Simple Nonconforming Quadrilateral Stokes Element

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A simple nonconforming quadrilateral Stokes element based on "rotated" multi-linear shape functions is analyzed. On strongly nonuniform meshes the usual *parametric* version of this element suffers from a lack of consistency, while its *nonparametric* counterpart turns out to be convergent with optimal orders. This theoretical result is confirmed by numerical tests.

INTRODUCTION

Nonconforming finite elements are attractive for discretizing the Stokes as well as the Navier-Stokes problem since they possess favorable stability properties and divergence-free nodal bases are easily constructed [1, 2]. This allows the elimination of the pressure variables, leading to positive definite systems for the velocity variables alone, which may be efficiently solved by preconditioned conjugate gradient methods [3], or by multigrid techniques [2, 4]. While the convergence properties of the *triangular* nonconforming elements are well studied in the literature (see, e.g., Ref. 5) the analysis of their *quadrilateral* counterparts is less complete. A low-order rectangular element with 5 local degrees of freedom was introduced and analyzed by Han [6], but no numerical tests were reported. This paper deals with another nonconforming Stokes element which is based on "rotated" multilinear shape functions and, due to its very simple structure, appears particularly attractive from the computational point of view. It turns out that the *parametric* version of this element, i.e., that which is defined via transformations to a reference configuration, works well only for certain types of weakly uniform meshes. However, with a *nonparametric* Ansatz a generally convergent method is obtained. These phenomena can be theoretically explained and have been verified by test calculations. For an extensive comparison of this new element with other low-order Stokes elements see Ref. 7, and for a multigrid implementation of its divergence-free form see Ref. 2, and also a forthcoming report.

I. THE STOKES PROBLEM

We consider the usual linear Stokes problem

$$-\Delta u + \nabla p = f, \qquad \nabla \cdot u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where the pair $\{u,p\}$ represents the velocity and the pressure of a viscous incompressible flow in a bounded region $\Omega \subset \mathbf{R}^n$, $n = 2$ or $n = 3$. For simplicity, we assume Ω to be convex polygonal respectively polyhedral. The inner product and norm in the Lebesgue space $L^2 \equiv L^2(\Omega)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, and the usual norm in the Sobolev space $H^m \equiv H^m(\Omega)$ by $\|\cdot\|_m$. Furthermore, H_0^1 is the completion in H^1 of the space of test functions $C_0^\infty(\Omega)$, and H^{-1} is its dual space. By L_0^2 we denote the subspace of all L^2-functions over Ω having mean value zero. These are all spaces of $\mathbf{R}$-valued functions. Spaces of $\mathbf{R}^n$-valued functions are denoted with boldface type, though no distinction is made in the notation of norms and inner products.

The usual weak formulation of Eq. (1.1) reads as follows:

(P) *Find a pair* $\{u,p\} \in \mathbf{H}_0^1 \times L_0^2$ *such that*

$$(\nabla u, \nabla \varphi) - (p, \nabla \cdot \varphi) - (\chi, \nabla \cdot u) = (f, \varphi), \quad \forall \{\varphi, \chi\} \in \mathbf{H}_0^1 \times L_0^2. \tag{1.2}$$

Problem (P) has a unique solution for any force $f \in \mathbf{H}^{-1}$. This is a consequence of the well known estimate (see, e.g., Ref. 5)

$$\inf_{q \in L_0^2} \sup_{v \in \mathbf{H}_0^1} \left(\frac{(q, \nabla \cdot v)}{\|q\| \|\nabla v\|} \right) \geq \beta_0 > 0. \tag{1.3}$$

If $f \in \mathbf{L}^2$, then the solution is in $\mathbf{H}^2 \times H^1$ and satisfies the *a priori* estimate

$$\|u\|_2 + \|p\|_1 \leq c\|f\|. \tag{1.4}$$

For approximating problem (P) by the finite element method one chooses appropriate spaces $\mathbf{H}_h \sim \mathbf{H}_0^1$ and $L_h \sim L_0^2$, consisting of piecewise polynomial functions, where $h > 0$ is a mesh size parameter tending to zero. Then, using corresponding "discrete" bilinear forms $a_h(v,w) \sim (\nabla v, \nabla w)$ and $b_h(v,w) \sim -(v, \nabla \cdot w)$, the discrete Stokes problem reads

(P_h) *Find a pair* $\{u_h, p_h\} \in \mathbf{H}_h \times L_h$ *such that*

$$a_h(u_h, \varphi_h) + b_h(p_h, \varphi_h) + b_h(\chi_h, u_h) = (f, \varphi_h), \quad \forall \{\varphi_h, \chi_h\} \in \mathbf{H}_h \times L_h. \tag{1.5}$$

This problem also has a unique solution in $\mathbf{H}_h \times L_h$, if $a_h(\cdot,\cdot)$ is definite on $\mathbf{H}_h$, and if $b_h(\cdot,\cdot)$ satisfies a discrete analog of the estimate (1.3) called the "uniform Babuska-Brezzi stability condition."

II. THE ROTATED MULTILINEAR STOKES ELEMENT

Let $\mathbf{T}_h$ be regular decompositions of the domain $\Omega \subset \mathbf{R}^n$ into (convex) quadrilaterals respectively hexahedrons denoted by T, where the mesh parameter $h > 0$ describes the maximum diameter of the elements of $\mathbf{T}_h$. By $\partial\mathbf{T}_h$ we denote the set of all $(n-1)$-faces Γ of the elements $T \in \mathbf{T}_h$. The family $\{\mathbf{T}_h\}$ is assumed to satisfy the usual "uniform shape condition." Accordingly, the generic constant c used below is always independent of h. In defining the "parametric" rotated multilinear element one uses the unit n-cube (with edges parallel to the coordinate axes) as a reference element $\hat{T}$. For each $T \in \mathbf{T}_h$, let $\psi_T: \hat{T} \to T$ be the corresponding n-linear 1 − 1-transformation. We set

$$\tilde{Q}_1(T) = \{q \circ \psi_T^{-1} : q \in \operatorname{span}\langle 1, x_i, x_i^2 - x_{i+1}^2, i = 1, \dots, n \rangle\}.$$

