A Monolithic ALE-FEM Technique for Numerical Benchmarking and Optimization of Fluid-Structure Interaction Problems

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Recent Developments in Fluid Mechanics

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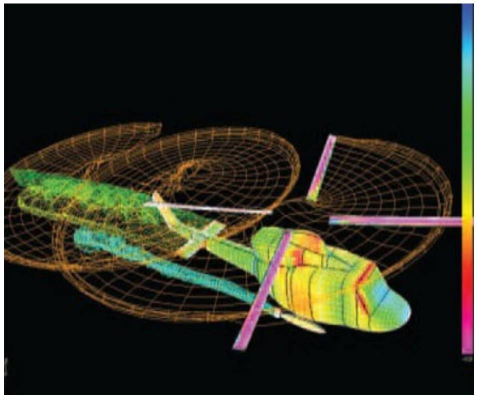
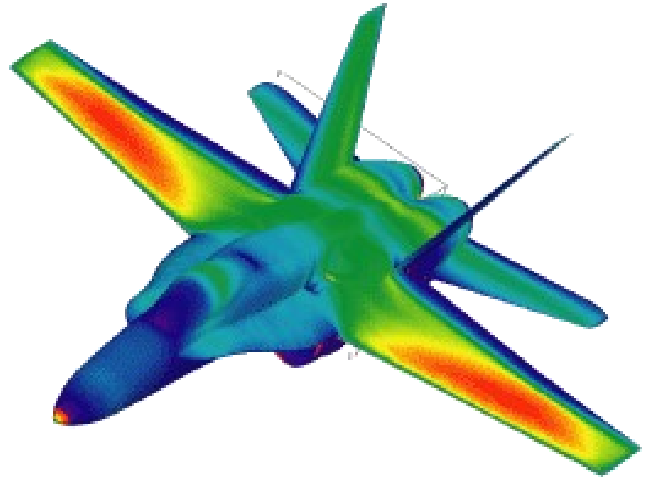
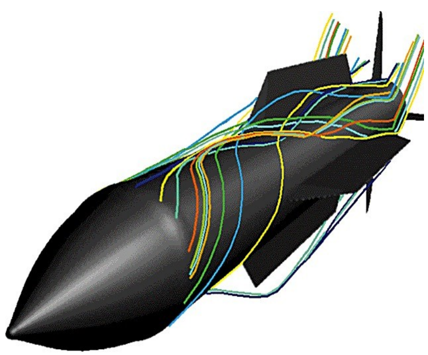
FEM-Multigrid for
Multiphase (FSI) Problems
With optimization



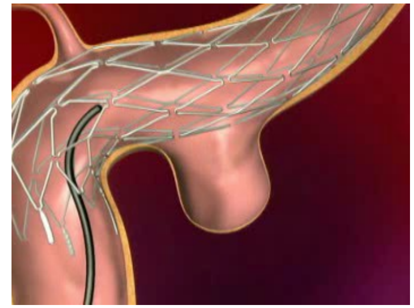


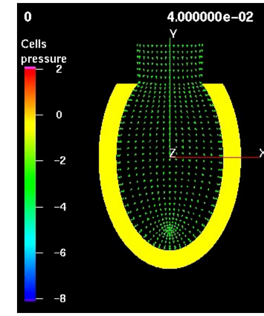


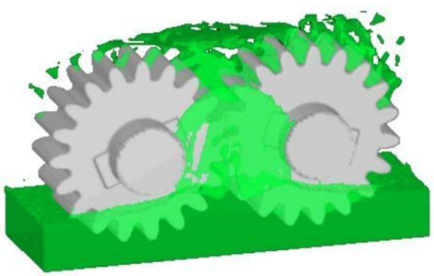
Aim:
FSI-model
Benchmark
Optimization
Extensions

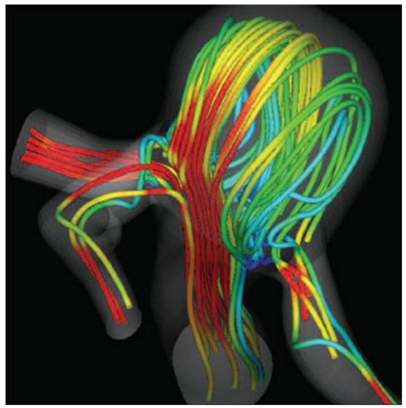
Fluid Structure Interaction (FSI)

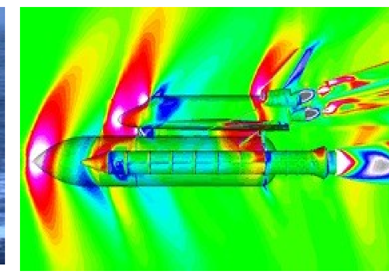
- large deformation of a structure in internal/external flow
  - aeorelasticity
  - bioengineering
  - polymer, food, paper processing
  - . . .









