Mixed FEM of Higher-Order for Time-Dependent Contact Problems

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Abstract

In this paper mixed finite element methods of higher-order for time-dependent contact problems are discussed. The mixed methods are based on resolving the contact conditions by the introduction of Lagrange multipliers. Dynamic Signorini problems with and without friction are considered involving thermomechanical and rolling contact. Rothe's method is used to provide a suitable time and space discretization. To discretize in time, a stabilized Newmark method is applied as an adequate time stepping scheme. The space discretization relies on finite elements of higher-order. In each time step the resulting problems are solved by Uzawa's method or, alternatively, by methods of quadratic programming via a suitable formulation in terms of the Lagrange multipliers. Numerical results are presented towards an application in production engineering. The results illustrate the performance of the presented techniques for a variety of problem formulations.

Keywords:

frictional contact, Signorini problem, thermomechanical contact, finite element method, higher-order

1. Introduction

Dynamic contact, including frictional and thermal effects, appears in many engineering processes and has an essential effect on the behavior of machines, tools, workpieces, etc. For instance, the main effects on the dynamic behavior of metal-cutting machines typically result from the contact of the tool and the workpiece in a small contact zone. One of the most decisive factors to control dynamic phenomena in milling and cutting processes is, therefore, the determination of appropriate quantities as contact forces or contact zones. Thus, an essential part of simulation tools coping with such processes consists in the application of appropriate numerical schemes for contact.

Modeling contact problems involves systems of partial differential equations with inequality conditions describing several aspects of contact as geometrical constraints, friction or thermal effects. In literature, a huge number of numerical schemes is given dealing with the specific phenomena of contact. We refer to the monographs [38, 53] and the survey articles [15, 36] for an overview. Numerical schemes for dynamic problems are usually based on a combination of time and spatial discretization approaches. A usual proceeding is to use Rothe's method in which the time variable and then the spatial variables are discretized. A well-established approach for the time discretization of hyperbolic problems with finite differences is the Newmark method [42]. An extended variant is the generalized- α method, cf. [8]. The Newmark method can also be applied to discretize dynamic contact problems which, however, requires the use of some special parameters, cf. [3, 9]. It is an easily realizable approach for unilateral contact problems, where the geometrical constraints are ensured in each time step. Finite elements or other Galerkin-type methods are applied for the spatial discretization.

Important issues arising in numerical schemes for dynamic contact problems are, for instance, resolving contact in time preserving energy and momentum [1, 39], stabilizitation to avoid numerical oscillations [12, 22, 30, 31, 32, 37, 40, 44], discretizations with adaptivity [5, 6] and the efficient implementation. Widely used discretization approaches for contact problems are described in [3, 10, 49, 54]. They rely, for instance, on special contact elements with Lagrange multipliers or on penalty methods to capture the geometrical contact conditions.

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Using the Newmark approach, one obtains a sequence of partial differential equations of which the solutions are discretized in time. In the framework of linear elasticity, this sequence, also known as the semi-discrete problem, admits static contact problems in each time step. Consequently, techniques for the static case can be applied directly. In literature many approaches for static contact problems are described, which can, in principle, be used to combine them with the Newmark scheme. Again, we refer to the monographs [38, 53]. Solution schemes for static contact problems are still an important subject of current research. We refer to the recent works [13, 27, 28, 35, 52]. Evidently, the use of them in dynamic contact problems a wide range of application.

A well-established approach to solve static contact problems is given by the application of mixed methods where the geometrical contact conditions and the frictional conditions are captured by Lagrange multipliers. It is widely studied and enhanced by Haslinger et al. [23, 25, 26] for many applications in frictional contact problems. In particular, efficient domain decomposition techniques are applied in the context of the FETI approach, cf. [14]. The discretization is based on a mixed variational formulation derived from a discretized saddle point formulation. The main advantage of this approach is that the Lagrange multipliers can be interpreted as normal and tangential contact forces. Moreover, the constraints for the Lagrange multipliers are sign conditions and box constraints which are simpler than the original contact conditions. The unique existence of a discrete saddle point is usually verified via an inf-sup condition associated to the discretization spaces. In the case of low-oder finite elements, the key to guarantee the inf-sup condition is to use a discretization of the Lagrange multipliers on boundary meshes with a larger mesh size than that of the primal variable, cf. [24]. But, the application of higher-order finite elements is possible as well, which may avoid the use of different mesh sizes by using different polynomial degrees. We refer to [47] for more details, in particular, with respect to the discrete inf-sup condition and solution schemes by some Schur complement techniques. Further benefits of higher-order discretizations are, for instance, the reduction of locking effects and, using hp-adaptivity, high convergence rates or even exponential convergence rates, cf. [45].

In this work, we combine the stabilized Newmark scheme proposed in [12, 37] and the mixed method with a higher-order discretization to obtain a numerical scheme for dynamic contact problems and consider several physical attributes such as damping, friction, thermoelastic coupling and rolling contact. A framework is proposed which enables to include all these attributes in a general setting. The stabilization of the Newmark scheme is based on an additional L^2 projection on the admissible set specified by the contact conditions, which can easily be realized for higher-order discretization in space. Stabilization techniques for the Newmark scheme are also proposed in [22, 32], which are, however, more complex to apply in this context, since the efficient construction of the redistributed mass matrix for higher-order basis functions is an open problem.

The physical interpretation of the Lagrange multipliers as contact forces exhibits several advantages. For instance, in thermoelasticity the modelling of the heat induction generated by the frictional contact can directly be realized, using the Lagrange multiplier associated to the frictional condition. It represents the tangential forces which are proportional to the heat induction.

One of the aims of this work is to show the applicability of the proposed approach with the help of several benchmark examples of dynamic contact. We focus on the stability of the Lagrange multipliers in space and time as well as the conservation of energy. Since the time stepping scheme is slightly dissipative due to the stabilization step, we especially study the dependence of the loss of energy w.r.t. the discretization parameters. Finally, as a realistic example of a dynamic contact problem in 3D, which includes all discussed effects, we consider an NC-shape grinding process of free formed surfaces with a toroid grinding wheel.

The article is organized as follows: In Section 2, we introduce some notations and the general mixed formulation of contact problems. Moreover, we propose a discretization of higher-order with finite elements and introduce some solution schemes to solve the resulting systems. In the remaining part of this paper, our aim is to capture contact, friction, damping and thermoelasticity using this general mixed formulation. We show that dynamic contact problems with all these different attributes have, in principle, the same structure in the setting of the mixed method. The first example is a dynamic contact problem of Signorini-type with damping and Tresca friction, which can easily be extended to Coulomb friction. It is discussed in Section 3. We introduce the time discretization using the stabilized Newmark approach and formulate the resulting contact problems in each time step in the sense of the mixed method. We present numerical experiments and show the applicability of the theoretical findings. Section 4 focuses on dynamic contact problems in thermoelasticity where heat generated by friction is fully coupled with linear elastic deformation. Again, we use the stabilized Newmark approach to discretize the contact problem and the Crank-Nicholson scheme for the heat propagation. As in the previous section, we use the mixed method in each time step. We examine numerical

experiments again, but now with a thermoelastic coupling. The NC-Shape grinding process is considered in Section 5. To model this process, we have to take friction, thermoelastic coupling and rotational effects into account. In particular, the rotational effects are included in the discretization scheme by an arbitrary Lagrangian Eulerian (ALE) ansatz. Even though this problem is highly complex and includes very different physical phenomena, it is, nevertheless, possible to bring it into the proposed framework of mixed methods. We conclude the article with a discussion of the results and an outlook to future works.

2. The general mixed method

In this section we present a general mixed method for problems with geometrical and frictional contact. The method is general in the sense that a general bilinear form a and a general linear form ℓ on some Sobolev spaces are introduced. Whenever a certain (sub-)problem has the specific form of an energy minimization problem or a variational inequality of second kind describing geometrical contact and/or friction, this general mixed method can be applied to obtain the subsequently proposed discretizations and solution schemes. We introduce a higher-order discretization based on finite elements for the underlying Sobolev spaces and solution schemes relying on Uzawa's method and, alternatively, on the reformulation of the problem in the Lagrange multipliers.

2.1. Notations

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a domain with sufficiently smooth boundary $\Gamma := \partial \Omega$. Moreover, let $\Gamma_D \subset \Gamma$ be closed with positive measure and let $\Gamma_C \subset \Gamma \setminus \Gamma_D$ with $\overline{\Gamma}_C \subsetneq \Gamma \setminus \Gamma_D$. $L^2(\Omega)$, $H^k(\Omega)$ with $k \ge 1$, and $H^{1/2}(\Gamma_C)$ denote the usual Sobolev spaces and we set $H_D^1(\Omega) := \{\varphi \in H^1(\Omega) \mid \gamma(\varphi) = 0 \text{ on } \Gamma_D\}$ with the trace operator γ . The space $H^{-1/2}(\Gamma_C)$ denotes the topological dual space of $H^{1/2}(\Gamma_C)$ with the norms $\|\cdot\|_{1/2,\Gamma_C}$ and $\|\cdot\|_{1/2,\Gamma_C}$. Let $(\cdot, \cdot)_{0,\omega}$, $(\cdot, \cdot)_{0,\Gamma'}$ be the usual L^2 -scalar products on $\omega \subset \Omega$ and $\Gamma' \subset \Gamma$, respectively. Note that the linear and bounded mapping $\gamma_C := \gamma_{|\Gamma_C} : H^1_D(\Omega) \to H^{1/2}(\Gamma_C)$ is surjective due to the assumptions on Γ_C , cf. [33, p.88]. For functions in $L^2(\Gamma_C)$, the inequality symbols \geq and \leq are defined as "almost everywhere". We set $H^{1/2}_+(\Gamma_C) := \{\mu \in H^{1/2}(\Gamma_C) \mid \mu \geq 0\}$ and $L^2_s(\Gamma_C) := \{\mu \in (L^2(\Gamma_C))^{d-1} \mid |\mu| \leq 1 \text{ on supp } s, \ \mu = 0 \text{ on } \Gamma_C \setminus \text{supp } s\}$ with the euclidian norm $|\cdot|$ and $s \in L^2(\Gamma_C), s \geq 0$. Furthermore, we define $H^{-1/2}_+(\Gamma_C) := (H^{1/2}_+(\Gamma_C))' := \{\mu \in H^{-1/2}(\Gamma_C) \mid \forall \nu \in H^{1/2}_+(\Gamma_C) : \langle \mu, \nu \rangle \geq 0\}$ as the dual cone of $H^{1/2}_+(\Gamma_C)$. For the displacement field φ we specify the linearized strain tensor as $\varepsilon(\varphi) := \frac{1}{2}(\nabla \varphi + (\nabla \varphi)^{\top})$ and the stress tensor as $\sigma(\varphi)_{ij} := C_{ijkl} \varepsilon(\varphi)_{kl}$ describing a linear-elastic material law where $C_{ijkl} \in L^{\infty}(\Omega)$ with $C_{ijkl} = C_{jilk} = C_{klij}$ and $C_{ijkl}\tau_{ij}\tau_{kl} \ge \kappa \tau_{ij}^2$ for $\tau \in L^2(\Omega)_{sym}^{d\times d}$ and a $\kappa > 0$. In the following, *n* denotes the vector-valued function describing the outer unit normal vector with respect to Γ_C and t the $d \times (d-1)$ -matrix-valued function containing the tangential vectors. We define $\sigma_{n,j} := \sigma_{ij}n_i$, $\sigma_{nn} := \sigma_{ij}n_in_j$, and $\sigma_{nt,l} := \sigma_{ij}n_it_{jl}$. Moreover, we set $\gamma_n(\varphi) := \gamma_C(\varphi_i)n_i$, $\gamma_t(\varphi)_j := \gamma_C(\varphi_i)t_{ij}$ and $\gamma_{N,i}(\varphi) := \gamma_{|\Gamma_N}(\varphi_i)$. We assume that Γ_C is parameterized by a sufficiently smooth function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ so that, without loss of generality, the geometrical contact condition for a displacement φ in the d-th component is given by $\phi(x) + \varphi_d(x, \phi(x)) \le \psi(x_1 + \varphi_1(x, \phi(x)), \dots, x_{d-1} + \varphi_{d-1}(x, \phi(x)))$ with $x := (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}$ and a sufficiently smooth function ψ describing the surface of an obstacle. Since this condition is non-linear in general, one usually applies the linearization $\gamma_n(\varphi) \leq g_{\psi}$ with $g_{\psi}(x) := (\psi(x) - \phi(x))(1 + (\nabla \phi(x))^\top \nabla \phi(x))^{-1/2}$, cf. [33, Ch.2]. Finally, we set $V := (H_D^1(\Omega))^d$, $W := (L^2(\Omega))^d$, $\Lambda_n := H_+^{-1/2}(\Gamma_C)$, $\Lambda_{t,s} := L_s^2(\Gamma_C)$, $j_s(\varphi) := \int_{\Gamma_C} s|\gamma_t(\varphi)| ds$ and