For defining the corresponding local interpolation operators $i_T: C(T) \to \tilde{Q}_1(T)$, there are two natural sets of nodal functionals $\{F^{(a/b)}_{\Gamma,i}, \Gamma \in \partial\mathbf{T}_h, i = 1,\ldots,n\}$,

$$(a)\quad F^{(a)}_{\Gamma,i}(v) = |\Gamma|^{-1}\int_\Gamma v_i(x)\,do_x, \qquad (b)\quad F^{(b)}_{\Gamma,i}(v) = v_i(b_\Gamma),$$

where b_Γ is the midpoint of the $(n-1)$-face Γ. Either choices are unisolvent with $\tilde{Q}_1(T)$, but lead to different finite element spaces. The corresponding finite element spaces are

$$L_h = \{q_h \in L^2_0: q_{h|T} \equiv \text{const}, \ \forall\, T \in \mathbf{T}_h\},$$

$$\mathbf{H}^{(a/b)}_h = \left\{ \begin{array}{l} v_h \in \mathbf{L}^2: v_{h|T} \in \tilde{Q}_1(T)^n, \ \forall\, T \in \mathbf{T}_h, v_h \text{ continuous with respect} \\ \text{to all the nodal functionals } F^{(a/b)}_{\Gamma,i}(\cdot), \text{ and } F^{(a/b)}_{\Gamma,i}(v_h) = 0 \text{ if } \Gamma \subset \partial\Omega \end{array} \right\}.$$

Clearly, $\mathbf{H}^{(a)}_h \neq \mathbf{H}^{(b)}_h$, while for the corresponding triangular element the spaces $\mathbf{H}^{(a)}_h$ and $\mathbf{H}^{(b)}_h$ coincide. Since the spaces $\mathbf{H}^{(a/b)}_h$ are nonconforming, $\mathbf{H}^{(a/b)}_h \not\subset \mathbf{H}^1_0(\Omega)$, we have to work with "piecewise" defined bilinear forms and corresponding norms

$$a_h(v,w) \equiv \sum_{T\in\mathbf{T}_h} (\nabla v, \nabla w)_T, \qquad b_h(q,v) \equiv -\sum_{T\in\mathbf{T}_h} (q, \nabla\cdot v)_T, \qquad \|v\|_h \equiv a_h(v,v)^{1/2}.$$

Due to the parametric definition of the spaces $\mathbf{H}^{(a/b)}_h$, the system matrices A_h and B_h corresponding to the discrete forms $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ may be calculated very efficiently by back-transformation to the reference element $\hat{T}$.

Let $j_h: L^2_0 \to L_h$ be the operator of piecewise constant interpolation (modified to preserve the zero-mean value property) that satisfies, for $q \in L^2_0 \cap H^1$,

$$\|q - j_h q\| \le ch\|q\|_1. \tag{2.1}$$

Further, let $i_h = i^{(a/b)}_h$ be the global interpolation operator in $\mathbf{H}^{(a/b)}_h$ generated by the local operators i_T. Unfortunately, on general nonuniform meshes the usual optimal order error estimates do not hold for $i^{(a/b)}_h$. This is due to the fact that the spaces $\mathbf{H}^{(a/b)}_h$ are not "isoparametric," i.e., the multilinear transformations $\psi_T: \hat{T} \to T$ are of another polynomial type than the shape functions on $\hat{T}$. In order to guarantee proper approximation properties for $\mathbf{H}^{(a/b)}_h$, we have to impose a certain weak uniformity condition on the meshes $\mathbf{T}_h$. For each element $T \in \mathbf{T}_h$, let $\alpha_T \in (0,\pi)$ denote the maximum angle enclosed between the normal unit vectors corresponding to any two opposite $(n-1)$-faces of T. Then, the quantity

$$\sigma_h \equiv \max\{|\pi - \alpha_T|, T \in \mathbf{T}_h\}$$

is a measure for the degeneration of the mesh $\mathbf{T}_h$.

Lemma 1. For the interpolation operators $i_h = i^{(a/b)}_h$, there holds the error estimate

$$\|v - i_h v\| + h\|v - i_h v\|_h \le ch(h + \sigma_h)\|v\|_2, \qquad v \in \mathbf{H}^1_0 \cap \mathbf{H}^2. \tag{2.2}$$

Proof. The proof uses the standard Bramble-Hilbert lemma (see, e.g., Ref. [8]). For any $T \in \mathbf{T}_h$, let $D\psi_T(\hat{x})$ denote the Jacobian matrix of the mapping $\psi_T: \hat{T} \to T$, which is a linear function of $\hat{x}$. The second gradient $D^2\psi_T$ is proportional to the coefficients of the multilinear terms in ψ_T. By elementary geometric arguments one obtains the estimates

$$|D\psi_T| \le ch, \quad |D\psi_T^{-1}| \le ch^{-1}, \quad |D^2\psi_T| \le ch|\pi - \alpha_T|, \quad |\det(D\psi_T)| \le ch^n. \tag{2.3}$$

To some function $v \in H^2(T)$, on T, we associate the function $\hat{v} \equiv v(\psi_T^{-1}(\cdot)) \in H^2(\hat{T})$, on $\hat{T}$. By the Bramble-Hilbert lemma, there holds

$$\|\hat{v} - i_{\hat{T}}\hat{v}\|_{\hat{T}} + \|\nabla(\hat{v} - i_{\hat{T}}\hat{v})\|_{\hat{T}} \leq c\|\hat{\nabla}^2\hat{v}\|_{\hat{T}}. \tag{2.4}$$

Hence, observing the relations $\widehat{i_T v} = i_{\hat{T}}\hat{v}$ and $\nabla v \circ \psi_T = D\psi_T^{-1}\hat{\nabla}\hat{v}$, and the bounds (2.3) it follows that

$$\begin{aligned}\|v - i_T v\|_T + h\|\nabla(v - i_T v)\|_T &\leq ch^{n/2}[\|\hat{v} - i_{\hat{T}}\hat{v}\|_{\hat{T}} + \|\hat{\nabla}(\hat{v} - i_{\hat{T}}\hat{v})\|_{\hat{T}}] \\ &\leq ch^{n/2}\|\hat{\nabla}^2\hat{v}\|_{\hat{T}}.\end{aligned} \tag{2.5}$$

To estimate the term on the right-hand side we note that

$$|\hat{\nabla}^2\hat{v}| \leq c|D\psi_T|^2|\nabla^2 v| + c|D^2\psi_T||\nabla v| \leq ch^2|\nabla^2 v| + ch|\pi - \alpha_T||\nabla v|. \tag{2.6}$$