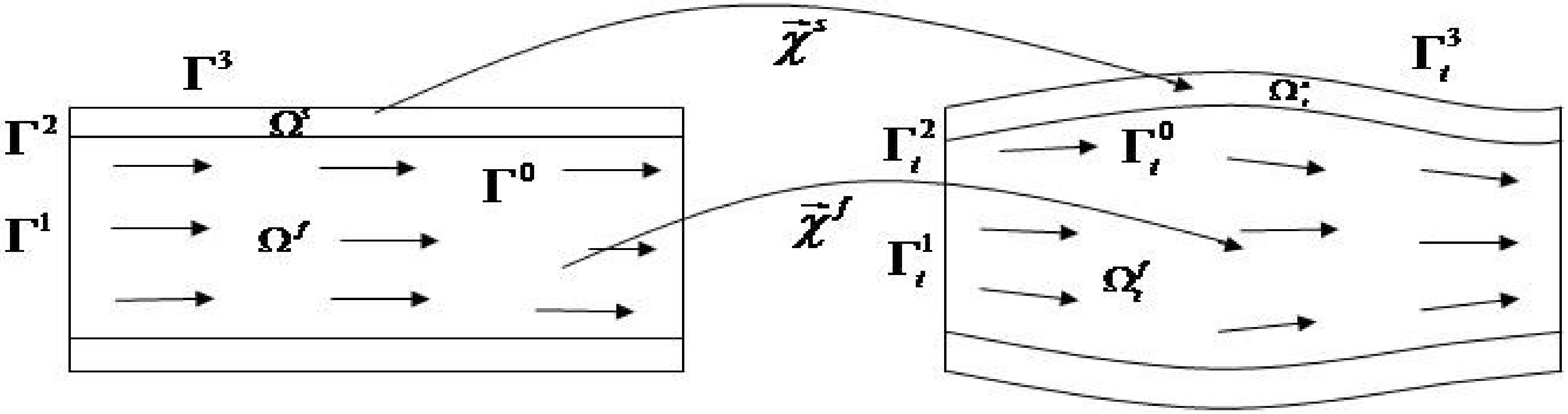


- physical models
  - viscous fluid flow
  - elastic body under large deformations
  - interaction between the two parts

- numerical tasks involved
  - space and time discretization
  - nonlinear system
  - large linear systems

- testing and validation
  - accuracy, efficiency, robustnes
  - benchmarking

Problem description



Structure part

$$\chi^s : \Omega^s \times [0,T] \mapsto \Omega_t^s$$

$$u^s = \chi(X,t) - X, \quad v^s = \frac{\partial u^s}{\partial t}$$

$$F = I + \nabla u^s, \quad J = det\,(F)$$

Fluid part

$$\chi^f : \Omega^f \times [0,T] \mapsto \Omega_t^f$$

$$u^f = \chi(X,t) - X$$

$$v^f : \Omega^f \times [0,T] \mapsto \mathbb{R}^n$$

Governing Equations

structure part

$$\frac{\partial v^s}{\partial t} = \operatorname{div}\left(J\sigma^s \mathrm{F}^{-T}\right) + f \quad \text{in } \Omega^s$$

$$\det(\mathrm{F}) = 1 \quad \text{in } \Omega^s$$

$$u^s = 0$$

$$\sigma^s n = 0 \quad \text{on } \Gamma^3$$

fluid part

$$\frac{\partial v^f}{\partial t} + \left(\nabla v^f\right) v^f = \operatorname{div} \sigma^f + f \quad \text{in } \Omega_t^f$$

$$\operatorname{div} v^f = 0$$

$$v^f = v_0$$

$$\text{or } \sigma^f n = 0$$

interface conditions

$$v^f = v^s \quad \text{on } \Gamma_t^0$$

$$\sigma^f n = \sigma^s n \quad \text{on } \Gamma_t^0$$

Arbitrary Lagrange-Euler Formulation

$$\chi : \Omega \times [0,T] \square\ \Omega_t \qquad v = \frac{\partial \chi}{\partial t}, \qquad \mathrm{F} = \frac{\partial \chi}{\partial X}, \qquad J = \det \mathrm{F}$$

$$\zeta_R : R \times [0,T] \square\ R_t \qquad R_t \subset \Omega_t \qquad \forall t \in [0,T], \qquad v_R = \frac{\partial \zeta_R}{\partial t} \quad \mathrm{F}_R = \frac{\partial \zeta_R}{\partial X} \qquad J_R = \det \mathrm{F}_R$$

$$\frac{\partial}{\partial t} \int_{R_t} \rho \, dv + \int_{\partial R_t} \rho (v - v_R) \cdot n_{R_t} \, da = 0$$

$$\frac{\partial}{\partial t}(\rho J_R) + \operatorname{div}\left(\rho J_R (v - v_R) \mathrm{F}_R^{-T}\right) = 0$$

- Lagrangian description:

$$\zeta_R = \chi \Rightarrow \mathrm{F}_R = \mathrm{F}, \; J_R = J, v_R = v$$

$$\frac{\partial}{\partial t}(\rho J) = 0$$

Mesh nodes are not fixed
Structure mechanics
Mesh tangle
expensive

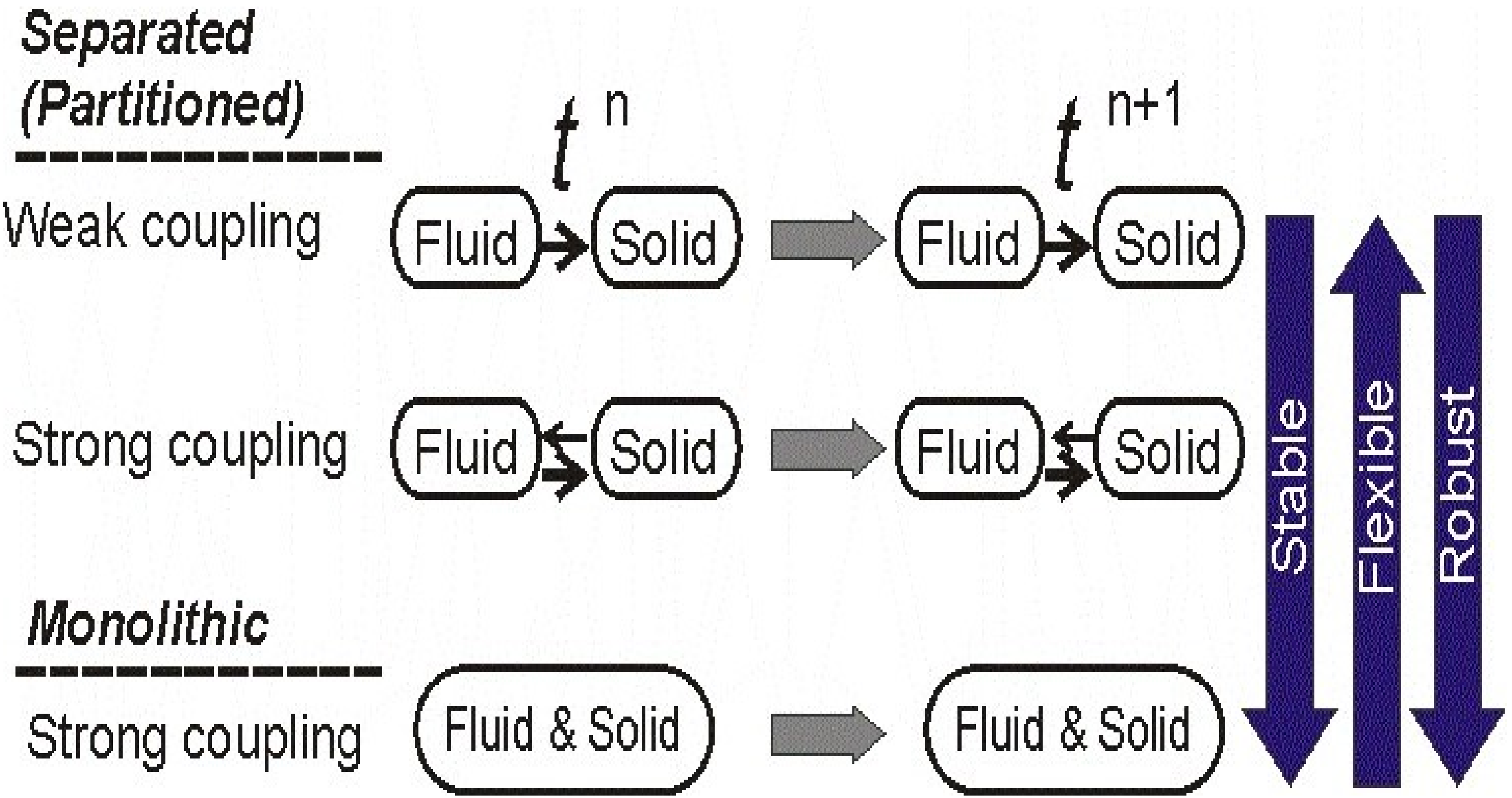
- Eulerian description :

$$\zeta_R = \mathrm{Id} \Rightarrow \mathrm{F}_R = \mathrm{I}, \; J_R = 1, v_R = 0$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0$$

Mesh nodes are fixed
Fluid mechanics
Large distortion handle
Numerical instable for convective term

Coupling strategies



Constitutive equations

- incompressible Newtonian fluid

$$\sigma^f = -pI + 2\nu D$$

- hyperelastic material, incompressible

$$\sigma^f = -pI + 2F\frac{\partial \Psi}{\partial F}F^T, \qquad \det F = 1$$

$$\Psi(F) = \alpha(I_C - 3)\,\text{Neo-Hook}$$

$$\Psi(F) = \alpha_1(I_C - 3) + \alpha_2(I_C - 3) + \alpha_3(|Fe| - 1)^2\ \text{Mooney-Rivlin + anisotropic}$$

where $C = FF^T$ and $I_C = \mathrm{tr}C,\quad I_c = \frac{1}{2}\left(\mathrm{tr}C^2 - (\mathrm{tr}C)^2\right)$

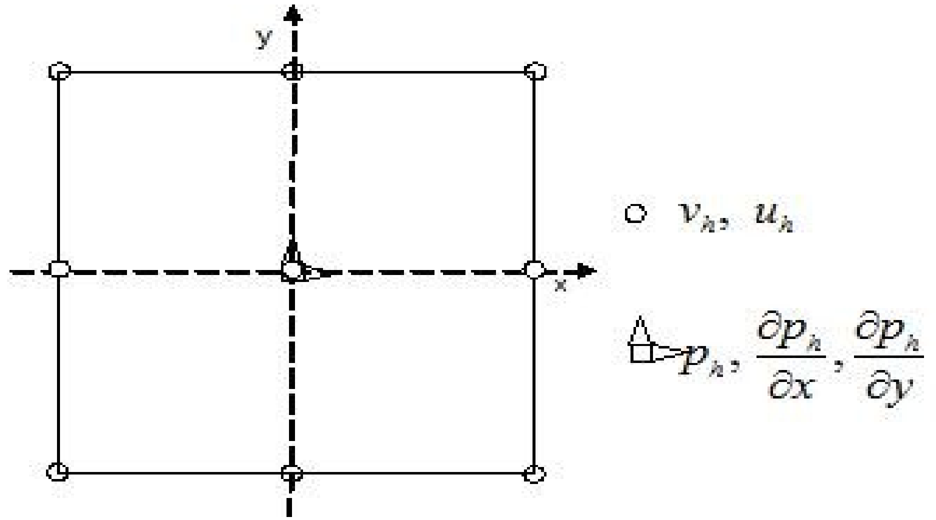
- or **St. Venant--Kirchhoff** material, compressible