 $K_{\psi} := \{ \varphi \in V \mid \gamma_n(\varphi) \le g_{\psi} \}.$

2.2. The mixed formulation

Let a be a symmetric, continuous and elliptic bilinear form on $V \times V$, ℓ be a continuous linear form and define $E(\varphi) := \frac{1}{2}a(\varphi,\varphi) - \langle \ell,\varphi \rangle$. Furthermore, let $s \in L^2(\Gamma_C)$ with $s \ge 0$ and ψ so that $g_{\psi} \in H^{1/2}(\Gamma_C)$. Using standard arguments of convex analysis, we conclude that the functional $E + j_s$ is weakly lower semicontinuous, coercive and strictly convex and due to the closedness and convexity of K_{ψ} there exists a unique minimizer $u \in K_{\psi}$ with

$$E(u) = \min_{\varphi \in K_{\psi}} (E + j_s)(\varphi).$$
(2.1)

Moreover, since E is Fréchet differentiable in u with the Fréchet derivative $\langle E'(u), \varphi \rangle = a(u, \varphi) - \langle \ell, \varphi \rangle$ and j_s is convex, the stationarity condition holds,

$$a(u,\varphi-u) - \langle \ell,\varphi-u \rangle + j(\varphi) - j(u) \ge 0$$
(2.2)

for all $\varphi \in K_{\psi}$. Due to the convexity of *E*, the solution of (2.2) is also a minimizer of (2.1). We refer to [47], [33, Prop. 3.1, p.33] and [17, Ch. II, Prop. 1.2, p.35] for the proofs of these elementary assertions. To derive a mixed variational formulation, we resolve the condition $\varphi \in K_{\psi}$ and the functional j_s via the introduction of Lagrange multipliers. Using the Theorem of Hahn-Banach, it can be shown that

$$\sup_{u_n \in \Lambda_n} \langle \mu_n, \gamma_n(\varphi) - g_{\psi} \rangle = \begin{cases} 0, & \varphi \in K_{\psi} \\ \infty, & \varphi \notin K_{\psi}, \end{cases}$$
(2.3)

cf. [47]. Furthermore, there holds $j_s(\varphi) = \sup_{\mu_t \in \Lambda_{ts}} (\mu_t, s\varphi_t)_{0,\Gamma_c}$ for all $\varphi \in V$. Therefore, we have

$$(E+j_s)(u) = \inf_{\varphi \in V} \sup_{\mu_n \in \Lambda_n, \mu_t \in \Lambda_{t,s}} \mathcal{L}(\varphi, \mu_n, \mu_t)$$

with the Lagrange functional $\mathcal{L}(\varphi, \mu_n, \mu_t) := E(\varphi) + \langle \mu_n, \gamma_C(\varphi) - g_{\psi} \rangle + (\mu_t, s\gamma_t(\varphi))_{0,\Gamma_C}$. Thus, *u* is a minimizer of (2.1), whenever the triple $(u, \lambda_n, \lambda_t) \in V \times \Lambda_n \times \Lambda_{t,s}$ is a saddle point,

$$\mathcal{L}(u,\lambda_n,\lambda_t) = \inf_{\varphi \in V} \sup_{\mu_n \in \Lambda_n, \mu_t \in \Lambda_{t,s}} \mathcal{L}(\varphi,\mu_n,\mu_t).$$
(2.4)

Again, using the stationarity condition, we obtain that the triple $(u, \lambda_n, \lambda_t) \in V \times \Lambda_n \times \Lambda_t$ is a saddle point if and only if,

$$a(u,\varphi) = \langle \ell,\varphi \rangle - \langle \lambda_n,\gamma_n(\varphi) \rangle - (\lambda_t,s\gamma_t(\varphi))_{0,\Gamma_C},$$

$$\langle \mu_n - \lambda_n,\gamma_n(u) - g_{\psi} \rangle + (\mu_t - \lambda_t,s\gamma_t(u))_{0,\Gamma_C} \le 0$$
(2.5)

for all $(\varphi, \mu_n, \mu_t) \in V \times \Lambda_n \times \Lambda_{t,s}$, cf. [47]. The existence of a saddle point is usually shown by assuming an inf-sup condition: there exists an $\alpha \in \mathbb{R}_{>0}$ such that

$$\alpha \|\mu_n\|_{-1/2,\Gamma_C} \le \sup_{\varphi \in V, \|\varphi\|=1} \langle \mu_n, \gamma_n(\varphi) \rangle$$
(2.6)

for all $\mu_n \in H^{-1/2}(\Gamma_C)$, cf. [33, Lem. 3.2, p.45] and [47]. Since the mapping γ_n is surjective, condition (2.6) directly follows from the closed range theorem so that the existence is, indeed, ensured, cf. [55, p.205]. It is easy to see, that the Lagrange multipliers λ_n and λ_t are unique which is a consequence of (2.6) and the fact that $\gamma_t(\ker \gamma_n)$ is dense in $L^2(\Gamma_C)$.

Remark 2.1. The Lagrange multipliers can be interpreted as contact forces which is an important advantage of the mixed formulation. Under certain regularity assumptions, there holds $\lambda_n = -\sigma_{nn}(u)$ and $s\lambda_t = -\sigma_{nt}(u)$.

2.3. Discretization of higher-order

We propose a higher-order finite element discretization based on quadrangles or hexahedrons as follows: Let \mathcal{T} be a finite element mesh of Ω with mesh size h and let \mathcal{E} be a finite element mesh of Γ_C with mesh size H. The number of elements in \mathcal{T} is denoted by $M_{\mathcal{T}}$ and in \mathcal{E} by $M_{\mathcal{E}}$. Furthermore, let $\Psi_T : [-1, 1]^d \to T \in \mathcal{T}$ and $\Phi_E : [-1, 1]^{d-1} \to E \in \mathcal{E}$ be bijective and sufficiently smooth transformations and let $p, q \in \mathbb{N}$. Using the polynomial tensor product space $\mathcal{P}_{r,d}$ of order r on the reference element $[-1, 1]^d$, we define

$$\mathcal{S}_h := \left\{ \varphi \in H_D^1(\Omega) \mid \forall T \in \mathcal{T} : \varphi_{|T} \circ \Psi_T \in \mathcal{P}_{p,d} \right\},$$
$$\mathcal{M}_H := \left\{ \mu \in L^2(\Gamma_C) \mid \forall E \in \mathcal{E} : \mu_{|E} \circ \Phi_E \in \mathcal{P}_{q,d-1} \right\}.$$

For a finite subset $C \subset [-1, 1]^{d-1}$, we define

$$\mathcal{M}_{H,+} := \{ \mu \in \mathcal{M}_H \mid \forall E \in \mathcal{E} : \forall x \in C : \ \mu(\Phi_E(x)) \ge 0 \},$$

$$\mathcal{M}_{H,s} := \{ \mu \in (\mathcal{M}_H)^{k-1} \mid \forall E \in \mathcal{E} : \forall x \in C : \ |\mu(\Phi_E(x))| \le 1 \land (\Phi_E(x) \notin \text{supp } s \Rightarrow \mu(\Phi_E(x)) = 0) \}$$

We set $V_h := (S_h)^d$, $\Lambda_{n,H} := \mathcal{M}_{H,+}$ and $\Lambda_{t,s,H} := \mathcal{M}_{H,s}$. The discrete saddle point problem is to find $(u_h, \lambda_{n,H}, \lambda_{t,H}) \in V_h \times \Lambda_{n,H} \times \Lambda_{t,s,H}$ such that

$$\mathcal{L}(u_h, \lambda_{n,H}, \lambda_{t,H}) = \inf_{\varphi_h \in V_h} \sup_{\mu_{n,H} \in \Lambda_{n,H}, \mu_{t,H} \in \Lambda_{t,s,H}} \mathcal{L}(\varphi_h, \mu_{n,H}, \mu_{t,H}).$$
(2.7)