This yields

$$\|v - i_T v\|_T + h\|\nabla(v - i_T v)\|_T \leq c(h^2\|\nabla^2 v\|_T + h|\pi - \alpha_T|\|\nabla v\|_T). \tag{2.7}$$

Summing this over all $T \in \mathbf{T}_h$ yields the desired result. ■

Remark 1. The interpolation estimate (2.2) is sharp in the respect that the dependence on the quantities σ_h is unavoidable. This can be seen, for instance, on meshes composed of quadrilaterals of the type shown in Fig. 1. The (conforming) *isoparametric* multilinear element does not have this defect since in this case one can employ a sharper version of the Bramble-Hilbert lemma yielding the estimate (2.2) with the right-hand side involving only the pure second derivatives of $\hat{v}$. Then, the estimate (2.6) does not contain the gradient term $ch|\pi - \alpha_T||\nabla v|$, since the mapping ψ_T is also multilinear. This leads to the usual optimal order error estimates for i_T independent of σ_h.

Remark 2. The *parametric* rotated multilinear element has a *nonparametric* counterpart. For any element $T \in \mathbf{T}_h$, let $\{\xi_i\}$ denote a coordinate system which is obtained by connecting the center points of any two opposite $(n-1)$-faces of T. Since the mesh family $\{\mathbf{T}_h\}$ is uniformly regular, the linear transformation between $\{\xi_i\}$ and the cartesian system $\{x_i\}$ is bounded independently of h. On each $T \in \mathbf{T}_h$, we set

$$\tilde{Q}_1(T) \equiv \text{span}\langle 1, \xi_i, \xi_i^2 - \xi_{i+1}^2, i = 1, \ldots, n\rangle.$$

The corresponding local interpolation operator $i_T: C(T) \to \tilde{Q}_1(T)$ is defined using the same sets of nodal functionals $\{F_{\Gamma,i}^{(a/b)}(\cdot)\}$ as in the parametric case. Then, there also holds

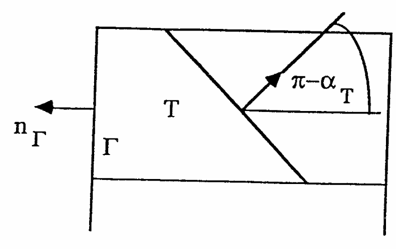

FIG. 1. Deterioration of Quadrilaterals.

$\mathbf{H}_h^{(a/b)} \not\subset \mathbf{H}_0^1$ and $\mathbf{H}_h^{(a)} \neq \mathbf{H}_h^{(b)}$. Since in this case $\tilde{Q}_1(T)$ automatically contains all linear polynomials in x, one may apply the Bramble-Hilbert lemma directly on each element $T \in \mathbf{T}_h$, without refering to the reference element $\hat{T}$. This yields the estimate (2.2),

$$\|v - i_h^{(a/b)}v\| + h\|v - i_h^{(a/b)}v\|_h \leq ch^2\|v\|_2, \qquad v \in \mathbf{H}_0^1 \cap \mathbf{H}^2, \tag{2.8}$$

without any dependency on σ_h. We omit the rather standard technical details.

Analogously, as in the triangular case (see Ref. [5]), it is easy to see that the "mean-value-oriented" interpolation operator $i_h^{(a)}: \mathbf{H}_0^1 \to \mathbf{H}_h^{(a)}$ has the following properties:

$$b_h(\chi_h, v - i_h^{(a)}v) = 0, \ \ \forall\, \chi_h \in L_h, \qquad v \in \mathbf{H}^1, \tag{2.9}$$

$$\|i_h^{(a)}v\|_h \leq \gamma\|v\|_1, \qquad v \in \mathbf{H}_0^1. \tag{2.10}$$

This, together with the continuous stability estimate (1.3), directly implies the uniform stability for the pairing $\{\mathbf{H}_h^{(a)}, L_h\}$,

$$\min_{q_h \in L_h} \max_{v_h \in \mathbf{H}_h^{(a)}} \left(\frac{b_h(q_h, v_h)}{\|q_h\|\,\|v_h\|_h}\right) \equiv \beta_h \geq \frac{\beta_0}{\gamma} > 0. \tag{2.11}$$

The "midpoint-oriented" interpolation operator $i_h^{(b)}: \mathbf{C}(\Omega) \to \mathbf{H}_h^{(b)}$ generally does not satisfy Eqs. (2.9) and (2.10). In order to guarantee the stability property (2.11) for the pairing $\{\mathbf{H}_h^{(b)}, L_h\}$, we have to require the meshes $\mathbf{T}_h$ to be sufficiently uniform.

Lemma 2. Suppose that the quantity $\sigma \equiv \sup_{h>0} \sigma_h$ is sufficiently small. Then, the uniform stability estimate (2.11) holds true also for the pairing $\{\mathbf{H}_h^{(b)}, L_h\}$.

Proof. We use the stability estimate (2.11) already known for the pairing $\{\mathbf{H}_h^{(a)}, L_h\}$. Let $q_h \in L_h$ be given. To any $v_h \in \mathbf{H}_h^{(a)}$ we associate a function $\bar{v}_h \in \mathbf{H}_h^{(b)}$, by requiring that

$$\bar{v}_h(b_\Gamma) = |\Gamma|^{-1} \int_\Gamma v_h\, do_x, \ \ \forall\ \Gamma \in \partial\mathbf{T}_h. \tag{2.12}$$

Since the mesh family $\{\mathbf{T}_h\}$ is uniformly regular, there holds

$$\|\bar{v}_h\|_h \leq c\|v_h\|_h. \tag{2.13}$$

Furthermore,

$$b_h(q_h, \bar{v}_h) = b_h(q_h, v_h) + b_h(q_h, \bar{v}_h - v_h), \tag{2.14}$$

where the second term on the right-hand side can be written in the form

$$b_h(q_h, \bar{v}_h - v_h) = -\sum_{T \in \mathbf{T}_h} \int_T q_h \nabla \cdot (\bar{v}_h - v_h)\, dx = \sum_{T \in \mathbf{T}_h} \int_{\partial T} q_h(\bar{v}_h - v_h) \cdot n\, do_x. \tag{2.15}$$

