

A flux-corrected RBF-FD method for convection dominated problems in domains and on manifolds

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**Workshop on
Multiscale modeling and simulations to bridge molecular
and cellular scales**

October 1-5, 2018



technische universität
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fakultät für
mathematik



Outline

- 1 Motivation
- 2 The (meshless) RBF-FD method for surface-PDEs of the reaction-diffusion-convection type
- 3 FCT stabilization for convection dominated problems
- 4 Outlook

Motivation: chemotaxis on a membrane

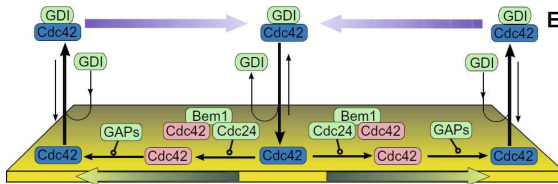


Figure: The membrane-cytoplasmic shuttling of Cdc42 (inactive form, blue; active, pink). Taken from [1].

System of
ODEs

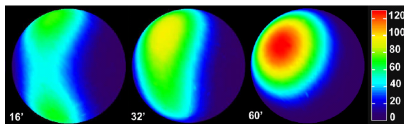


Figure: A 3D view on the surface of a yeast cell shows the distribution of the activated Cdc42. Taken from [1].

Motivation: Γ -applications for chemotaxis models

Charles M. Elliott, Björn Stinner and Chandrasekhar Venkataraman
"Modelling cell motility and chemotaxis with evolving surface finite
elements", J. R. Soc. Interface, published online, 2012.

parametric finite-elements method

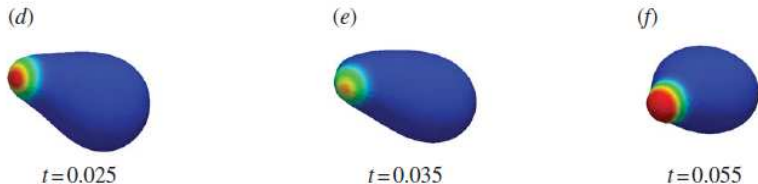
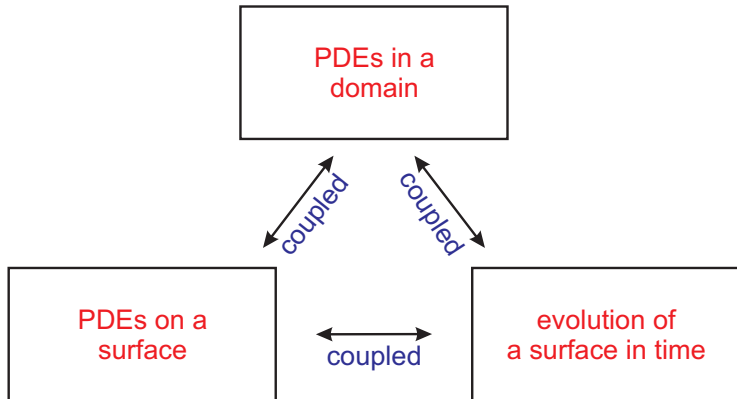


Figure: Migration of cells.

Framework



Framework

$$\frac{\partial^* \rho}{\partial t} + \nabla_{\Gamma(t)} \cdot (w \rho) = D \Delta_{\Gamma(t)} \rho + s(\cdot, \rho), \text{ on } \Gamma(t) \times T,$$

Framework

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where

$$\partial_t^* \rho = \partial_t \rho + \mathbf{v} \cdot \nabla \rho + \rho \nabla_{\Gamma(t)} \cdot \mathbf{v}$$

and

$$\mathbf{v} = V \mathbf{n} + \mathbf{v}_S$$

is the velocity of the surface $\Gamma(t)$.

Framework

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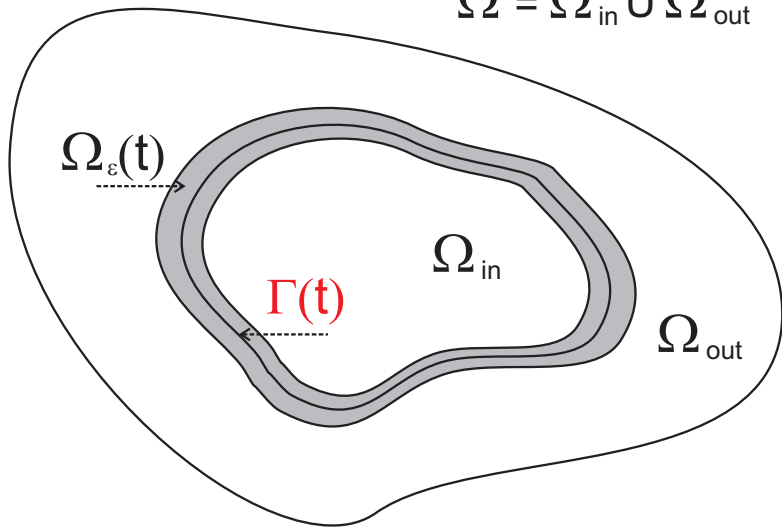
$$\mathbf{v} = V \mathbf{n} + \mathbf{v}_S$$

is the velocity of the surface $\Gamma(t)$.

analytical prescription of $\Gamma = \Gamma(t)$.