It is easy to see that the first component is the unique minimizer of the minimization problem $(E + j_{s,hH})(u_h) = \min_{\varphi_h \in K_{hH}}(E + j_{s,hH})(\varphi_h)$ with $K_{hH} := \{\varphi_h \in V_h \mid \forall \mu_{n,H} \in \Lambda_{n,H} : \langle \mu_{n,H}, \gamma_n(\varphi_h) - g_{\psi} \rangle \leq 0\}$ and $j_{s,hH}(\varphi_h) := \sup_{\mu_{t,H} \in \Lambda_{t,s,H}} (\mu_{t,H}, s\gamma_t(\varphi_h))_{0,\Gamma_c}$. Again by stationarity, it follows that $(u_h, \lambda_{n,H}, \lambda_{t,H}) \in V_h \times \Lambda_{n,H} \times \Lambda_{t,s,H}$ is a discrete saddle point if and only if

$$a(u_{h},\varphi_{h}) = \langle \ell,\varphi_{h} \rangle - \langle \lambda_{n,H},\gamma_{n}(\varphi_{h}) \rangle - (\lambda_{t,H},s\gamma_{t}(\varphi_{h}))_{0,\Gamma_{C}},$$

$$\langle \mu_{n,H} - \lambda_{n,H},\gamma_{n}(u_{h}) - g_{\psi} \rangle + (\mu_{t,H} - \lambda_{t,H},s\gamma_{t}(u_{h}))_{0,\Gamma_{C}} \leq 0$$
(2.8)

for all $(\varphi_h, \mu_{n,H}, \mu_{t,H}) \in V_h \times \Lambda_{n,H} \times \Lambda_{t,s,H}$. Similarly to the non-discrete case, there exists a unique discrete saddle point $(u_h, \lambda_{n,H}, \lambda_{t,H}) \in V_h \times \Lambda_{n,H} \times \Lambda_{t,s,H}$, if there exists an $\alpha \in \mathbb{R}_{>0}$ such that

$$\alpha(\|\mu_{n,H}\|_{-1/2,\Gamma_{C}} + \|\mu_{t,H}\|_{-1/2,\Gamma_{C}}) \leq \sup_{\varphi_{h} \in V_{h}, \|\varphi_{h}\| = 1} (\mu_{n,H}, \gamma_{n}(\varphi_{h}))_{0,\Gamma_{C}} + (\mu_{t,H}, s\gamma_{t}(\varphi_{h}))_{0,\Gamma_{C}}$$
(2.9)

for all $(\mu_{n,H}, \mu_{t,H}) \in \mathcal{M}_H \times (\mathcal{M}_H)^{d-1}$. The condition (2.9) means that spaces V_h and \mathcal{M}_H have to be balanced appropriately. Obviously, this balance depends neither on the definition of $\mathcal{M}_{H,+}$ nor on the definition of $\mathcal{M}_{H,s}$. Under certain regularity assumptions, it can be shown that (2.9) is valid, if the value given by $\Pi(h, H, p, q) := hH^{-1} \max\{1, q\}^2 p^{-1}$ is sufficiently small, cf. [47]. Obviously, we can vary h and H or p and q or both to reduce $\Pi(h, H, p, q)$. It is noted that varying h and H implies that the Lagrange multiplier is possibly defined on a coarser mesh which may lead to a higher implementational complexity. Using a surface mesh \mathcal{E} , which is inherited from the interior mesh \mathcal{T} , the implementational effort is essentially smaller. However, in this case we have h/H = 1 and can only vary p and q to keep $\Pi(h, H, p, q)$ small. It can be observed in numerical experiments that the Lagrange multiplier oscillates for an inappropriate choice of h, H, p and q which suggests that the Lagrange multiplier is not unique or, in other words, the mixed discretization is not stable. In this case, the Lagrange multiplier is not a reasonable approximation of the contact forces and has no physical meaning. In general it is not clear when $\Pi(h, H, p, q)$ is small enough such that (2.9) holds. Nevertheless, it justifies the modification of the discretization scheme by coarsening the mesh \mathcal{E} or by decreasing the polynomial degree q to obtain a stable scheme. For more details on the subject of stability with respect to the introduced mixed method we refer to [47] and [48].

Remark 2.2. The convergence of the scheme depends on the choice of the discrete set *C* and the validity of (2.9). It can be shown that u_h strongly converges to u and $\lambda_{n,H}$ as well as $\lambda_{s,t,H}$ weakly converge to λ_n and λ_t , respectively, if (2.9) holds and *C* is chosen as the set of $(q + 1)^{d-1}$ Gauss points. We refer to [46, 48] for more details on the convergence of mixed schemes with higher-order discretizations.

2.4. Solution schemes for higher-order discretizations

To solve the discrete formulation (2.8), we apply a basis $\{\varphi_i\}_{0 \le i < \bar{n}}$ of S_h with $\bar{n} := \dim S_h$, which is constructed via standard techniques of H^1 -conforming hp-finite elements, cf., e.g., [2, 11, 50]. The construction of a basis of \mathcal{M}_H is much simpler, since no continuity requirements have to be taken into account. However, the basis has to be compatible with the restrictions (sign conditions, boundedness) of the Lagrange multipliers. For this reason, we look at the construction of the basis and their consequences to the resulting system and solution processes in more detail.

The basic idea is to construct a basis via Lagrange basis functions. For this purpose, let *C* be the tensor product of the q + 1 points $\xi_0, \ldots, \xi_q \in [-1, 1]$, i.e. $C := \{\xi_\alpha \mid \alpha \in N\}$ with $\xi_\alpha := (\xi_{\alpha_1}, \ldots, \xi_{\alpha_{d-1}})$ and $\mathcal{N} := \{0, \ldots, q\}^{d-1}$. Furthermore, let $\chi : \mathcal{E} \times \mathcal{N} \to \{0, \ldots, \overline{m} - 1\}$ be a bijective numbering with $\overline{m} := (q + 1)^{d-1} M_{\mathcal{E}}$. Without loss of generality, we assume a number $0 \le \widehat{m} \le \overline{m}$ such that $\chi(E, \alpha) \ge \widehat{m}$ implies $\Phi_E(\xi_\alpha) \notin$ supp *s* and vice versa for $E \in \mathcal{E}$ and $\alpha \in \mathcal{N}$. After these preparations, we define the Lagrange basis functions $\{\psi_i\}_{0 \le i < \overline{m}}$ of \mathcal{M}_H by

$$\psi_{\chi(E,\alpha)}(\Phi_E(x)) := \prod_{l=1}^{d-1} \prod_{\nu=0\atop\nu\neq\alpha_l}^q \frac{x_{\alpha_l} - \xi_\nu}{\xi_{\alpha_l} - \xi_\nu}$$

for $E \in \mathcal{E}$, $\alpha \in \mathcal{N}$ and $x \in [-1, 1]^{d-1}$ as well as $\psi_{\chi(\tilde{E}, \alpha)}(\Phi_{\tilde{E}}(x)) := 0$ for $\tilde{E} \in \mathcal{E} \setminus E$. The compatibility to the restrictions of the Lagrange multipliers is then easily realized in the following way: There holds $\mu_{n,H} = z_{n,j}\psi_j \in \Lambda_{n,H}$ iff $z_n \in \bar{\Lambda}_n := \mathbb{R}^{\tilde{m}}_{\geq 0}$ and $\mu_{t,H} = z_{t,j(d-1)+i}\psi_j e_i^{d-1} \in \Lambda_{t,s,H}$ iff $z_t \in \bar{\Lambda}_t := \{z \in \mathbb{R}^{(d-1)\hat{m}} \mid \omega_{\chi(E,\alpha)}(z) \leq 1\}$ where $\omega_j(z) := \sum_{l=0}^{d-2} z_{j(d-1)+l}^2$,

and $e_i^d \in \mathbb{R}^d$ is the unit vector given by $(e_i^d)_j := \delta_{ij}$ with Kronecker's delta δ . Thus, the discretization (2.8) is equivalent to find $(x, y_n, y_t) \in \mathbb{R}^{\bar{n}} \times \bar{\Lambda}_n \times \bar{\Lambda}_t$ such that

$$Ax + B_n^{\top} y_n + B_t^{\top} y_t = L, (y_n - z_n)^{\top} (B_n x - G) + (y_t - z_t)^{\top} B_t x \le 0$$
(2.10)

for all $(z_n, z_t) \in \overline{\Lambda}_n \times \overline{\Lambda}_t$. Here, $A \in \mathbb{R}^{\overline{n} \times \overline{n}}$, $L \in \mathbb{R}^{\overline{n}}$, $B_n \in \mathbb{R}^{\overline{m} \times \overline{n}}$, $B_t \in \mathbb{R}^{(d-1)\overline{m} \times \overline{n}}$ and $G \in \mathbb{R}^{\overline{m}}$ are defined as

$$\begin{split} A_{jd+i,ld+r} &:= a(\varphi_l e_r^d, \varphi_j e_i^d), \quad L_{jd+i} := \langle \ell, \varphi_j e_i^d \rangle, \quad G_l := (\psi_l, g_{\psi})_{0,\Gamma_C}, \\ B_{n,l,jd+i} &:= (\psi_l, \gamma_n(\varphi_j e_i^d))_{0,\Gamma_C}, \quad B_{t,l(d-1)+r,jd+i} := (\psi_l e_r^{d-1}, \gamma_l(\varphi_j e_i^d))_{0,\Gamma_C}. \end{split}$$

Obviously, the solution of (2.8) is then given by $u_h = x_{jd+i}\varphi_j e_i^d$, $\lambda_{n,H} = y_{n,j}\psi_j$ and $\lambda_{s,t,H} = y_{t,j(d-1)+i}\psi_j e_i^{d-1}$. A simple iterative scheme to solve the system (2.10) is Uzawa's method with projections. In each iteration step the projections $P_n : \mathbb{R}^{\bar{m}} \to \mathbb{R}^{\bar{m}}$ and $P_t : \mathbb{R}^{(d-1)\hat{m}} \to \mathbb{R}^{(d-1)\hat{m}}$ are applied so that the coefficient vectors y_n^r and y_t^r are in $\bar{\Lambda}_n$ and $\bar{\Lambda}_t$, respectively. Due to the Lagrange-like structure of the basis functions, the projections P_n and P_t are very simple:

$$P_{n,j}(z) := \begin{cases} z_j, & z_j \ge 0\\ 0, & \text{else} \end{cases} \quad j = 0, \dots, \bar{m}$$

$$P_{t,j(d-1)+i}(z) := \begin{cases} z_{j(d-1)+i}, & \omega_j(z) \le 1, \\ (\omega_j(z))^{-1/2} z_{j(d-1)+i}, & \text{else} \end{cases} \quad j = 0, \dots, \hat{m}, \ i = 0, \dots, d-2.$$

$$(2.11)$$

Uzawa's method reads

$$\begin{aligned} x^{r+1} &= x^r - \rho_1 S^{-1} (A x^r + B_n^{\top} y_n^r + B_t^{\top} y_t^r - L), \\ y_n^{r+1} &= P_n (y_n^r + \rho_2 (B_n x^{r+1} - G)), \\ y_t^{r+1} &= P_t (y_t^r + \rho_2 B_t x^{r+1}) \end{aligned}$$
(2.12)

with some parameters $\rho_1, \rho_2 > 0$. Usually, $S^{-1} \in \mathbb{R}^{n \times n}$ is chosen as A^{-1} or as an appropriate approximation of A^{-1} . We refer to [21] for some convergence results with respect to general Uzawa's methods.