On any fixed $(n-1)$-face Γ, there holds

$$\int_\Gamma [\bar{v}_h - v_h] \cdot n_\Gamma do_x = \int_\Gamma \bar{v}_h \cdot n_\Gamma do_x - |\Gamma|\bar{v}_h(b_\Gamma) \cdot n_\Gamma = \int_\Gamma \omega D_t^2 \bar{v}_h \cdot n_\Gamma do_x, \tag{2.16}$$

with a weight $\omega(x) \approx \text{diam}(\Gamma)^2$ and a sum D_t^2 of second tangential derivatives on Γ. Therefore, the term on the left-hand side in Eq. (2.15) can be written as the sum of terms of the form

$$A_T \equiv q_{h|T}\left(\int_{\Gamma} \omega D_t^2 \bar{v}_h \cdot n_\Gamma \, do_x + \int_{\Gamma'} \omega' D_t^2 \bar{v}_h \cdot n_{\Gamma'} \, do_x\right), \tag{2.17}$$

where Γ and Γ' are two opposite $(n-1)$-faces of some $T \in \mathbf{T}_h$. Using the relation

$$|n_\Gamma + n_{\Gamma'}| \leq c|\pi - \alpha_T|, \tag{2.18}$$

and the "inverse" inequality

$$|D_t^2 \bar{v}_{h|\Gamma}| \leq c \operatorname{diam}(T)^{-2} \|\nabla \bar{v}_h\|_{L^2(T)}, \tag{2.19}$$

we conclude that

$$|A_T| \leq c|\pi - \alpha_T| \, \|q_h\|_{L^2(T)} \|\nabla \bar{v}_h\|_{L^2(T)}. \tag{2.20}$$

Collecting these estimates for all $T \in \mathbf{T}_h$ yields

$$|b_h(q_h, \bar{v}_h - v_h)| \leq c\sigma_h \|q_h\| \, \|\bar{v}_h\|_h. \tag{2.21}$$

Then, we use this together with Eqs. (2.13) and (2.14) to obtain

$$\beta\|q_h\| \leq \sup_{v_h \in \mathbf{H}_h^{(a)}} \left(\frac{b_h(q_h, v_h)}{\|v_h\|_h}\right) + c\sigma_h \|q_h\|, \tag{2.22}$$

from which the assertion follows, provided that the σ_h are sufficiently small. ∎

On the basis of the stability estimate (2.11) and the approximation properties (2.1) and (2.2) we can now derive asymptotic error estimates. We begin with the *parametric* case.

Theorem 1. *Suppose that the foregoing assumptions hold. Then, for* $\mathbf{H}_h = \mathbf{H}_h^{(a)}$, *and if the quantity* $\sigma \equiv \sup_{h>0} \sigma_h$ *is sufficently small, also for* $\mathbf{H}_h = \mathbf{H}_h^{(b)}$, *the discrete Stokes problems* (P_h) *have unique solutions* $\{u_h, p_h\} \in \mathbf{H}_h^{(a/b)} \times L_h$, *and there holds*

$$\|u - u_h\|_h + \|p - p_h\| \leq c(h + \sigma_h)\{\|u\|_2 + \|p\|_1\}, \tag{2.23}$$

$$\|u - u_h\| + \|p - p_h\|_{-1} \leq c(h + \sigma_h)^2\{\|u\|_2 + \|p\|_1\} \tag{2.24}$$

(*Here* $\|\cdot\|_{-1}$ *denotes the norm of the dual space of* $L_0^2 \cap H^1$).

Proof. The argument is similar to that for the triangular case (see, e.g., Ref. [5]). Clearly, the bilinear form $a_h(\cdot,\cdot)$ defines an inner product on $\mathbf{H}_h$. Hence, in view of the stability property (2.11), the problems (P_h) are uniquely solvable in $\mathbf{H}_h \times L_h$. Next, combining Eqs. (1.2) and (1.5), we obtain the error identity

$$a_h(u - u_h, \varphi_h) + b_h(p - p_h, \varphi_h) + b_h(\chi_h, u - u_h) = \Gamma_u(\varphi_h) - \Gamma_p(\varphi_h), \tag{2.25}$$

for all $\{\varphi_h, \chi_h\} \in \mathbf{H}_h \times L_h$, where

$$\Gamma_u(\varphi_h) \equiv \sum_{T \in \mathbf{T}_h} \int_{\partial T} \partial_n u \varphi_h \, do_x, \qquad \Gamma_p(\varphi_h) \equiv \sum_{T \in \mathbf{T}_h} \int_{\partial T} p\varphi_h \cdot n \, do_x. \tag{2.26}$$

The functionals $\Gamma_u(\cdot)$ and $\Gamma_p(\cdot)$ are also well defined on the direct sum $\mathbf{H}_h \oplus \mathbf{H}_0^1$. For functions $\varphi_h \in \mathbf{H}_h \oplus \mathbf{H}_0^1$, there holds

$$\Gamma_u(\varphi_h) = \sum_{\Gamma \subset \partial \mathbf{T}_h} \int_\Gamma (\partial_n u - \overline{\partial_n u})[\varphi_h - \overline{\varphi_h}]_\Gamma \, do_x + \sum_{\Gamma \subset \partial \mathbf{T}_h} \int_\Gamma \overline{\partial_n u}[\varphi_h]_\Gamma \, do_x, \tag{2.27}$$

where $[\cdot]_\Gamma$ denotes the jump of a function accross Γ, and the bar indicates the mean value over Γ. By a standard argument using a Poincaré-type inequality it follows that

$$|\Sigma_1| \equiv \left| \sum_{\Gamma\subset\partial\mathbf{T}_h} \int_\Gamma (\partial_n u - \overline{\partial_n u})[\varphi_h - \overline{\varphi_h}]_\Gamma \, do_x \right| \le ch\|u\|_2\|\varphi_h\|_h. \tag{2.28}$$

Analogous relations also hold for the functional $\Gamma_p(\cdot)$; we omit the obvious details. Next, we consider the cases $\mathbf{H}_h = \mathbf{H}_h^{(a)}$ and $\mathbf{H}_h = \mathbf{H}_h^{(b)}$ separately.