Geometrical illustration

$$\Omega = \Omega_{\text{in}} \cup \Omega_{\text{out}}$$



Surface PDE (evolving Γ)

$$\partial_t \rho + \mathbf{v} \cdot \nabla \rho + \rho \nabla_{\Gamma(t)} \cdot \mathbf{v} + \nabla_{\Gamma(t)} \cdot (\mathbf{w} \rho) = D \Delta_{\Gamma(t)} \rho + s(\cdot, \rho)$$

Surface PDE (evolving Γ)

$$\partial_t \rho + \mathbf{v} \cdot \nabla \rho + \rho \nabla_{\Gamma(t)} \cdot \mathbf{v} + \nabla_{\Gamma(t)} \cdot (\mathbf{w} \rho) = D \Delta_{\Gamma(t)} \rho + s(\cdot, \rho)$$

The level-set function:

$$\phi(\mathbf{x}) = \begin{cases} < 0 & \text{if } \mathbf{x} \text{ is inside } \Gamma \\ = 0 & \text{if } \mathbf{x} \in \Gamma \\ > 0 & \text{if } \mathbf{x} \text{ is outside } \Gamma \end{cases}$$

Surface PDE (evolving Γ)

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Then

$$P_\Gamma = I - \frac{\nabla \phi}{\|\nabla \phi\|} \otimes \frac{\nabla \phi}{\|\nabla \phi\|} \text{ is a projection onto } \mathcal{T}_x \Gamma.$$

If ϕ is a signed distance, then $|\nabla \phi| = 1$.

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If ϕ is a signed distance, then $|\nabla \phi| = 1$.

$$\Gamma_{\mathbf{c}}(t) = \{\mathbf{x} : \phi(t, \mathbf{x}) = \mathbf{c}\}.$$

Radial Basis Functions (RBFs)

$$s(x) = \sum_{j=1}^n \lambda_j \varphi(\|x - x_j\|),$$

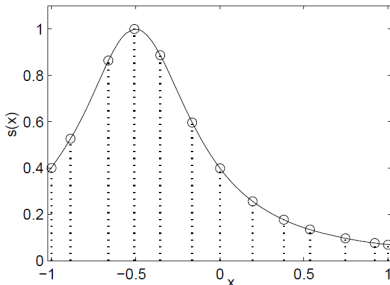
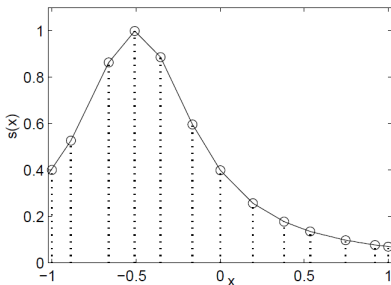
- dimension independent,
- sufficiently smooth,
- easy to find (multiple-) derivatives,
- the corresponding SLE is well-posed, i.e. $\det(\mathbf{A}) \neq 0$,
- meshfree,
- other advantages.

Interpolation with RBFs

Linear vs Multiquadratic RBF:

$$s(x) = \sum_{j=1}^N \lambda_j \|x - x_j\|$$

$$s(x) = \sum_{j=1}^N \lambda_j \sqrt{c^2 + \|x - x_j\|^2}$$

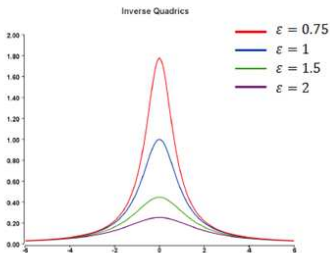
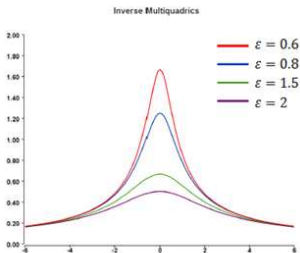
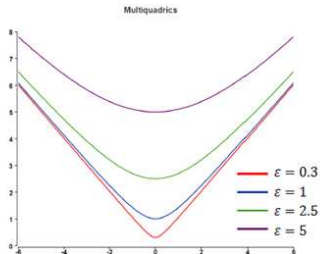
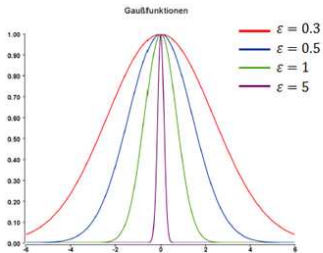