Remark 2.3. It is widely known that the number of iteration steps in Uzawa's method highly depends on the mesh size if the projections P_n and P_t solely ensure the restrictions pointwisely (as in (2.11)). We present Uzawa's method because of its simplicity and direct applicability to higher-order discretizations.

An alternative scheme is based on the dual formulation of (2.10). The basic idea is to reformulate (2.10) into a minimization problem in terms of the Lagrange multipliers. It is easy to see that (x, y_n, y_i) fulfills (2.10) if and only if $x = A^{-1}L - B_n^{\top}y_n - B_t^{\top}y_t$ and

$$F(y_n, y_t) = \min_{(z_n, z_t) \in \bar{\Lambda}_n \times \bar{\Lambda}_t} F(z_n, z_t),$$

$$F(z_n, z_t) := \frac{1}{2} (z_n^{\top} B_n + z_t^{\top} B_t) A^{-1} (B_n^{\top} z_n + B_t^{\top} z_t) + (z_n^{\top} B_n + z_t^{\top} B_t) (G - A^{-1} L).$$
(2.13)

The Problem (2.13) can be solved using optimization schemes of quadratic programming, for instance, standard optimization tools based on QP- or SQP-techniques. We refer to the SQP-package Snopt by Gill et. al [19, 20] for some usable implementations. The fact that the dimension \bar{m} of the optimization variable given by the Lagrange multipliers is, in general, much smaller than the dimension of the discrete displacement variable \bar{n} makes this approach very applicable. For low-order finite elements, the reformulation in the dual variables is widely studied and enhanced for many applications in frictional contact problems. We refer to [14, 23, 25, 26] for more details, in particular, concerning splitting type algorithms and domain decomposition techniques. Moreover, there is, of course, a huge number of other very efficient approaches for solving contact problems in the case of low-order finite elements. We refer to some recent works [13, 27, 28, 35, 52].

Remark 2.4. The set C should be chosen so that the additional numerical error is minimized. For instance, Chebycheff points may be a good choice to ensure the additional error to be small, cf, e.g., [16]. However, the use of Gauss points seems to be more natural as their number matches the number of Lagrange basis functions. Moreover, this set of points enables us to prove convergence of the discretization scheme, see Remark 2.2. Using Gauss quadrature rules and Lagrange basis functions the entries of B_n , B_t and G can easily be computed. If $q + 1 \le p$ and Φ_E is affine linear, there holds for $E \in \mathcal{E}$, $\alpha \in \mathcal{N}$ and $l := \chi(E, \alpha)$

$$B_{n,l,jd+i} = \omega_{\alpha} n_i \varphi_j(\Phi_E(\xi_{\alpha})) |E|, \quad B_{t,l(d-1)+r,jd+i} \approx \omega_{\alpha} t_{ir}(s\varphi_j) (\Phi_E(\xi_{\alpha})) |E|, \quad G_l \approx \omega_{\alpha} g_{\psi}(\Phi_E(\xi_{\alpha})) |E|$$

with $\omega_{\alpha} := \prod_{l=1}^{d-1} \omega_{\alpha_l}$ and the weights ω_j of the Gauss quadrature rule.

Remark 2.5. If s = 0, the contact model is reduced to the frictionless case. All terms concerning λ_t can be omitted in the mixed formulation (2.5) and its discretization (2.7). The resulting system (2.10), Uzawa's method (2.12) and the minimization approach (2.13) can also be simplified by omitting all terms concerning y_t .

Remark 2.6. Unfortunately, the introduced framework of Section 2.2 is not directly applicable to Coulomb friction where the frictional function *s* is defined as $s := \mathcal{F}|\sigma_{nn}(u)|$ with some frictional coefficient $\mathcal{F} > 0$. However, Coulomb friction can be embedded into the framework using a simple fix point scheme where we exploit that the Lagrange multiplier λ_n coincides with the normal contact stress $-\sigma_{nn}(u)$. For an arbitrary frictional function $s \in L^2(\Gamma_C)$ with $s \ge 0$, we define $(u(s), \lambda_n(s), \lambda_t(s))$ as the unique saddle point of the Signorini problem with Tresca friction, and furthermore, the operator \mathcal{H} as $\mathcal{H}(s) := \mathcal{F}|\lambda_n(s)|$. Assuming that \mathcal{H} has a fix point, i.e., $\mathcal{H}(\bar{s}) = \bar{s}$, the saddle point $(u(\bar{s}), \lambda_n(\bar{s}), \lambda_t(\bar{s}))$ fulfills the Coulomb friction law. Transfering this concept to the discrete mixed variational formulation, we obtain $(x(s), y_n(s), y_t(s))$ as the solution of (2.10) and define $\tilde{\mathcal{H}}(s) := \mathcal{F}[y_{n,j}(s)\psi_j]$. Again, a fix point \tilde{s} of $\tilde{\mathcal{H}}$ (or a suitable approximation) implies solution vectors $(x(\bar{s}), y_n(\tilde{s}), y_t(\tilde{s}))$ and a discrete saddlepoint $(u_h(\tilde{s}), \lambda_{n,H}(\tilde{s}), \lambda_{t,H}(\tilde{s}))$ which approximatively fulfills the Coulomb friction law. We refer to [23, 25] and reference therein for more details on this well-known proceeding.

3. Dynamic Signorini problems with friction

As the first contact problem, we arrange frictional contact problems with Rayleigh damping into the general framework introduced in the last section. In contrast to static problems, the frictional constraint is defined with respect to the velocity and not to the displacements.

3.1. Continuous problem formulation

Let $I := [0, T] \subset \mathbb{R}$ be a time interval. The density of the material is given by ρ . The initial displacement is specified by u_s and the initial velocity by v_s . The surface of the obstacle at time *t* is described by the function $\psi(t)$, and the bound for the tangential stress by s(t).

We assume that the damping is proportional to the velocity and use the approach of Rayleigh to describe this proportionality. The damping effects are splitted into a mass proportional and a stiffness proportional part. The term $a_d\rho \dot{u}$ represents the damping depending on the mass, where a_d is a positive material constant. The part proportional to the stiffness is given by $\sigma(b_d \dot{u})$ with a material constant $b_d \ge 0$.

The strong formulation of the dynamic Signorini problem with friction is to find a solution u in the space $W^{2,\infty}(I; \{v \in V | \sigma(v) \in H(\text{div}, \Omega)\})$ such that

$$\rho \ddot{u} + a_d \rho \dot{u} - \operatorname{div} \left(\sigma \left(u + b_d \dot{u} \right) \right) = f \text{ in } \Omega \times I, \tag{3.1}$$

$$u = 0 \text{ on } \Gamma_D \times I, \quad \sigma_n(u) = 0 \text{ on } \Gamma_N \times I,$$
(3.2)

$$u(0) = u_s, \ \dot{u}(0) = v_s \ in \ \Omega,$$
 (3.3)

$$\gamma_n(u) - g_{\psi} \le 0, \ -\sigma_{nn}(u) \ge 0, \ \sigma_{nn}(u) \left(\gamma_n(u) - g_{\psi}\right) = 0 \text{ on } \Gamma_C \times I, \tag{3.4}$$

$$\sigma_{nn}(u)\left(\gamma_n(\dot{u}) - g_{\dot{\psi}}\right) = 0 \text{ on } \Gamma_C \times I, \qquad (3.5)$$

$$|\sigma_{nt}(u)| \le s \text{ with } \left\{ \begin{array}{l} |\sigma_{nt}(u)| < s \Rightarrow \gamma_t(\dot{u}) = 0\\ |\sigma_{nt}(u)| = s \Rightarrow \exists \zeta \in \mathbb{R}_{\ge 0} : \gamma_t(\dot{u}) = -\zeta \sigma_{nt} \end{array} \right\} \text{ on } \Gamma_C \times I, \tag{3.6}$$

with $f \in L^{\infty}(I; W)$ and $0 \le s \in L^{\infty}(I; L^2(\Gamma_C))$. Here, (3.1) specifies the balance of momentum. The homogeneous Dirichlet and Neumann boundary conditions are given in (3.2). The equations (3.3) represent the initial conditions.

The geometrical contact conditions are incorporated by (3.4). The persistency condition of dynamic contact, which ensures energy conservation under the usual assumptions, is denoted in (3.5). Finally, the frictional conditions are represented by (3.6).

Using integration by parts in space, we obtain the weak formulation

Problem 3.1. Find a function $u \in \mathcal{V} := W^{2,\infty} \left(I; \left(L^2(\Omega) \right)^d \right) \cap L^\infty(I; V)$ with $u(t) \in K_{\psi(t)}, u(0) = u_s \in V$, and $\dot{u}(0) = v_s \in W$ for which

$$(\rho\ddot{u} + a_d\rho\dot{u}, \dot{\varphi} - \dot{u}) + (\sigma(u + b_d\dot{u}), \varepsilon(\dot{\varphi} - \dot{u})) + j(\dot{\varphi}) - j(\dot{u}) \ge (f, \dot{\varphi} - \dot{u})$$

holds for all $\dot{\varphi} \in K_{\psi(t)}$ and all $t \in I$.

Note that the existence of a solution in \mathcal{V} can not be proven, even in the contact free case, see [18], Section 7.2. A detailed derivation of the weak formulation can be found in [43], Section 6.4, and [10, 12].

3.2. Discretization

We apply Rothe's method to obtain a discretization of the dynamic Signorini problem given in Problem 3.1. The problem is discretized in temporal direction by the Newmark method, see [42]. The resulting spatial problems are discretized by finite elements. Note that the classical Newmark method does not lead to a stable discretization scheme, see, e.g., [38]. Here, we apply a stabilization approach originally presented in [12, 37] and improved in [34]. For this scheme, it is known that the Lagrange multipliers are stable in time and that it is slightly energy disspative.