(i) In the case $\mathbf{H}_h = \mathbf{H}_h^{(a)}$, the second sum, Σ_2, in Eq. (2.27) vanishes and we obtain

$$|\Gamma_u(\varphi_h)| \le ch\|u\|_2\|\varphi_h\|_h, \qquad |\Gamma_p(\varphi_h)| \le ch\|p\|_1\|\varphi_h\|_h, \tag{2.29}$$

for $\varphi_h \in \mathbf{H}_h^{(a)} \oplus \mathbf{H}_0^1$. Furthermore, using additionally Eq. (2.2), it follows that, for $v \in \mathbf{H}_0^1 \cap \mathbf{H}^2$,

$$\begin{aligned} |\Gamma_u(v - i_h^{(a)}v)| &\le ch(h + \sigma_h)\|u\|_2\|v\|_2, \\ |\Gamma_p(v - i_h^{(a)}v)| &\le ch(h + \sigma_h)\|p\|_1\|v\|_2. \end{aligned} \tag{2.30}$$

(ii) In the case $\mathbf{H}_h = \mathbf{H}_h^{(b)}$, the second sum Σ_2 does not vanish. In virtue of the continuity properties of functions in $\mathbf{H}_h^{(b)}$, there holds (see the proof of Lemma 2)

$$\int_\Gamma \overline{\partial_n u} \cdot [\varphi_h]_\Gamma \, do_x = \int_\Gamma \overline{\partial_n u} \cdot [\varphi_h - |\Gamma|\varphi_h(b_\Gamma)]_\Gamma \, do_x = \int_\Gamma \omega\overline{\partial_n u} \cdot D_t^2[\varphi_h]_\Gamma \, do_x, \tag{2.31}$$

with a weight function $\omega(x) \approx \text{diam}(\Gamma)^2$ and a certain sum D_t^2 of second-order tangential derivatives on Γ. Therefore, the term Σ_2 can be written as a sum of terms

$$A_T \equiv \int_\Gamma \omega\overline{\partial_n u} \cdot D_t^2\varphi_h \, do_x + \int_{\Gamma'} \omega\overline{\partial_{n'} u} \cdot D_t^2\varphi_h \, do_x, \tag{2.32}$$

where Γ and Γ' are two opposite $(n-1)$-faces of some $T \in \mathbf{T}_h$. Using again the relations (2.18) and (2.19) we conclude that, for $\varphi_h \in \mathbf{H}_h^{(b)}$,

$$|A_T| \le c|\pi - \alpha_T|\,\|u\|_{H^2(T)}\|\nabla\varphi_h\|_{L^2(T)}, \tag{2.33}$$

and collecting these estimates

$$|\Sigma_2| \equiv \left| \sum_{\Gamma\subset\partial\mathbf{T}_h} \int_\Gamma \overline{\partial_n u}[\varphi_h]_\Gamma \, do_x \right| \le c\sigma_h\|u\|_2\|\varphi_h\|_h. \tag{2.34}$$

Consequently, for $\varphi_h \in \mathbf{H}_h^{(b)}$, there holds

$$|\Gamma_u(\varphi_h)| \le c(h + \sigma_h)\|u\|_2\|\varphi_h\|_h, \qquad |\Gamma_p(\varphi_h)| \le c(h + \sigma_h)\|p\|_1\|\varphi_h\|_h. \tag{2.35}$$

For $\varphi_h = v - i_h^{(b)}v \in \mathbf{H}_h^{(b)} \oplus \{\mathbf{H}_0^1 \cap \mathbf{H}^2\}$, we have to modify the argument. Notice, that the relations (2.28) and (2.31) remain valid. Observing that $[\varphi_h]_\Gamma = [i_h v]_\Gamma$, Eq. (2.32) takes the form

$$A_T \equiv \int_\Gamma \omega\overline{\partial_n u} \cdot D_t^2 i_h v \, do_x + \int_{\Gamma'} \omega\overline{\partial_{n'} u} \cdot D_t^2 i_h v \, do_x. \tag{2.36}$$

By the transformation argument of the proof of Lemma 1 one easily shows that

$$|D_t^2 i_h v_{|\Gamma}| \le ch^{-1}\|\nabla^2 v\|_{L^2(T)} + ch^{-2}|\pi - \alpha_T|\,\|\nabla v\|_{L^2(T)}. \tag{2.37}$$

Using this in Eq. (2.36) and summing again over all $T \in \mathbf{T}_h$ then results in

$$\left|\sum_{\Gamma\subset\partial\mathbf{T}_h}\int_\Gamma \overline{\partial_n u}\,[v - i_h^{(b)}v]_\Gamma\,do_x\right| \le c(h + \sigma_h)^2\|u\|_2\|v\|_2. \tag{2.38}$$

Hence, by the interpolation estimates (2.1) and (2.2) it follows that, for $v \in \mathbf{H}_0^1 \cap \mathbf{H}^2$,

$$\begin{aligned} |\Gamma_u(v - i_h^{(b)}v)| &\le c(h + \sigma_h)^2\|u\|_2\|v\|_2, \\ |\Gamma_p(v - i_h^{(b)}v)| &\le c(h + \sigma_h)^2\|p\|_1\|v\|_2. \end{aligned} \tag{2.39}$$

We will now continue the proof simultaneously for $\mathbf{H}_h = \mathbf{H}_h^{(a/b)}$. From the interpolation estimate (2.2), we obtain that

$$\begin{aligned} \|u - u_h\|_h^2 \le c(h + \sigma_h)^2\|u\|_2^2 + 2|b_h(p - p_h, i_h u - u_h) \\ - \Gamma_u(i_h u - u_h) + \Gamma_p(i_h u - u_h)|. \end{aligned} \tag{2.40}$$

Further, combining Eqs. (2.25) and (2.29), respectively (2.35), a standard argument yields

$$|b_h(p - p_h, i_h u - u_h)| \le ch\|p\|_1\|u - u_h\|_h + c(h + \sigma_h)\|u\|_2\|p - p_h\|. \tag{2.41}$$

This implies as a first result

$$\|u - u_h\|_h^2 \le c(h + \sigma_h)\|u\|_2\|p - p_h\| + c[(h + \sigma_h)\|u\|_2 + h\|p\|_1]^2. \tag{2.42}$$

Next, we use the stability estimate (2.11) and (2.1) to obtain

$$\|p - p_h\| \le ch\|p\|_1 + \max_{\varphi_h\in\mathbf{H}_h}\left(\frac{b_h(p - p_h, \varphi_h)}{\|\varphi_h\|_h}\right). \tag{2.43}$$

In view of Eqs. (2.25) and (2.29), respectively (2.35), it follows that

$$\|p - p_h\| \le ch(\|p\|_1 + \|u\|_2) + c\|u - u_h\|_h. \tag{2.44}$$

Combining this with Eq. (2.42) we obtain the desired estimate (2.23).

