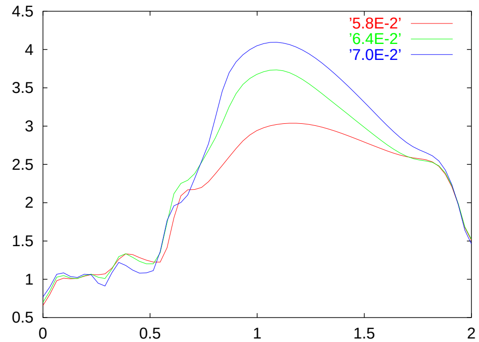
Pressure



Viscosity

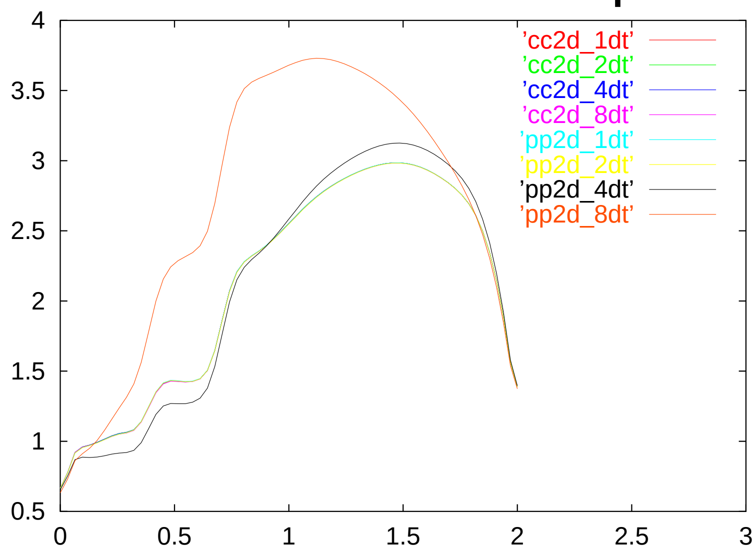


Stress11



Pressure wave

- sensitivity of the solution with the time step of the operator splitting method



Conclusion and Outlook

We have to come with the conclusion that using finite element methods for numerical simulation of incompressible granular material is a useful tool

- The complete picture of the flow is involved, i.e. the velocity, the pressure as well as the stress
- The decoupled solves
  - lead to the same solution as the coupled solves, but smaller time steps are required
  - is at least ten time faster than the coupled solves per time step
- The computer simulation confirm the well known physical behavior
  - The independence of the drag force with the velocity grain
  - The propagation of a pressure wave down the hopper, which leads to the shear banding instability.

Clearly, shear band is not true physical discontinuity, rather than a change in the involved physical system. So further treatment is the handle **compressible granular material**.