$$\sigma^s = \frac{1}{J}F\left(\lambda^s(\mathrm{tr}E)I + \mu^s E\right)F^T$$

where $E = \frac{1}{2}\left(F^T F - I\right)$

Discretization in space and time

- Discretization in space: FEM $Q_2/Q_2/P_1^{disc}$



$$U_h = \{u_h \in [C(\Omega_h)]^2, u_h|_T \in [Q_2(T)]^2 \forall T \in \tau_h, u_h = 0 \text{ on } \Gamma^1\},$$

$$V_h = \{v_h \in [C(\Omega_h)]^2, v_h|_T \in [Q_2(T)]^2 \forall T \in \tau_h, v_h = 0 \text{ on } \Gamma^1\},$$

$$P_h = \{p_h \in L^2(\Omega_h), p_h|_T \in P_1(T) \forall T \in \tau_h\}.$$

- **Discretization in time:** Crank-Nicholson, BE, FS, schemes

Discrete nonlinear system

$$R(\mathrm{x}) = 0 \qquad x = (u_h, v_h, p_h) \in U_h \times V_h \times P_h$$

$$Mu_h - \frac{k}{2}\left(M^s v_h + L^f u_h\right) = \mathrm{rhs}\left(u_h^n, v_h^n\right)$$

$$\left(M^f + \beta M^s\right)v_h + \frac{k}{2}N_1(v_h, u_h) + \frac{1}{2}N_2(v_h, u_h) + \frac{k}{2}\left(S^s(u_h) + S^f(v_h)\right) - kBp_h = \mathrm{rhs}\left(u_h^n, v_h^n, p_h^n\right)$$

$$C(u_h) + B^{f^T} v_h = 1$$

$$\Downarrow$$

$$\frac{\partial R}{\partial X}(X) = \begin{pmatrix} M - \frac{k}{2}L^f & \frac{k}{2}M^s & 0 \\ \frac{1}{2}\frac{\partial\left(N_1 + S^s + S^f\right)}{\partial u_h} + k\frac{\partial B}{\partial u_h}p_h & M^s + \beta M^f + \frac{1}{2}\frac{\partial N_2}{\partial v_h} + \frac{k}{2}\frac{\partial\left(N_1 + S_f^2\right)}{\partial v_h} & kB \\ B^{s^T} + \frac{\partial B^{f^T}}{\partial u_h}v_h & B^{f^T} & 0 \end{pmatrix}$$

Discrete nonlinear system

$$R(\mathrm{x}) = 0 \qquad x = (u_h, v_h, p_h) \in U_h \times V_h \times P_h$$

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$$C(u_h) + B^{f^T} v_h = 1$$

$$\Downarrow$$

$$\begin{bmatrix} S_{uu} & S_{uv} & 0 \\ S_{vu} & S_{vv} & kB \\ c_u B_s^T & c_v B_f^T & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ p \end{bmatrix} = \begin{bmatrix} \mathrm{f_u} \\ \mathrm{f_u} \\ f_p \end{bmatrix}$$

Typical discrete saddle-point problem

Challenges Nonlinear Solvers

Solve for the residual of the nonlinear system algebraic equations

$$R(\mathrm{x}) = 0, \quad \mathrm{x} = (\mathrm{u}, \mathrm{v}, p)$$

Use Newton method with damping results in iterations of the form

$$\mathrm{x}^{n+1} = \mathrm{x}^{n} + \omega^{n} \left[\frac{\partial R(\mathrm{x}^{n})}{\partial \mathrm{x}} \right]^{-1} R(\mathrm{x}^{n})$$

- Continuous Newton: on variational level (before discretization)

→ The continuous Frechet operator can be **analytically calculated**

- Inexact Newton: on matrix level (after discretization)

→ The Jacobian matrix is **approximated** using finite differences as

$$\left[\frac{\partial R(\mathrm{x}^{n})}{\partial x} \right]_{ij} \approx \frac{R_i(\mathrm{x}^{n} + \varepsilon \mathrm{e}_j) - R_i(\mathrm{x}^{n} - \varepsilon \mathrm{e}_j)}{2\varepsilon}$$

Multigrid Solver

- standard geometric multigrid approach
- smoother by local MPSC-Ansatz (Vanka-like smoother

$$\begin{bmatrix} u^{l+1} \\ v^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} u^{l} \\ v^{l} \\ p^{l} \end{bmatrix} - \omega \sum_{\text{Patch}\Omega_i} \begin{bmatrix} S_{uu|\Omega_i} & S_{uv|\Omega_i} & 0 \\ S_{vu|\Omega_i} & S_{vv|\Omega_i} & kB_{|\Omega_i} \\ c_u B^T_{s|\Omega_i} & c_v B^T_{f|\Omega_i} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{def}^l_u \\ \mathrm{def}^l_v \\ def^l_p \end{bmatrix}$$

- full inverse of the local problems by LAPACK (39 x39 systems)
- alternatives: simplified local problems (3x3 systems) or ILU(k)
- combination with GMRES/BiCGStab methods possible
- full (canonical) FEM prolongation, restriction

Very accurate, flexible and highly efficient FSI solver

(→ FSI Benchmarks)

1st step: *Identification of appropriate FSI settings for numerical benchmarking and calculate*

If done → *2nd step*

2nd step: *Extension to FSI-Optimisation benchmark settings*

- Data files are available on

http://featflow.de/beta/en/benchmarks.html

Numerical FSI

- *Realistic materials*
  - **incompressible Newtonian fluid**, laminar flow regime
  - **elastic solid**, large deformations
- *Comparative evaluation*
  - setup with periodical oscillations
  - non-graphically based quantities
- *Computable configurations*
  - laminar flow
  - reasonable aspect ratios
  - simple geometry (2D)
- Mainly based on validated CFD benchmarks, but also close to experimental set-up