Temporal discretization. The time interval *I* is split into *M* equidistant subintervals $I_m := (t_{m-1}, t_m]$ of length $k = t_m - t_{m-1}$ with $0 =: t_0 < t_1 < ... < t_{M-1} < t_M := T$. The value of a function *w* at a time instance t_m is approximated by w^m . We use the notation v = u for the velocity. The semi-discrete problem then reads as follows:

Problem 3.2. Find u_{pred} , u, and v with $u^0 = u_s$ and $v^0 = v_s$, such that in every time step $m \in \{1, 2, ..., M\}$, the functions $u_{\text{pred}}^m \in K_{\psi^m}$, $u^m \in K_{\psi^m}$, and $v^m \in W$ are the solution of

$$b\left(u_{\text{pred}}^{m}, \chi - u_{\text{pred}}^{m}\right) \geq b\left(u^{m-1} + kv^{m-1}, \chi - u_{\text{pred}}^{m}\right), \tag{3.7}$$

$$c\left(u^{m}, \varphi - u^{m}\right) + j\left(\varphi\right) - j\left(\Delta u^{m}\right) \geq \langle l^{m}, \varphi - u^{m}\rangle, \tag{3.8}$$

$$b(v^{m},\omega) = \frac{1}{k}b(2u^{m}-u^{m-1}-u^{m}_{\text{pred}},\omega).$$
(3.9)

for all $\chi, \varphi \in K_{\psi^m}$ and all $\omega \in W$, where $\Delta u^m := k^{-1} (u^m - u^{m-1})$,

$$\begin{split} b\left(\omega,\varphi\right) &:= \left(\rho\omega,\varphi\right), \\ c\left(\omega,\varphi\right) &:= \left(1+\frac{1}{4}ka_d\right)b\left(\omega,\varphi\right) + \frac{1}{4}k\left(k+b_d\right)\left(\sigma(\omega),\varepsilon(\varphi)\right) \end{split}$$

and

$$\begin{split} \langle l^m, \varphi \rangle &:= \frac{1}{4} k^2 \left(f^m + f^{m-1}, \varphi \right) + b \left(u^m_{\text{pred}}, \varphi \right) - \frac{1}{4} k \left(k - b_d \right) \left(\sigma \left(u^{m-1} \right), \varepsilon(\varphi) \right) + \frac{1}{4} k a_d b \left(u^{m-1}, \varphi \right) \\ &- \frac{1}{4} k^2 a_d b \left(v^{m-1}, \varphi \right) - \frac{1}{4} k^2 b_d \left(\sigma \left(v^{m-1} \right), \varepsilon(\varphi) \right). \end{split}$$

The set $K_{\psi^m} := \left\{ \varphi \in V \mid \gamma_n(\varphi) \le g_{\psi^m} \right\}$ contains the admissible displacements at time *m*. In the classical Newmark scheme, the predictor $u_{\text{pred}}^m = u^{m-1} + kv^{m-1}$ is used. Here, inequality (3.7) represents the projection of the predictor step of the Newmark scheme on the set of admissible displacements K_{ψ^m} , which ensures the stability of the scheme.

The bilinearforms b and c are uniformly elliptic, continuous, and symmetric. Thus, in each time step the variational inequalities (3.7) and (3.8) directly match the general framework presented in Section 2.2. Using the equivalent mixed problem formulation, the mixed semi-discrete problem reads

Problem 3.3. Find $(u_{\text{pred}}, u, v, \lambda_{\text{pred}}, \lambda_n, \lambda_t)$ with $u^0 = u_s$ and $v^0 = v_s$, such that $(u_{\text{pred}}^m, u^m, v^m, \lambda_{\text{pred}}^m, \lambda_n^m, \lambda_t^m) \in V \times V \times W \times \Lambda_n \times \Lambda_n \times \Lambda_n \times \Lambda_t$, so the solution of the system

$$b\left(u_{\text{pred}}^{m},\chi\right) + \left\langle\lambda_{\text{pred}}^{m},\chi\right\rangle = b\left(u^{m-1} + kv^{m-1},\chi\right),\tag{3.10}$$

$$\left\langle \mu_{\text{pred}} - \lambda_{\text{pred}}^{m}, \gamma_{n} \left(u_{\text{pred}}^{m} \right) - g_{\psi^{m}} \right\rangle \leq 0, \qquad (3.11)$$

$$C(u^{n},\varphi) + \langle \lambda_{n}^{n},\varphi \rangle + \langle \lambda_{t}^{n},s^{n}\gamma_{t}(\varphi) \rangle_{0,\Gamma_{C}} = \langle l^{n},\varphi \rangle, \qquad (3.12)$$

$$\left\langle \mu_n - \lambda_n^m, \gamma_n \left(u^m \right) - g_{\psi^m} \right\rangle + \left(\mu_t - \lambda_t^m, s^m \gamma_t \left(\Delta u^m \right) \right)_{0, \Gamma_C} \leq 0, \tag{3.13}$$

$$b(v^{m},\omega) = \frac{1}{k}b(2u^{m}-u^{m-1}-u^{m}_{\text{pred}},\omega).$$
(3.14)

for all $\chi, \varphi \in V$, all $\mu_{\text{pred}}, \mu_n \in \Lambda_n$, all $\mu_t \in \Lambda_{t,s^m}$, all $\omega \in W$, and all $m \in \{1, 2, \dots, M\}$.

Remark 3.4. The existence and uniqueness of the semi-discrete solution directly follows from the results in Section 2.2.

Spatial discretization. Using the finite element approach of Section 2.3, one eventually obtains the full discrete problem

Problem 3.5. Find $(u_{\text{pred},h}, u_h, v_h, \lambda_{\text{pred},H}, \lambda_{n,H}, \lambda_{t,H})$ with $u_h^0 = \pi_h u_s$ and $v_h^0 = \pi_h v_s$, where π_h denotes the L^2 -projection onto V_h , such that $(u_{\text{pred},h}^m, u_h^m, v_h^m, \lambda_{\text{pred},H}^m, \lambda_{t,H}^m) \in V_h \times V_h \times V_h \times \Lambda_{n,H} \times \Lambda_{t,s^m,H}$ is the solution of the system

$$b\left(u_{\text{pred},h}^{m},\chi_{h}\right) + \left\langle\lambda_{\text{pred},H}^{m},\chi_{h}\right\rangle = b\left(u_{h}^{m-1} + kv_{h}^{m-1},\chi_{h}\right), \qquad (3.15)$$

$$\left\langle \mu_{\text{pred},H} - \lambda_{\text{pred},H}^{m}, \gamma_n \left(u_{\text{pred},h}^{m} \right) - g_{\psi^m} \right\rangle \leq 0, \qquad (3.16)$$

$$c\left(u_{h}^{m},\varphi_{h}\right)+\left\langle\lambda_{n,H}^{m},\varphi_{h}\right\rangle+\left(\lambda_{t,H}^{m},s^{m}\gamma_{t}\left(\varphi_{h}\right)\right)_{0,\Gamma_{C}} = \left\langle l_{h}^{m},\varphi_{h}\right\rangle,$$
(3.17)

$$\left\langle \mu_{n,H} - \lambda_{n,H}^{m}, \gamma_{n} \left(u_{h}^{m} \right) - g_{\psi^{m}} \right\rangle + \left(\mu_{t,H} - \lambda_{t,H}^{m}, s^{m} \gamma_{t} \left(\Delta u_{h}^{m} \right) \right)_{0,\Gamma_{C}} \leq 0, \qquad (3.18)$$

$$b(v_{h}^{m},\omega_{h}) = \frac{1}{k}b(2u_{h}^{m}-u_{h}^{m-1}-u_{\text{pred},h}^{m},\omega_{h}). \quad (3.19)$$

for all $\chi_h, \varphi_h, \omega_h \in V_h$, all $\mu_{\text{pred},H}, \mu_{n,H} \in \Lambda_{n,H}$, all $\mu_{t,H} \in \Lambda_{t,s^m,H}$, and all $m \in \{1, 2, \dots, M\}$.

Here, l_h^m is an approximation tp l^m and is defined by

$$\begin{split} \left\langle t_h^m, \varphi_h \right\rangle &:= \frac{1}{4} k^2 \left(f^m + f^{m-1}, \varphi_h \right) + b \left(u_{\text{pred},h}^m, \varphi_h \right) - \frac{1}{4} k \left(k - b_d \right) \left(\sigma \left(u_h^{m-1} \right), \varepsilon(\varphi_h) \right) + \frac{1}{4} k a_d b \left(u_h^{m-1}, \varphi_h \right) \\ &- \frac{1}{4} k^2 a_d b \left(v_h^{m-1}, \varphi_h \right) - \frac{1}{4} k^2 b_d \left(\sigma \left(v_h^{m-1} \right), \varepsilon(\varphi_h) \right). \end{split}$$

Using the solution scheme as introduced in Section 2.4, we solve the system (3.15-3.16) and then the mixed problem (3.17-3.18). Note that, the equation (3.19) reduces to a linear combination of vectors, if no adaptivity in space is used.

We obtain the existence of a unique discrete solution $(u_{\text{pred},h}, u_h, v_h, \lambda_{\text{pred},H}, \lambda_{n,H}, \lambda_{t,H})$ of Problem 3.5 provided that the discrete inf-sup condition (2.9) holds, see Section 2.3. To this end, we have to ensure that the number $\Pi(h, H, p, q)$ is sufficiently small. In the next section, we discuss several variants to reduce $\Pi(h, H, p, q)$ appropriately.

3.3. Numerical Examples

In this section, we investigate the numerical properties of the presented discretization method of higher-order for dynamic contact problems with friction and damping. We begin with a simple example, where the solution is known so that the numerical convergence rates and the stability of the Lagrange multiplier in time can be studied. The stability in space and time is afterwards explored considering a more complex example. The section concludes with the discussion of a complex frictional contact problem.