To prove the estimate (2.24), we employ a duality argument. Let $\{v, q\} \in \mathbf{H}_0^1 \times L_0^2$ be the unique solution of the auxiliary Stokes problem

$$(\nabla v, \nabla\varphi) - (q, \nabla\cdot\varphi) - (\chi, \nabla\cdot u) = (u - u_h, \varphi), \quad \forall\{\varphi, \chi\} \in \mathbf{H}_0^1 \times L_0^2, \tag{2.45}$$

which, in view of the *a priori* estimate (1.4), satisfies

$$\|v\|_2 + \|q\|_1 \le c\|u - u_h\|. \tag{2.46}$$

Using the above notation, there holds

$$\begin{aligned} \|u - u_h\|^2 = (u - u_h, -\Delta v + \nabla q) = a_h(u - u_h, v) + b_h(q, u - u_h) \\ -\Gamma_v(u - u_h) + \Gamma_q(u - u_h). \end{aligned} \tag{2.47}$$

Using the identity (2.25), the first two terms on the right-hand side can be written in the form

$$\begin{aligned} a_h(u - u_h, v) = a_h(u - u_h, v - i_h v) - b_h(p - p_h, v - i_h v) \\ - \Gamma_u(v - i_h v) + \Gamma_p(v - i_h v), \end{aligned} \tag{2.48}$$

and

$$b_h(q, u - u_h) = b_h(q - j_h q, u - u_h). \tag{2.49}$$

Hence, using the foregoing results we conclude that

$$\|u - u_h\| \le c(h + \sigma_h)^2\{\|u\|_2 + \|p\|_1\}. \tag{2.50}$$

To prove the negative norm estimate for the pressure, we recall that the divergence operator is a homeomorphism from $\mathbf{H}_0^1 \cap \mathbf{H}^2$ onto $L_0^2 \cap H^1$ (see Ref. [5]). This implies that

$$\|p - p_h\|_{-1} = \sup_{q \in L_0^2 \cap H^1} \left(\frac{(p - p_h, q)}{\|q\|_1} \right) \le c \sup_{v \in \mathbf{H}_0^1 \cap \mathbf{H}^2} \left(\frac{b_h(p - p_h, v)}{\|v\|_2} \right). \tag{2.51}$$

For any $v \in \mathbf{H}_0^1 \cap \mathbf{H}^2$, we find, in view of Eq. (2.25),

$$a_h(u - u_h, v) + b_h(p - p_h, v) = a_h(u - u_h, v - i_h v) + b_h(p - p_h, v - i_h v) \\ - \Gamma_u(v - i_h v) - \Gamma_p(v - i_h v), \tag{2.52}$$

and, by integration by parts,

$$a_h(u - u_h, v) = -(u - u_h, \Delta v) + \Gamma_v(u - u_h). \tag{2.53}$$

Hence, using again the foregoing results, we conclude that

$$|b_h(p - p_h, v)| \le c(h + \sigma_h)\|v\|_2\{\|p - p_h\| + \|u - u_h\|_h + (h + \sigma_h)\|u\|_2 \\ + h\|p\|_1\}. \tag{2.54}$$

Inserting this into Eq. (2.52) eventually yields the desired result,

$$\|p - p_h\|_{-1} \le c(h + \sigma_h)^2\{\|u\|_2 + \|p\|_1\}. \tag{2.55}$$

This completes the proof of the theorem. ■

Remark 3. The preceding analysis indicates that the convergence of the *parametric* rotated multilinear Stokes element, for $h \to 0$, requires the underlying meshes $\{\mathbf{T}_h\}$ to be asymptotically uniform in the sense that $\sigma_h \equiv \max\{|\pi - \alpha_T|, T \in \mathbf{T}_h\} \to 0$, as $h \to 0$. This conclusion is supported by our numerical tests. However, the condition for convergence is not very restrictive. It is, for instance, automatically satisfied for "weakly" uniform meshes that are obtained from an arbitrary macrodecomposition by the usual systematic refinement process. Also, the approximation of curved boundaries and certain types of regular local refinements are allowed, but self-adaptive mesh generation is excluded.

Remark 4. In working with the "point-value-oriented" finite element spaces $(\mathbf{H}_h^{(b)}, L_h\}$ it is convenient to replace the bilinear form $b_h(\cdot,\cdot)$ by its numerically integrated version

$$\tilde{b}_h(q,v) = \sum_{\Gamma \in \partial \mathbf{T}_h} |\Gamma| q(b_\Gamma) v(b_\Gamma) \cdot n_\Gamma. \tag{2.56}$$

In this case the uniform stability condition (2.11) is satisfied without any additional condition on the meshes $\mathbf{T}_h$. This immediately follows by the argument used in the proof of Lemma 2, observing that

$$b_h(\chi_h, v_h) = \tilde{b}_h(\chi_h, \tilde{v}_h), \quad \forall \chi_h \in L_h, v_h \in \mathbf{H}_h^{(a)}. \tag{2.57}$$

Then, by a standard perturbation argument (see, e.g., Ref. [8]) the estimates of Theorem 1 carry over to this case without any condition on the size of $\sigma \equiv \sup_{h>0} \sigma_h$. This variant of the scheme (P_h) has been used in some of the test calculations described below.

Remark 5. In view of Remark 2, the *nonparametric* versions of the spaces $\mathbf{H}_h^{(a/b)}$ have satisfactory approximation properties on general regular meshes. The stability properties are the same as those of their parametric counterparts, i.e., the convergence of the "midpoint-oriented" element $\mathbf{H}_h = \mathbf{H}_h^{(b)}$ can be guaranteed only under the assumption that $\sigma_h \to 0$. However, the nonparametric "mean-value-oriented" element $\mathbf{H}_h = \mathbf{H}_h^{(a)}$ is stable and convergent also on nonuniform meshes. In fact, the optimal order convergence estimates

$$\|u - u_h\|_h + \|p - p_h\| \le ch\{\|u\|_2 + \|p\|_1\}, \tag{2.58}$$

$$\|u - u_h\| + \|p - p_h\|_{-1} \le ch^2\{\|u\|_2 + \|p\|_1\}, \tag{2.59}$$

follow directly by the argument used in the proof of Theorem 1. This result is supported by our numerical tests. It should be noted that for this nonparametric element the system matrices A_h and B_h have to be calculated locally element by element as no reference configuration is available. This is possible without significant loss in computational efficiency but it somewhat conflicts with the basic concept of most of the FEM-packages.