Material parameters

solid

- ρ^s density
- ν^s Poisson ratio
- μ^s shear modulus

fluid

- ρ^f density
- ν^f kinematic viscosity

Parameter	polybutadiene & glycerine		**polypropylene & glycerine**	
$\rho^s [10^3 \mathrm{kg/m^3}]$ ν^s $\mu^s [10^6 \mathrm{kg/ms^2}]$	incompressible	0.91 0.50 0.53	compressible	1.1 0.42 317
$\rho^f [10^3 \mathrm{kg/m^3}]$ $\nu^f [10^{-3} \mathrm{m^2}/s]$		1.26 1.13		1.26 1.13

Parameter	FSI1	FSI2	FSI3
$\rho^s [10^3 \mathrm{kg/m^3}]$ ν^s $\mu^s [10^6 \mathrm{kg/ms^2}]$	1 0.4 0.5	1 0.4 0.5	1 0.4 2.0
$\rho^f [10^3 \mathrm{kg/m^3}]$ $\nu^f [10^{-3} \mathrm{m^2}/s]$	1 1	1 1	1 1
$\bar{U} [\mathrm{m}/s]$	0.2	1	2

Parameter	FSI1	FSI2	FSI3
$\beta = \frac{\rho^s}{\rho^f}$ $\mathrm{Ae} = \frac{E^s}{\rho^f \bar{U}^2}$	1 0.4	1 0.4	1 0.4
$\mathrm{Re} = \frac{\bar{U} d}{\nu^f}$ $\bar{U} [\mathrm{m/s}]$	20 0.2	100 1	200 2

Quantities of interest

- The position $A(t)=(x(t),y(t))$ of the end of the structure
- Pressure difference between the points *A(t)* and *B*

$$\Delta p^{AB}(t)=p^{B}(t)-p^{A(t)}(t)$$

- Forces exerted by the fluid on the *whole body*, i.e. lift and drag forces acting on the cylinder and the structure together

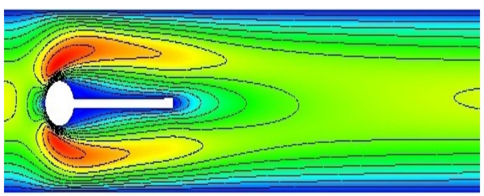
$$(F_D,F_L)=\int_S \sigma\, ndS=\int_{S_1}\sigma^{f}\, ndS+\int_{S_2}\sigma^{f|S} ndS=\int_{S_0}\sigma\, ndS$$

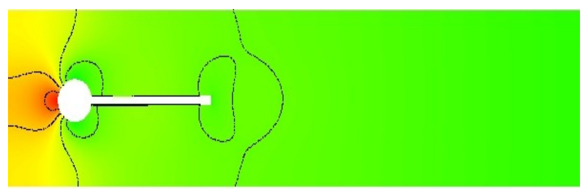
- Frequency and maximum amplitude
- Compare results for *one* full period and 3 different levels of spatial discretization *h* and 3 time step sizes Δt

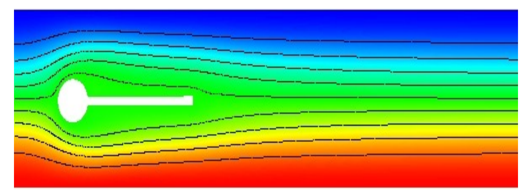
FSI1: steady, small deformations

Parameter	FSI1	FSI2	FSI3
$\rho^s[10^3 \mathrm{kg/m^3}]$ ν^s $\mu^s[10^6 \mathrm{kg/ms^2}]$	1 0.4 0.5	1 0.4 0.5	1 0.4 2.0
$\rho^s[10^3 \mathrm{kg/m^3}]$ $\nu^s[10^{-3} \mathrm{m^2/s}]$	1 1	1 1	1 1
$\bar{U}[\mathrm{m/s}]$	0.2	1	2

Parameter	FSI1	FSI2	FSI3
$\beta = \frac{\rho^s}{\rho^f}$ ν^s $\mathrm{Ae} = \frac{E^s}{\rho^f \bar{U}^2}$	1 0.4	1 0.4	1 0.4
$\mathrm{Re} = \frac{\bar{U} d}{\nu^f}$ $\bar{U}[\mathrm{m/s}]$	20 0.2	100 1	200 2