Figure 1: Analytic stress distribution in Ω for different time instances

Stability in time and convergence analysis. To analyze the stability in time, we consider the following example, which is a 2d version of an example given in [15]: The domain is $\Omega := [-h_0 - L, -h_0] \times [0, 2]$, $h_0 := 5$, L := 10, and the time interval is I := [0, 2]. We choose E := 900, v := 0, and $\rho \equiv 1$. The possible contact boundary is given by $\Gamma_C := \{-5\} \times [0, 2]$ and we set $\Gamma_D := \emptyset$ as well as $\Gamma_N := \partial \Omega \setminus \Gamma_C$. The initial conditions are $u_s \equiv 0$ and $v_s \equiv (v_0, 0)^{\top}$, $v_0 := 10$. The rigid foundation is given by $\psi \equiv 0$. We consider no friction, i.e. s = 0. From the specific velocity $c_0 = \sqrt{E/\rho} = 30$, we can determine the time $\tau_w = v_0/c_0 = 1/3$, which means that the contact lasts from $t_1 = 5/10 = 0.5$ to $t_2 = t_1 + 2\tau_w = 7/6$. With these values we can state the analytical solution of this problem: It holds for the displacement $u := (u_1, 0)$ with

$$u_1(x_1, x_2, t) := \begin{cases} v_0 t, & t \le t_1, \\ h_0 + v_0 \min\left\{-\frac{h_0 + x_1}{c_0}, \tau_w - |t - t_1 - \tau_w|\right\}, & t_1 < t \le t_2, \\ h_0 - v_0 (t - t_2), & t_2 < t, \end{cases}$$

and for the velocity $v := (v_1, 0)$ with

$$v_{1}(x_{1}, x_{2}, t) := \begin{cases} v_{0}, & t \leq t_{1}, \\ 0, & t_{1} < t \leq t_{2}, -\frac{h_{0} + x_{1}}{c_{0}} \leq \tau_{w} - |t - t_{1} - \tau_{w}|, \\ -v_{0} \operatorname{sign}(t - t_{1} - \tau_{w}), & t_{1} < t \leq t_{2}, -\frac{h_{0} + x_{1}}{c_{0}} > \tau_{w} - |t - t_{1} - \tau_{w}|, \\ -v_{0}, & t_{2} < t, \end{cases}$$

as well as for the normal contact stress

$$\sigma_{nn}(x_2, t) = \lambda(x_2, t) = \begin{cases} 0, & t < t_1, t > t_2 \\ -\frac{Ev_0}{c_0} = -300, & \text{else.} \end{cases}$$
(3.20)

In Figure 1, the stress distribution in Ω is depicted for different time instances. It shows the propagation of the stress wave emerging from the contact through the elastic body.

First, we consider the stability of the Lagrange multiplier in time. As it is widely known, cf. e.g. [12, 22], the use of the classical Newmark scheme leads to an instable Lagrange multiplier in time, cf. Figure 2(a). To circumvent this behavior, we have added the L^2 -projection onto the admissible set specified in (3.10-3.11) in the time stepping scheme described in Problem 3.3. For different values of the polynomial degree p, the Lagrange multiplier is plotted in Figure 2 (b-d). For all polynomial degrees p, we only observe small oscillations especially in the time steps directly after the first contact. Comparing the results in Figure 2 (b-d) with the known solution (3.20), we notice that the overshoot of the discrete Lagrange multiplier decreases with higher polynomial degrees significantly.

It is well known that the total energy is conserved, c.f. [38]. Consequently, we address the question, how this property is reproduced by the numerical scheme. The time stepping scheme specified in Problem 3.5 is energy



Figure 2: Plot of the lagrange multiplier $\lambda_{n,kH}$ for different p, k = 0.01, and h = 0.25



Figure 3: Convergence rate w.r.t. different variables and norms



Figure 4: Plot of the numerical solution for p = 3, h = 0.03125, and k = 0.00025

dissipative and the energy loss occurs in the time step directly after the first contact, cf. [34, 37]. These theoretical findings are approved in Figure 3(a) for high polynomial degrees. The amount of energy, which is lost, depends in general on the polynomial degree p, the mesh width h, and the time step length k. Due to the special structure of this example it does not depend on k in this case. The dependence on p and h is illustrated in Figure 3(b). We observe that the lost energy decreases with growing p and that it converges of order O(h) to zero for all polynomial degrees p.

Using the known solution of this example, we are able to determine the discretization error and convergence rates. In Figure 3(c), the convergence rate of the presented scheme w.r.t. *k* is depicted for fixed *h* and *p*. The convergence rate $O(k^2)$ in the unconstrained case is reduced to $O(k^{0.77})$ due to the presence of the contact conditions. In [34], the consistency order of $O(k^{0.5})$ is proven for viscoelastic problems, which is too pessimistic for this example.

Results concerning the spatial convergence rate are illustrated in Figure 3 (d-f) for different polynomial degrees p and different norms using a fixed time step length k. In Figure 3(d), the convergence w.r.t. the L^2 -norm is depicted. In the unconstrained case, we expect a convergence rate of $O(h^{p+1})$, if u is smooth. In contact problems, we cannot expect that u possesses high regularity due to the jumps in the stress distribution. Indeed, we notice the convergence rate $O(h^{0.69})$. The convergence rate in the H^1 -norm also decreases from $O(h^p)$ in the unconstrained case to $O(h^{0.33})$, c.f. Figure 3(e). For the L^{∞} -norm, we find a reduction from $O(h^2 \log h)$, p = 1, respectively $O(h^{p+1})$, p > 1, to $O(h^{0.71})$, see Figure 3(f). The reduced convergence rates are the same for all polynomial degrees p. However, the absolute error decreases for higher polynomial degrees.

Stability in space. To investigate the stability in space, we consider the following example: The domain Ω is given by $\Omega = [-2, 0] \times [0, 2]$ and the time interval by I = [0, 0.1]. The contact boundary is $\Gamma_C = \{0\} \times [0, 2]$. We prescribe inhomogeneous Neumann boundary conditions

$$q_n(x,t) := \begin{cases} \left(-10x_2^2 + 20x_2, 0\right)^\top, & x = (x_1, x_2) \in \Gamma_{N_1}, \ t < 0.01\\ 0, & \text{else}, \end{cases}$$
(3.21)

on $\Gamma_N = \partial \Omega \setminus \Gamma_C = \Gamma_{N_1} \cup \Gamma_{N_2}$ with $\Gamma_{N_1} := \{-2\} \times [0.875, 1.125]$ and $\Gamma_{N_2} = \Gamma_N \setminus \Gamma_{N_1}$. As above, we choose E = 900, $\nu = 0$, $\rho \equiv 1$, and $\psi \equiv 0$. A numerical solution for this example is depicted in Figure 4. We observe a stress wave, which emerges from the inhomogeneous Neumann boundary conditions on Γ_{N_1} , hits the obstacle in the middle of Γ_C , goes towards the boundary, and is eventually reflected.



Figure 5: Plot of the Lagrange multiplier for p = 3, h = 0.03125, k = 0.00025

p $M = 400, M_{\mathcal{T}} = 2$	Energy loss 960	M $p = 1, M_{\mathcal{T}} = 153$	Energy loss 60	$M_{\mathcal{T}}$ p = 1, M = 1600	Energy loss
1	3.114%	100	3.125 %	960	3.061 %
2	2.600 %	200	2.713 %	3840	2.589 %
3	2.469 %	400	2.501 %	15360	2.362 %
4	2.428 %	800	2.404 %		
5	2.405 %	1600	2.362 %		
6	2.393 %				

Table 1: Relative energy loss in % due to the numerical stabilization w.r.t. p, M, and M_T

In this section, we exemplarily consider the case p = 3 and how to reduce the value $\Pi(h, H, p, q)$ appropriately. For other polynomial degrees, we obtain similar results. In Figure 5(a), the Lagrange multiplier is depicted for H = 2h and q = 2. We observe a stable behavior in space and time, i.e. we do not notice any oscillations. For a single time step, the Lagrange multiplier is illustrated in Figure 5(b). Another stable discretization which avoids mesh coarsening for the Lagrange multiplier is achieved by the choice H = h and q = 1, see Figure 5(c). Finally, we choose q = 2 and H = h and obtain an unstable discretization, i.e. we observe strong oscillations in the Lagrange multiplier, c.f. Figure 5(d).

Contact with friction. In the following, we consider an example with friction. The domain is $\Omega = [-4, -1] \times [-6, -1]$, $\Gamma_D = \emptyset$, $\Gamma_C = [-4, -1] \times -1$, and I = [0, 0.4]. We assume homogeneous Neumann boundary conditions on $\Gamma_N = \partial \Omega \setminus \Gamma_C$. The obstacle is parametrized by the function $\psi(x_2) := -0.00625 (x_2^2 + 5x_2 + 6.25)$. We set the material properties to E = 900, $\nu = 0.3$, and $\rho \equiv 1$. The initial conditions are $u_s \equiv 0$ and $v_s \equiv (10, 5)^{T}$. We consider Coulomb friction with $\mathcal{F} = 0.05$, cf. Remark 2.6. In Figure 6, the numerical solution is illustrated. The stable Lagrange multiplier is plotted in Figure 7(a).

It is well known that energy dissipation is caused by friction. In Figure 7(b), we compare the amount of energy, which is lost due to friction, with the energy loss because of the numerical stabilization. Even with this low coefficient of friction, more energy is dissipated due to friction than to numerical reasons. While the energy loss due to friction converges to a fixed value and does not largely change varying p, h, and k, the amount of energy lost because of the



Figure 6: Plot of the von Mises equivalent stress distribution for p = 4, M = 400, $M_T = 15360$, and $M_E = 80$



Figure 7: Plot of the Lagrange multiplier and the relative energy loss for p = 4, M = 400, $M_T = 15360$, and $M_E = 80$

numerical stabilization strongly depends on the discretization and converges to zero. In Table 1, the development of the energy loss due to the numerical stabilization is described. We observe that for fixed *h* and *k* the energy loss decreases with increasing *p*. For fixed *p* and *h*, a decrease with an increasing number of time steps *M* is noticed. The same holds w.r.t. *h*. The results in Table 1 are exemplarily chosen, similar behavior is found for different fixed parameters. This results substantiate the conclusion of the first example that the use of higher polynomial degrees *p* reduces the loss of energy as a result of the numerical stabilization, which converges towards zero for $h, k \rightarrow 0$.

4. Dynamic thermomechanical contact problems

A portion of the energy, dissipated by frictional effects, generates heat, which is induced into the bodies in contact. In this section, we introduce a model for this physical process and extend the discretization techniques presented in the previous sections.

4.1. Continuous Problem Formulation

In the following, we introduce the strong formulation for thermomechanical contact and give a short overview of the linear theory of thermoelasticity, which describes the effects of heat on an elastic body. A detailed presentation of thermoelasticity may be found in [7]. Furthermore, we discuss the coupling of friction and heat.

We extend the linear elastic model described in Section 2 and 3.1 by thermal effects. The heat of the body is given by the function $\theta \in V_{\theta} := \left\{ \varphi \in L^2(I; H_D^1(\Omega)) | \dot{\varphi} \in L^2(I; H^{-1}(\Omega)) \right\}$. For notational simplicity, we assume that homogeneous Dirichlet boundary conditions hold on Γ_D for the heat distribution. Inhomogeneous Neumann boundary conditions are prescribed on Γ_C given by the function $\varpi \in L^2(I; L^2(\Gamma_C))$ and homogeneous Neumann boundary conditions on Γ_N . Inner heat sources are described by $f_{\theta} \in L^2(I; L^2(\Omega))$. The initial temperature is $\theta_s \in H_D^1(\Omega)$. The specific heat is given by the constant ζ and the constant κ denotes the conductivity. Eventually, the heat equation reads:

Problem 4.1. Find a function $\theta \in V_{\theta}$ with $\theta(0) = \theta_s$, which fulfills the variational equation

$$\left\langle \zeta \dot{\theta}, \varphi \right\rangle + (\kappa \nabla \theta, \nabla \varphi) = (f_{\theta}, \varphi) + (\varpi, \varphi)_{\Gamma_{C}}$$

$$\tag{4.1}$$

for all $\varphi \in H^1_D(\Omega)$ and all $t \in I$.