III. NUMERICAL TESTS

For the numerical verification of our theoretical results we have chosen one of the usual artificial test problems on the unit square, $\Omega = (0,1) \times (0,1)$, with the exact solution

$$u_1(x_1,x_2) = -256x_1^2(x_1 - 1)^2x_2(x_2 - 1)(2x_2 - 1),$$

$$u_2(x_1,x_2) = -u_1(x_2,x_1), \qquad p(x_1,x_2) = 150(x_1 - 1/2)(x_2 - 1/2).$$

In Figs. 2–5, four types of quadrilateral meshes are shown (mesh width $h = 1/16$) for which the calculations have been carried through.

As measures for the quality of the various Stokes elements we take the normalized relative L^2-errors of the velocity and the pressure approximations

$$\epsilon_u(h) = \frac{\|u - u_h\|}{h^2\|f\|}, \qquad \epsilon_p(h) = \frac{\|p - p_h\|}{h\|f\|}.$$

Another significant quantity is the constant β_h in the discrete stability estimate (2.11), which directly affects the accuracy of the pressure approximation. Let $\{\phi^j\}$ and $\{\chi^k\}$ be

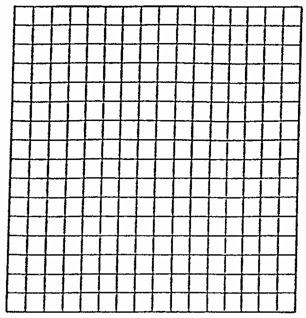

FIG. 2. Uniform rectangular mesh.

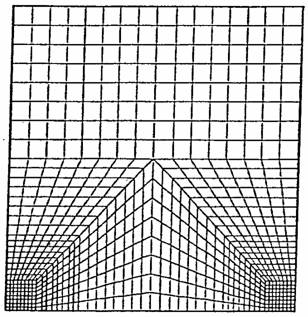

FIG. 3. Locally refined mesh.

the usual nodal bases of the finite element spaces $\mathbf{H}_h$ and L_h and let x and y denote the corresponding nodal vectors of the discrete velocity u_h and pressure p_h, respectively. We introduce the matrices $A \in \mathbf{R}^{n\times n}$, $M \in \mathbf{R}^{m\times m}$, $B \in \mathbf{R}^{n\times m}$, its transpose $B^* \in \mathbf{R}^{m\times n}$, and the vector $b \in \mathbf{R}^n$, by setting

$$A_{ij} = a_h(\phi^i, \phi^j), \qquad M_{ij} = (\chi^i, \chi^j), \qquad B_{ij} = b_h(\chi^j, \phi^i), \qquad b_j = (f, \phi^j).$$

Then, the discrete Stokes problem (1.5) can be written in the form

$$\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}, \tag{3.1}$$

which may be transformed equivalently to

$$B^*A^{-1}By = B^*A^{-1}b, \qquad Ax = b - By, \tag{3.2}$$

where the pressure is separated from the velocity. Clearly, the coefficient matrix $B^*A^{-1}B$ is positive definite on $\mathbf{R}^m/\mathbf{R}$ and has a condition number that is normally expected to be independent of h; in fact it is directly related to the size of $1/\beta_h$. Consequently, the first

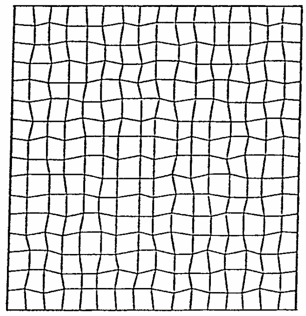

FIG. 4. Perturbed mesh (10%).

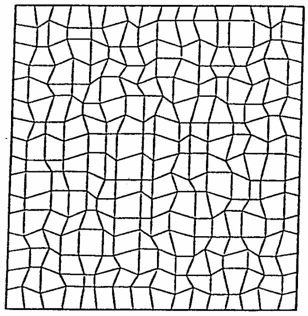

FIG. 5. Perturbed mesh (20%).

equation in (3.2) may be efficiently solved by the cg-method, provided that a fast solver (e.g., multigrid) for the evaluation of A^{-1} is available. This solution technique was used in computing the numbers listed below. Using the above notation, the relation (2.11) reads

$$\inf_{y\in\mathbf{R}^m} \sup_{x\in\mathbf{R}^n} \left(\frac{\langle y, B^*x\rangle}{\langle y, My\rangle \langle x, Ax\rangle} \right) = \beta_h, \tag{3.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the euclidian inner products of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. In terms of the transformed variables $\xi = A^{1/2}x$ and $\eta = M^{1/2}y$, this takes the form

$$\inf_{\eta\in\mathbf{R}^m} \sup_{\xi\in\mathbf{R}^n} \left(\frac{\langle A^{-1/2}BM^{-1/2}\eta, \xi\rangle}{|\eta|\,|\xi|} \right) = \beta_h, \tag{3.4}$$

which is equivalent to

$$\inf_{\eta\in\mathbf{R}^m} \left(\frac{\langle M^{-1/2}B^*A^{-1}BM^{-1/2}\eta, \eta\rangle^{1/2}}{|\eta|} \right) = \beta_h. \tag{3.5}$$

Hence, β_h is determined as the square root of the smallest (positive) eigenvalue of the generalized eigenvalue problem

$$B^*A^{-1}B\eta = \lambda M\eta. \tag{3.6}$$

For monitoring the behavior of β_h as $h \to 0$, it suffices to consider the asymptotic rate of convergence $\kappa_\infty(h)$ of the cg-method for solving the first equation in (3.2), as there holds

$$\kappa_\infty(h) \equiv \overline{\lim_{k\to\infty}} \left(|r^{(k)}|/|r^{(0)}|\right)^{1/k} \approx \frac{1-\beta_h}{1+\beta_h}. \tag{3.7}$$

Here, $r^{(k)}$ is the residuum at the kth iteration step.

Tables I–IV contain the results of a series of test calculations on the meshes shown in Figs. 2–5. The two *parametric* variants (*a*) and (*b*) of the rotated bilinear element have nearly the same quantitative stability and convergence behavior. In particular, both approximation schemes fail to converge on strongly perturbed meshes. The results obtained for the corresponding *nonparametric* versions (see Remark 2) are indicated by superscripts (a'), (b'). As is predicted by the theory (see Remark 5), only the "mean-value-oriented" nonparametric element (a') behaves well on strongly perturbed meshes.