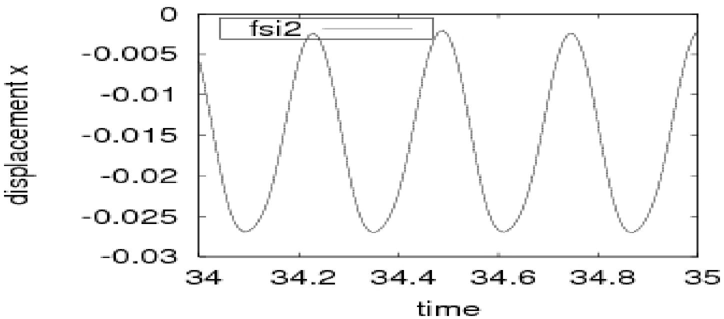


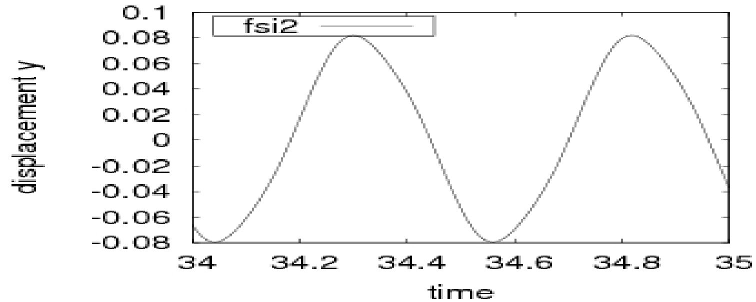


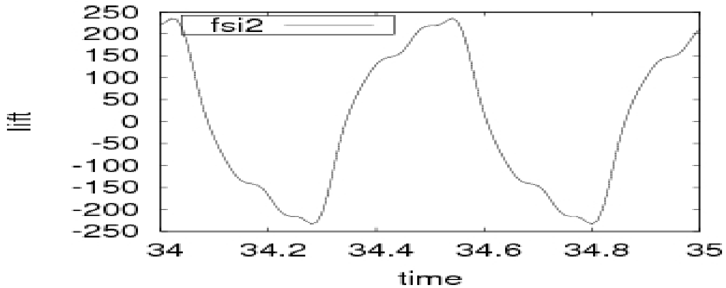


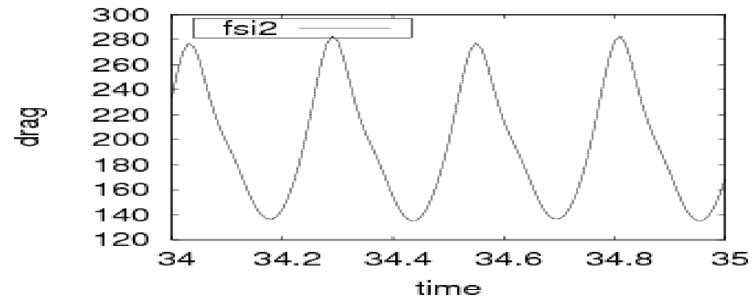
	ux of A $[\times 10^{-5} \mathrm{m}]$	uy of A $[\times 10^{-4} \mathrm{m}]$	drag	lift
FSI1	2.270493	8.208773	14.2942	0.76374

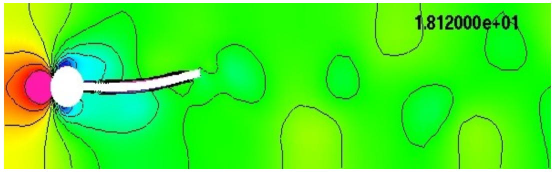
FSI2: large deformations

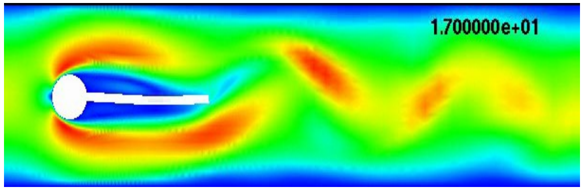


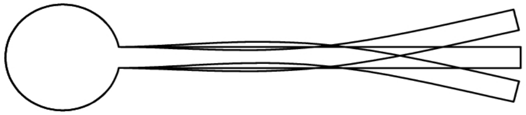








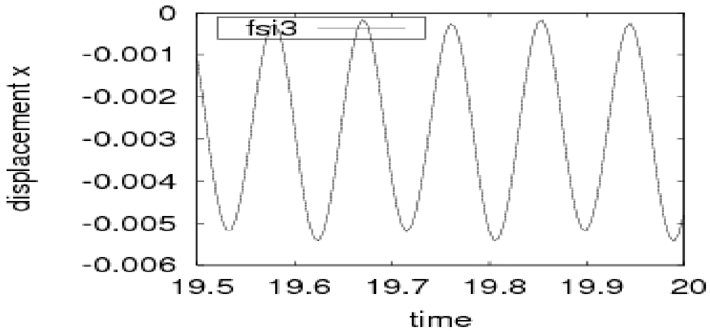


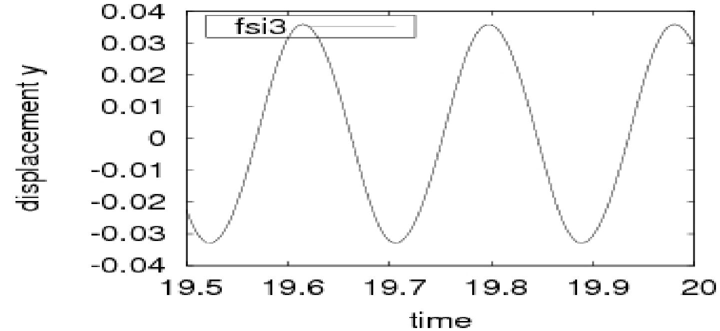


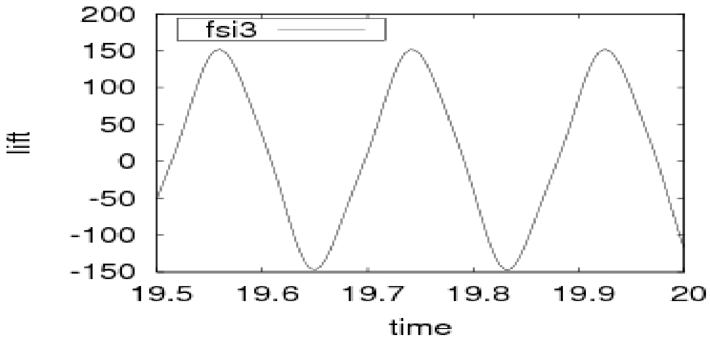
Test	ux of A $[\times 10^{-3}\text{m}]$	uy of A $[\times 10^{-3}\text{m}]$	drag	lift
FSI2	$-14.85 \pm 12.70[3.86]$	$1.30 \pm 81.7[1.93]$	$215.06 \pm 77.65[3.86]$	$0.61 \pm 237.8[1.93]$