Heat and displacement are usually connected by the coefficient of thermal expansion α . For a more convenient description, we use the stress-temperature modulus ρ , which is defined as

$$\varrho = -\frac{\alpha E}{1+\nu} \left(\frac{3\nu}{1-2\nu} + 1 \right).$$

As a consequence of the heating, thermal stresses occur in the elastic body. They are specified by the thermal stress tensor

$$\sigma_{\theta}(\theta)_{ij} := \varrho \left(\theta - \theta_s\right) \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

Due to the elastic deformation, an additional heat source specified by the term $\rho \theta_s \text{tr} \dot{\epsilon}(u)$ has to be included in the heat equation.

During the frictional contact energy is dissipated, which is transfered into heat mostly. This energy is described by $\int_I \int_{\Gamma_c} \sigma_{nt} \gamma_t(\dot{u}) \, dx \, dt$. The heat is transferred into the elastic body, the obstacle and the environment. We assume that a fixed portion of the generated energy enters the elastic body, where the proportionality factor is denoted by $K_W \in [0, 1]$. Moreover, we neglect the heat transfer between the obstacle and the rigid foundation in the contact zone. The generated heat is incorporated in the heat equation as an inhomogeneous Neumann boundary condition on Γ_C , i.e. $\varpi = K_W \sigma_{nt} \gamma_t(\dot{u})$. Together with the formulation of the frictional contact problem in Section 3.1, the strong formulation of the thermoelastic contact problem is given by

$$\begin{split} \rho\ddot{u} - \operatorname{div}\left(\sigma(u) + \sigma_{\theta}(\theta)\right) &= f, \quad \zeta\dot{\theta} - \operatorname{div}\left(\kappa\nabla\theta\right) + \varrho\theta_{s}\operatorname{tr}\dot{\varepsilon}(u) = f_{\theta} \quad \text{in } \Omega \times I, \\ u &= \theta = 0 \text{ on } \Gamma_{D} \times I, \quad \sigma_{nn}(u) = \frac{\partial\theta}{\partial n} = 0 \text{ on } \Gamma_{N} \times I, \quad u(0) = u_{s}, \ \dot{u}(0) = v_{s}, \ \theta(0) = \theta_{s} \text{ in } \Omega, \\ \gamma_{n}(u) - g_{\psi} &\leq 0, \quad -\sigma_{nn}(u) \geq 0, \quad \sigma_{nn}(u) \left(\gamma_{n}(u) - g_{\psi}\right) = \sigma_{nn}(u) \left(\gamma_{n}(\dot{u}) - g_{\dot{\psi}}\right) = 0 \text{ on } \Gamma_{C} \times I \\ |\sigma_{nt}(u)| &\leq s \text{ with } \begin{cases} |\sigma_{nt}(u)| < s \Rightarrow \gamma_{t}(\dot{u}) = 0 \\ |\sigma_{nt}(u)| = s \Rightarrow \exists \zeta \in \mathbb{R}_{\geq 0} : \gamma_{t}(\dot{u}) = -\zeta\sigma_{nt} \end{cases} \text{ on } \Gamma_{C} \times I, \\ \frac{\partial\theta}{\partial n} - K_{W}\sigma_{nt}\gamma_{t}(\dot{u}) = 0 \text{ on } \Gamma_{C} \times I. \end{split}$$

We omit the Rayleigh damping here to ease the notation. Using the same arguments as in Section 3.1, the weak formulation reads:

Problem 4.2. Find a function $(u, \theta) \in \mathcal{K}_{\psi} \times V_{\theta}$ with $u(0) = u_s$, $\dot{u}(0) = v_s$, and $\theta(0) = \theta_s$, such that

$$(\rho\ddot{u},\dot{\varphi}-\dot{u}) + (\sigma(u) + \varrho(\theta - \theta_s)\mathbf{I}, \varepsilon(\dot{\varphi}-\dot{u})) + j(\dot{\varphi}) - j(\dot{u}) \geq (f,\dot{\varphi}-\dot{u}) \langle \zeta\dot{\theta},\chi \rangle + (\kappa\nabla\theta,\nabla\chi) + (\varrho\theta_s \mathrm{tr}\dot{\varepsilon}(u),\chi) - (K_W\sigma_{nt}\gamma_t(\dot{u}),\chi)_{\Gamma_c} = (f_\theta,\chi)$$

holds for all $\dot{\varphi} \in K_{\psi(t)}$, all $\chi \in V_{\theta}$ and $t \in I$.

4.2. Discretization

The discretization of the frictional part of the thermoelastic contact problem is carried out as described in Section 3.2, see Problem 3.3 and 3.5. Here, we focus on the discretization of the heat part and on the solution of the discrete system. For the temporal discretization of the heat equation, we use the Cranck-Nicholson scheme, see, e.g., [29]. The time discretization of equation (4.1) reads

$$c_{\theta}\left(\theta^{m},\varphi\right) = l_{\theta}^{m}\left(\varphi\right) + \frac{1}{2}k\left(\varpi^{m} + \varpi^{m-1},\varphi\right)_{\Gamma_{C}},\tag{4.2}$$

where

$$\begin{aligned} c_{\theta}\left(\omega,\varphi\right) &:= \left(\zeta\omega,\varphi\right) + \frac{1}{2}k\left(\kappa\nabla\omega,\nabla\varphi\right),\\ \left\langle l_{\theta}^{m},\varphi\right\rangle &:= \left(\zeta\theta^{m-1},\varphi\right) + \frac{1}{2}k\left(\kappa\nabla\theta^{m-1},\nabla\varphi\right) + \frac{1}{2}k\left(f_{\theta}^{m} + f_{\theta}^{m-1},\varphi\right) \end{aligned}$$

Using equation (4.2), Problem 3.3, and again the spatial discretization techniques presented in Section 2.3, we obtain the space and time discrete problem:

Problem 4.3. Find $(u_{\text{pred},h}, u_h, v_h, \theta_h, \lambda_{\text{pred},H}, \lambda_{n,H}, \lambda_{t,H})$ with $u_h^0 = \pi_h u_s$, $v_h^0 = \pi_h v_s$, and $\theta_h^0 = \tilde{\pi}_h \theta_s$ such that the function $(u_{\text{pred},h}^m, u_h^m, v_h^m, \theta_h^m, \lambda_{\text{pred},H}^m, \lambda_{t,H}^m) \in V_h \times V_h \times S_h \times \Lambda_{n,H} \times \Lambda_{n,H} \times \Lambda_{t,s^m,H}$ is the solution of the system

$$b\left(u_{\text{pred},h}^{m},\chi_{h}\right) + \left\langle\lambda_{\text{pred},H}^{m},\chi_{h}\right\rangle = b\left(u_{h}^{m-1} + kv_{h}^{m-1},\chi_{h}\right), \tag{4.3}$$

$$\left\langle \mu_{\text{pred},H} - \lambda_{\text{pred},H}^{m}, \gamma_{n} \left(u_{\text{pred},h}^{m} \right) - g_{\psi^{m}} \right\rangle \leq 0, \tag{4.4}$$

$$\bar{z} \left(u^{m}, \varrho^{m}, \varphi_{n} \right) + \left(\lambda^{m}, \varphi_{n}^{m} \right) + \left(\lambda^{m}, \varphi_{n}^{m}, \varphi_{n}^{m} \right) = - \left(\bar{\ell}_{n}^{m}, \varphi_{n}^{m} \right) = 0, \tag{4.4}$$

$$\left\langle \mu_{n,H} - \lambda_{n,H}^{m}, \gamma_{n}\left(u_{h}^{m}\right) - g_{\psi^{m}} \right\rangle + \left\langle \mu_{t,H} - \lambda_{t,H}^{m}, s^{m}\gamma_{t}\left(\Delta u_{h}^{m}\right) \right\rangle_{0,\Gamma_{C}} \leq 0,$$

$$\left\langle \mu_{n,H} - \lambda_{n,H}^{m}, \gamma_{n}\left(u_{h}^{m}\right) - g_{\psi^{m}} \right\rangle + \left\langle \mu_{t,H} - \lambda_{t,H}^{m}, s^{m}\gamma_{t}\left(\Delta u_{h}^{m}\right) \right\rangle_{0,\Gamma_{C}} \leq 0,$$

$$(4.6)$$

$$\bar{c}_{\theta}\left(\theta_{h}^{m}, u_{h}^{m}, \vartheta_{h}\right) + \left\langle \alpha_{h}^{m}\left(u_{h}^{m}, \lambda_{t,H}^{m}\right), \vartheta_{h} \right\rangle = \left\langle \bar{l}_{\theta,h}^{m}, \vartheta_{h} \right\rangle,$$

$$(4.7)$$

$$b(v_{h}^{m},\omega_{h}) = \frac{1}{k}b(2u_{h}^{m}-u_{h}^{m-1}-u_{\text{pred},h}^{m},\omega_{h}).$$
(4.8)

for all $\chi_h, \varphi_h, \omega_h \in V_h$, all $\vartheta_h \in S_h$, all $\mu_{\text{pred},H}, \mu_{n,H} \in \Lambda_{n,H}$, all $\mu_{t,H} \in \Lambda_{t,s^m,H}$, and all $m \in \{1, 2, \dots, M\}$.