TABLE I. Convergence and stability properties on uniform rectangular meshes.

h	$\epsilon_u^{(a)}(h)$	$\epsilon_p^{(a)}(h)$	$\kappa_\infty^{(a)}(h)$	$\epsilon_u^{(b)}(h)$	$\epsilon_p^{(b)}(h)$	$\kappa_\infty^{(b)}(h)$	$\epsilon_u^{(c)}(h)$	$\epsilon_p^{(c)}(h)$	$\kappa_\infty^{(c)}(h)$
$\frac{1}{8}$	0.0401	0.0137	0.16	0.0602	0.0162	0.12	0.0724	0.0142	0.29
$\frac{1}{16}$	0.0428	0.0130	0.26	0.0728	0.0145	0.26	0.0859	0.0128	0.33
$\frac{1}{32}$	0.0437	0.0127	0.27	0.0776	0.0133	0.28	0.0910	0.0114	0.36
$\frac{1}{64}$	0.0440	0.0125	0.28	0.0793	0.0128	0.29	0.0934	0.0109	0.38

TABLE II. Convergence properties on locally refined meshes.

h	$\epsilon_u^{(a)}(h)$	$\epsilon_p^{(a)}(h)$	$\kappa_\infty^{(a)}(h)$	$\epsilon_u^{(b)}(h)$	$\epsilon_p^{(b)}(h)$	$\kappa_\infty^{(b)}(h)$	$\epsilon_u^{(c)}(h)$	$\epsilon_p^{(c)}(h)$	$\kappa_\infty^{(c)}(h)$
$\frac{1}{8}$	0.0286	0.0113	0.24	0.0301	0.0125	0.22	0.0326	0.0085	0.31
$\frac{1}{16}$	0.0334	0.0106	0.35	0.0447	0.0122	0.33	0.0426	0.0081	0.37
$\frac{1}{32}$	0.0351	0.0102	0.39	0.0537	0.0112	0.39	0.0494	0.0074	0.41
$\frac{1}{64}$	0.0358	0.0100	0.41	0.0576	0.0104	0.39	0.0524	0.0068	0.43

TABLE III. Convergence and stability properties on nonuniform meshes (10% stochastical perturbation of the corresponding uniform mesh).

h	$\epsilon_u^{(a)}(h)$	$\epsilon_p^{(a)}(h)$	$\kappa_\infty^{(a)}(h)$	$\epsilon_u^{(b)}(h)$	$\epsilon_p^{(b)}(h)$	$\kappa_\infty^{(b)}(h)$	$\epsilon_u^{(c)}(h)$	$\epsilon_p^{(c)}(h)$	$\kappa_\infty^{(c)}(h)$
$\frac{1}{8}$	0.0420	0.0138	0.32	0.0604	0.0162	0.30	0.0763	0.0145	0.37
$\frac{1}{16}$	0.0501	0.0133	0.39	0.0798	0.0149	0.37	0.0915	0.0130	0.39
$\frac{1}{32}$	0.0810	0.0130	0.39	0.1151	0.0142	0.39	0.0975	0.0118	0.42
$\frac{1}{64}$	0.2348	0.0129	0.41	0.2753	0.0150	0.41	0.1027	0.0112	0.40

h	$\epsilon_u^{(a')}(h)$	$\epsilon_p^{(a')}(h)$	$\kappa_\infty^{(a')}(h)$	$\epsilon_u^{(b')}(h)$	$\epsilon_p^{(b')}(h)$	$\kappa_\infty^{(b')}(h)$
$\frac{1}{8}$	0.0431	0.0139	0.32	0.0646	0.0164	0.30
$\frac{1}{16}$	0.0493	0.0133	0.39	0.0987	0.0154	0.38
$\frac{1}{32}$	0.0515	0.0130	0.39	0.1741	0.0159	0.42
$\frac{1}{64}$	0.0519	0.0129	0.40	0.5022	0.0201	0.44

TABLE IV. Convergence properties on perturbed quadrilateral meshes (0%–25% stochastical perturbation of the corresponding uniform mesh).

%	$\epsilon_u^{(a')}(\frac{1}{32})$	$\epsilon_p^{(a')}(\frac{1}{32})$	$\kappa_\infty^{(a')}(h)$	$\epsilon_u^{(b')}(\frac{1}{32})$	$\epsilon_p^{(b')}(\frac{1}{32})$	$\kappa_\infty^{(b')}(h)$
0	0.0437	0.0127	0.27	0.0776	0.0133	0.28
5	0.0484	0.0128	0.38	0.1070	0.0139	0.39
10	0.0515	0.0130	0.39	0.1741	0.0159	0.42
15	0.0567	0.0134	0.41	0.2850	0.0191	0.42
20	0.0638	0.0140	0.42	0.4405	0.0235	0.42
25	0.0729	0.0148	0.42	0.6414	0.0292	0.42

This behavior appears to be merely a problem of consistency. In fact, in all cases the stability constants β_h are of moderate size and nearly independent of the mesh size h. For comparison, also some results for the usual nonconforming linear *triangular* element are included; all quantities refering to this element are marked with a superscript (c). It should be noted that in rating the approximation results one has to take into account that, on the same mesh, the triangular element (c) has about 60% more unknowns than the corresponding quadrilateral elements.

Table I shows that, on uniform meshes, the two *parametric* versions (a) and (b) of the "rotated" bilinear element and the linear element (c) are of the same quality. (Notice that in this case there is no difference between *parametric* and *nonparametric*.)

Table II shows that the *parametric* elements (a) and (b) as well as the linear element (c) are nearly of the same quality also on locally refined meshes as long as a quasi-uniform structure is preserved.

Table III shows that for both *parametric* elements (a) and (b) as well as for the "mid-point-oriented" *nonparametric* element (b') the accuracy deteriorates on perturbed meshes, while the stability is preserved. However, only the quality in the velocity approximation appears to be effected. So far we have no rigorous explanation for this phenomenon. The "mean-value-oriented" *nonparametric* element (a') proves to be robust as is predicted by the theory. To better illustrate this effect we show in Table IV the corresponding results for the fixed mesh size $h = 1/32$ and a varying degree of mesh distorsion.

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