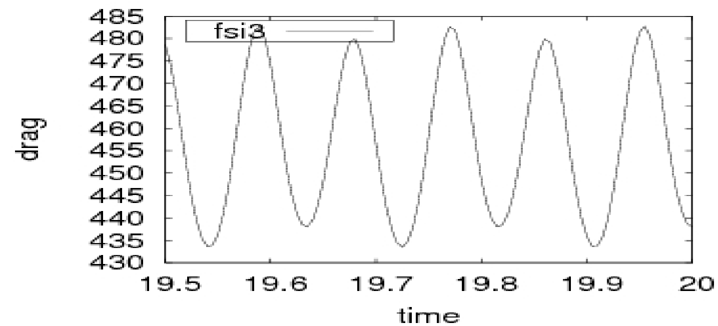
Mean ± amplitude[frequency]

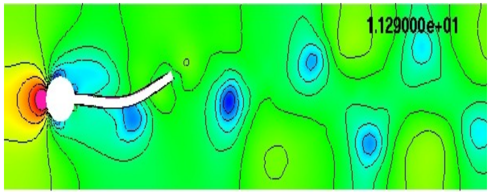
FSI3: large deformations

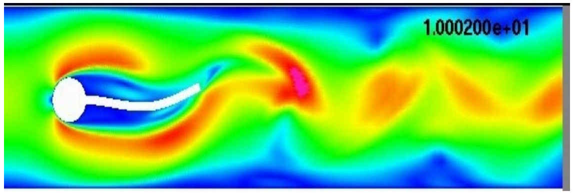


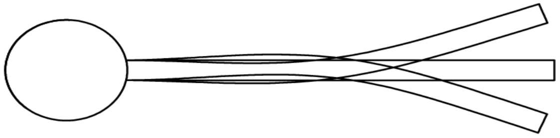












Test	ux of A $[\times 10^{-3}\text{m}]$	ux of A $[\times 10^{-3}\text{m}]$	drag	lift
FSI3	$-2.88 \pm 2.72[10.93]$	$1.47 \pm 34.99[5.46]$	$460.5 \pm 27.74[10.93]$	$2.50 \pm 153.91[5.46]$

Mean ± amplitude[frequency]

FSI Optimization

- The main design aims could be
  - I) Drag/Lift minimization
  - II) Minimal pressure loss
  - III) Minimal nonstationary oscillations
- To reach these aims, we might allow
  1. Boundary control of inflow section
  2. Change of geometry: elastic channel walls or length/thickness of elastic beam
  3. Optimal control of volume forces
- Optimal control of nonstationary flow might be hard for the starting
- Results for the moment are combination of I)-III) with 1)-3).