Here, we set

$$\begin{split} \bar{c}\left(\omega,\chi,\varphi\right) &:= \left(\rho\omega,\varphi\right) + \frac{1}{4}k^{2}\left(\sigma(\omega) + \sigma_{\theta}\left(\chi\right),\varepsilon(\varphi)\right), \\ \bar{c}_{\theta}\left(\omega,\chi,\varphi\right) &:= \left(\zeta\omega,\varphi\right) + \frac{1}{2}k\left(\kappa\nabla\omega,\nabla\varphi\right) + \frac{1}{2}k\left(\varrho\theta_{s}\mathrm{tr}\dot{\varepsilon}(\chi),\varphi\right), \\ \left\langle\bar{l}_{h}^{m},\varphi_{h}\right\rangle &:= \left\langle l_{h}^{m},\varphi_{h}\right\rangle - \frac{1}{2}k^{2}\left(\sigma_{\theta}\left(\theta_{h}^{m-1}\right),\varepsilon(\varphi_{h})\right), \\ \left\langle\alpha_{h}^{m}\left(u_{h}^{m},\lambda_{t,H}^{m}\right),\vartheta_{h}\right\rangle &:= \frac{1}{2}K_{W}k\left(s^{m}\lambda_{t,H}^{m}\gamma_{t}\left(\Delta u_{h}^{m}\right) + s^{m-1}\lambda_{t,H}^{m-1}\gamma_{t}\left(\Delta u_{h}^{m-1}\right),\vartheta_{h}\right)_{\Gamma_{c}}, \\ \left\langle l_{\theta,h}^{m},\vartheta_{h}\right\rangle &:= \left(\zeta\theta_{h}^{m-1},\vartheta_{h}\right) + \frac{1}{2}k\left(\kappa\nabla\theta_{h}^{m-1},\nabla\vartheta_{h}\right) + \frac{1}{2}k\left(f_{\theta}^{m} + f_{\theta}^{m-1},\vartheta_{h}\right), \\ \left\langle\bar{l}_{\theta,h}^{m},\vartheta_{h}\right\rangle &:= \left\langle l_{\theta,h}^{m},\vartheta_{h}\right\rangle - \frac{1}{2}k\left(\varrho\theta_{s}\mathrm{tr}\dot{\varepsilon}\left(u_{h}^{m-1}\right),\vartheta_{h}\right). \end{split}$$

In every time step m of the scheme specified in Problem 4.3, an highly coupled discrete problem has to be solved involving several iterative procedures. In the case of Coulomb friction, we present an iterative scheme on the basis of the procedure described in Remark 2.6.

Algorithm 4.4. *Given a stopping tolerance* tol > 0*:*

- (1) Determine the solution $u_{\text{pred},h}^m \in V_h$, $\lambda_{\text{pred},H}^m \in \Lambda_{n,H}$ of (4.3-4.4) (see Section 2.4).
- (2) Set $\lambda_{t,H}^{m,0} = \lambda_{t,H}^{m-1}$, $\lambda_{n,H}^{m,0} = \lambda_{n,H}^{m-1}$, $u_h^{m,0} = u_h^{m-1}$, i = 1.
- (3) Determine $\theta_h^{m,i} \in S_h$, such that for all $\vartheta_h \in S_h$

$$\bar{c}_{\theta}\left(\theta_{h}^{m,i}, u_{h}^{m,i-1}, \vartheta_{h}\right) + \left\langle \alpha_{h}^{m}\left(u_{h}^{m,i-1}, \lambda_{t,H}^{m,i-1}\right), \vartheta_{h} \right\rangle = \left\langle \bar{l}_{\theta,h}^{m}, \vartheta_{h} \right\rangle.$$

(4) Set $s^{m,i}(x) = \mathcal{F} \left| \lambda_{n,H}^{m,i-1}(x) \right|$.

(5) Determine the solution $u_h^{m,i} \in V_h$, $\lambda_{n,H}^{m,i} \in \Lambda_{n,H}$, $\lambda_{t,H}^{m,i} \in \Lambda_{t,s^{m,i},H}$ of

$$\bar{c} \left(u_h^{m,i}, \theta_h^{m,i}, \varphi_h \right) + \left\langle \lambda_{n,H}^{m,i}, \varphi_h \right\rangle + \left(\lambda_{t,H}^{m,i}, s^m \gamma_t \left(\varphi_h \right) \right)_{0,\Gamma_C} = \left\langle \bar{l}_h^m, \varphi_h \right\rangle + \left\langle \mu_{n,H} - \lambda_{n,H}^{m,i}, \gamma_n \left(u_h^{m,i} \right) - g_{\psi^m} \right\rangle + \left(\mu_{t,H} - \lambda_{t,H}^{m,i}, s^{m,i} \gamma_t \left(\Delta u_h^{m,i} \right) \right)_{0,\Gamma_C} \leq 0,$$

for all $\varphi_h \in V_h$, all $\mu_{n,H} \in \Lambda_{n,H}$, all $\mu_{t,H} \in \Lambda_{t,s^{m,i},H}$ (see Section 2.4).

(6) If

$$\max\left\{\left|u_{h}^{m,i}-u_{h}^{m,i-1}\right|,\left|\theta_{h}^{m,i}-\theta_{h}^{m,i-1}\right|,\left|\lambda_{n,H}^{m,i}-\lambda_{n,H}^{m,i-1}\right|,\left|\lambda_{t,H}^{m,i}-\lambda_{t,H}^{m,i-1}\right|\right\} > \text{tol},$$

set $i \leftarrow i + 1$ and go to (3).

(7) Determine $v_h^m \in V_h$ as solution of (4.8).

4.3. Numerical Example

Here, we consider the frictional example from Section 3.3 again. The additional material parameters are $\zeta \equiv 1$, $\kappa \equiv 5$, and $\alpha = 0.002$. Furthermore, we set $K_W = 1$ and assume homogeneous Neumann boundary conditions on Γ_N for the heat equation. This means, the body is perfectly isolated and the friction is the only heat source. Consequently, we expect the thermal energy to be equal to the energy dissipated by friction. In Figure 8, the temperature distribution in the body is depicted for different time steps. The stability of the Lagrange multiplier is illustrated in Figure 9(a). As a result of the thermal stresses, we obtain larger values for the Lagrange multiplier as in the case without thermal effects shown in Figure 7(a). We compare the thermal energy with the energy dissipated by friction in Figure 9(b). In theory, the energies has to be equal, which we observe in the discrete case as well. It should be remarked that the energy loss due to numerical stabilization shows the same behavior as described in the pure frictional case, cf. Section 3.3.



Figure 8: Temperature distribution for p = 4, $M_T = 3840$, and M = 400.



Figure 9: Plot of the Lagrange multiplier and comparison of the energy dissipated due to friction and thermal energy for p = 4, M = 400, $M_T = 3840$



Figure 10: Geometry of the spindle grinding wheel system and of the workpiece

5. An example from production engineering

To show the applicability of the proposed discretization schemes, we discuss a realistic process in production engineering: The NC-shape grinding process of free formed surfaces with a toroid grinding wheel. A subproblem in the simulation of the grinding process is to simulate dynamic thermomechanical contact with Rayleigh damping. A detailed survey of the engineering process and its simulation is given in [51]. Here, we extend the model by frictional and thermal effects. The simulation of such thermal effects enables the prediction of workpiece errors, for instance grinding burn.

The main difficulty in the realistic simulation of the grinding process is the rotation of the grinding wheel. The contact situation changes in every time step and the possible contact boundary is large. Furthermore, we need to work with an adaptively refined contact boundary to resolve the contact conditions accurately. To avoid remeshing in every time step, we use an arbitrary Lagrangian Eulerian (ALE) approach, which means that the mesh is fixed and the material is rotated through the mesh. The approach is decribed in detail, for instance, in [41] and [53].

The grinding wheel and the spindle are explicitly represented in the finite element analysis. The stiffness of the other parts of the grinding machine is included via elastic bearings. The geometry of the spindle grinding wheel system is depicted in Figure 10(a). The length of the spindle is 658 mm, the radius of the grinding wheel is 100 mm, and the radius of the torus is 4.2 mm. This values show the different length scales, which occur in this problem. In particular, the depth of cut is in the range of 0.05 mm to 0.5 mm, which, indeed, requires the application of methods with high accuracy as higher-order finite elements. The mesh consisting of 27984 cells is shown in Figure 10(a). Homogeneous Dirichlet boundary conditions are assumed on the surface of the bearings. Furthermore, all initial functions are set to zero. The moduli of elasticity are $E_1 = 2.1 \cdot 10^{11} \frac{\text{kg}}{\text{ms}^2}$ for the grinding wheel, and $E_3 = 10^9 \frac{\text{kg}}{\text{ms}^2}$ for the spindle and for the grinding wheel receiver, $E_2 = 2.1 \cdot 10^{13} \frac{\text{kg}}{\text{ms}^2}$ for the grinding wheel, and $E_3 = 10^9 \frac{\text{kg}}{\text{ms}^2}$ for the bearings. The other material parameters are constant throughout the domain and are set to $\nu = 0.29$, $\rho = 7.85 \frac{\text{kg}}{\text{dm}^3}$, $\alpha = 10.8 \cdot 10^{-6} \text{ K}^{-1}$, $\kappa = 16.7 \frac{\text{kg} \text{m}}{\text{Ks}^3}$, $\zeta = 450 \frac{\text{m}^2}{\text{s}^2 \text{K}}$, $a_d = 0.075$, and $b_d = 0$. The coefficient of friction is chosen as $\mathcal{F} = 0.3$ and the heat distribution coefficient as $K_W = 0.05$. Furthermore, we assume homogeneous Neumann boundary conditions. The rotational speed of the grinding wheel is $\omega = 170\pi \text{ s}^{-1}$. We select T = 0.02 s and $k = 10^{-5} \text{ s}$. The geometry of the workpiece, which has a free formed sinusoidal profile, is shown in Figure 10(b). The vertical and horizontal infeed is set to 0.5 mm. For the discretization of the displacement and the temperature, trilinear basis functions are used. The discrete Lagrange multipliers are based on piecewise constant shape functions with mesh size H = 2h.

Figure 11 shows the displacement in the center of the grinding wheel orthogonal to the plane, in which the workpiece lies. The sinusoidal profile of the workpiece is represented in the displacement of the grinding wheel, as expected. The heat distribution in the contact zone between grinding wheel and workpiece is depicted in Figure 12 for different time steps. The heat diffuses mainly in the direction of the rotation. Furthermore, the location of the highest



Figure 11: Displacement in the center of the grinding wheel orthogonal to the workpiece-plane



Figure 12: Heat distribution in the contact zone

temperature moves according to the contact zone. The value of the heat inflow depends on the tangential stress and consequently on the normal stress due to the friction. This dependence is also observed in the heat distribution.

6. Conclusions and outlook

In this paper, we have presented a discretization scheme for dynamic contact problems including damping, frictional, thermal, as well as rotational effects using higher-order finite element methods in space. We show the applicability of the mixed schemes, in particular, yielding stable Lagrange multipliers. The application of higher-order schemes is advantageous in the sense of higher accuracy, better numerical stabilization and possibly avoiding locking effects. However, we do not obtain the optimal exponential convergence rates because of the low regularity of the solution. To recover this, adaptive methods as presented, e.g., in [6] have to be applied, where h-adaptive methods for time-dependent contact problems are introduced. The extension to hp-adaptivity is, however, an open task. Beside higher-order methods in space, adaptive methods are needed in time as well. For the time discretization scheme used in this article, an approach for adaptive time stepping has been introduced in [34]. The extension of the time stepping schemes to higher-order is also an open question.

In this work, the contact between an elastic body and a rigid obstacle is considered. Currently, the discretization scheme is carried over to contact problems including nonlinear material laws and nonlinear as well as multibody contact conditions.

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