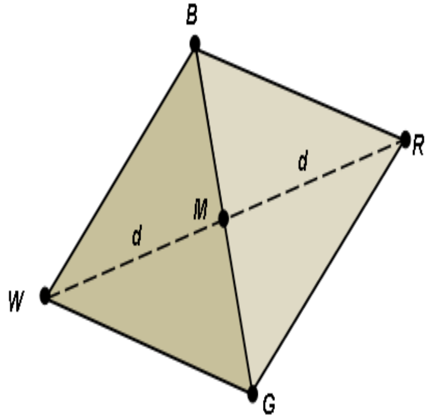
Nelder Mead algorithm

B= Best

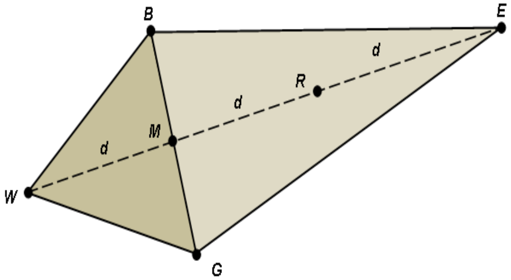
G=Good

W=Worst

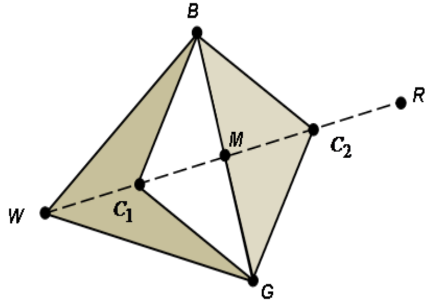
Mid, reflect point



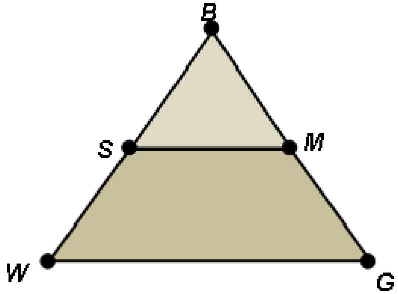
Reflect, extend point



Contraction point

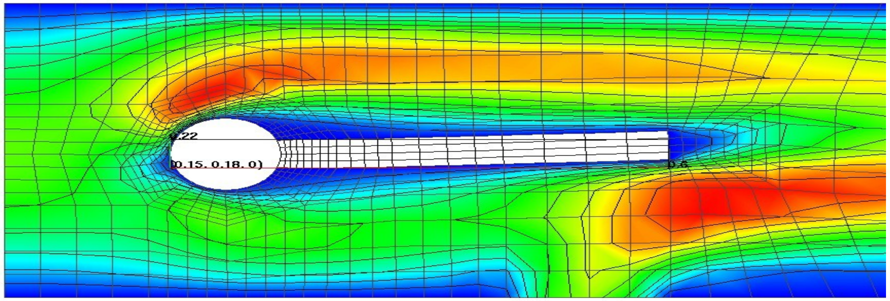


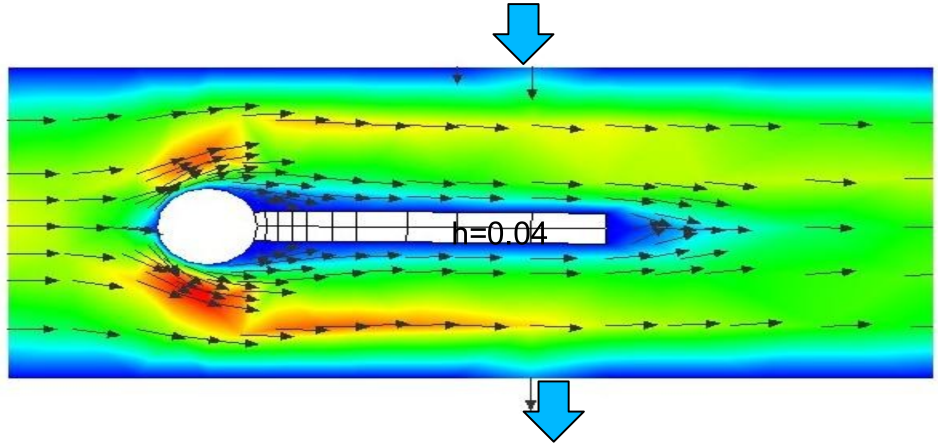
shrink



FSI Optimization

- controlled flow





	ux of A $[\times 10^{-3}\text{m}]$	uy of A $[\times 10^{-3}\text{m}]$	drag	lift
FSI1	0.0227	0.8209	14.295	0.7638

- Aim: $\text{minimize}\left(lift^2 + \alpha V^2\right)$

w.r.t V1, V2.

V1 velocity from top

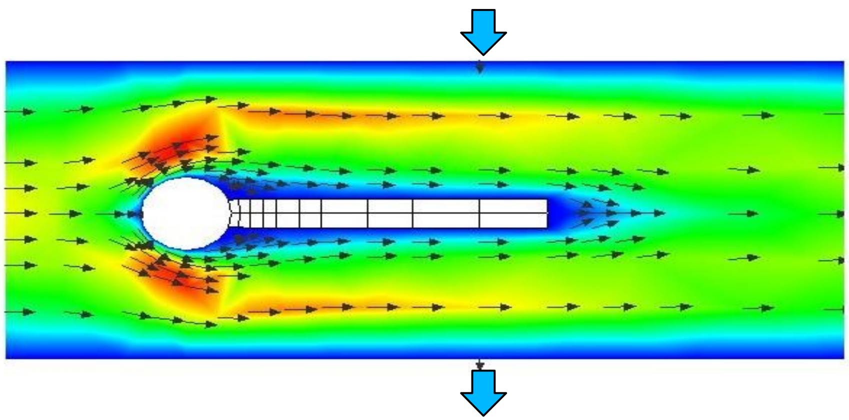
V2 velocity from below

FSI Opt1

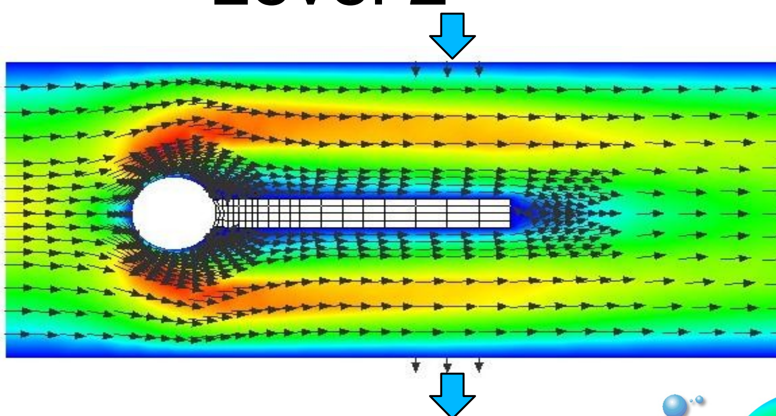
- TESTS for FSI 1 (Boundary control)

	level 1				level 2			
	Iter steps	extreme point	drag	Lift	Iter steps	extreme point	drag	Lift
1e0	57	(3.74e-1,3.88e-1)	1.5471e+01	8.1904e-1	59	(3.66e-1,3.79e-1)	1.5550e+01	7.8497e-1
1e-2 α	60	(1.04e0,1.06e0)	1.5474e+01	2.2684e-2	59	(1.02e0,1.04e0)	1.5553e+01	2.1755e-2
1e-4	73	(1.06e0,1.08e0)	1.5474e+01	2.3092e-4	71	(1.04e0,1.05e0)	1.5553e+01	2.2147e-4
1e-6	81	(1.06e0,1.08e0)	1.5474e+01	2.3096e-6	86	(1.04e0,1.05e0)	1.5553e+01	2.2151e-6

Level 1



Level 2



Summary-Outlook

- Numerical benchmarks (tests, comparisons)
- Optimization test case definition and results

- Biomedical application

like

Hemodynamics(aneurysm)