Interpolation with RBFs

Types of RBFs:

Infinitely smooth RBFs	$\varphi(\mathbf{r}) \ (\mathbf{r} \geq 0)$
Gaussian (GA)	$\mathbf{e}^{-(\varepsilon \mathbf{r})^2}$
Inverse quadratic (IQ)	$\frac{1}{1 + (\varepsilon \mathbf{r})^2}$
Inverse multiquadratic (IMQ)	$\frac{1}{\sqrt{1 + (\varepsilon \mathbf{r})^2}}$
Multiquadratic (MQ)	$\sqrt{1 + (\varepsilon \mathbf{r})^2}$
Piecewise smooth RBFs	
Linear	$\mathbf{r}$
Cubic	$\mathbf{r}^3$
Thin plate spline (TPS)	$\mathbf{r}^2 \log \mathbf{r}$

Interpolation with RBFs



Following Wright et al. 2012, 2014:

$$\nabla_{\Gamma(t)} \rho = \begin{pmatrix} (\mathbf{e}^x - n^x \mathbf{n}) \cdot \nabla \\ (\mathbf{e}^y - n^y \mathbf{n}) \cdot \nabla \\ (\mathbf{e}^z - n^z \mathbf{n}) \cdot \nabla \end{pmatrix} \rho = \begin{pmatrix} \mathbf{p}^x \cdot \nabla \\ \mathbf{p}^y \cdot \nabla \\ \mathbf{p}^z \cdot \nabla \end{pmatrix} \rho = \begin{pmatrix} \mathcal{G}^x \\ \mathcal{G}^y \\ \mathcal{G}^z \end{pmatrix} \rho$$

$$\begin{aligned} (\mathcal{G}^x I_\varphi \rho(\mathbf{x}))|_{\mathbf{x}=\mathbf{x}_i} &= \sum_{j=1}^N c_j (\mathcal{G}^x \varphi(\mathbf{r}_j(\mathbf{x})))|_{\mathbf{x}=\mathbf{x}_i} \\ &= \sum_{j=1}^N c_j [((1 - n_i^x n_i^x)(x_i - x_j) - \\ &\quad - n_i^y n_i^y (y_i - y_j) \\ &\quad - n_i^z n_i^z (z_i - z_j)) \frac{\varphi'_{\mathbf{r}}(\mathbf{r}_j(\mathbf{x}_i))}{\mathbf{r}_j(\mathbf{x}_i)}], \end{aligned}$$

where φ is a radial basis function (Gaussian, polyharmonic).

Operator assembly: Laplace-Beltrami

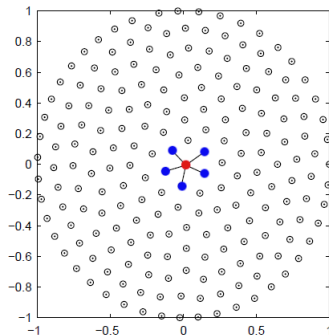
$$\partial_t \rho + \underbrace{\mathbf{v} \cdot \nabla \rho}_{\approx V(t, \Gamma(t)) \rho_h} + \underbrace{\rho \nabla_{\Gamma(t)} \cdot \mathbf{v}}_{\approx G(t, \Gamma(t)) \rho_h} + \underbrace{\nabla_{\Gamma(t)} \cdot (\mathbf{w} \rho)}_{\approx K(t, \mathbf{w}, \Gamma(t)) \rho_h} = \underbrace{D \Delta_{\Gamma(t)} \rho}_{\approx L(t, \Gamma(t)) \rho_h} + s(\cdot, \rho)$$

(generalized) Finite Difference Radial Basis Function (FD-RBF)

Approximation of a (linear) differential operator:

$$\mathcal{L}u = f, \quad \Omega \subset \mathbb{R}^d$$

$$\mathcal{L}u|_{\zeta} \approx \sum_{\zeta_j \in \Xi_{\zeta}} \omega_{\zeta_j} u(\zeta_j), \quad \Xi_{\zeta} = \{\zeta = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_K\}.$$



(generalized) Finite Difference Radial Basis Function (FD-RBF)

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$$\mathcal{L}u = f, \quad \Omega \subset \mathbb{R}^d$$

$$\mathcal{L}u|_{\zeta} = \sum_{j=0}^K \omega_j u(\zeta_j),$$

$$\mathcal{L} \sum_{i=0}^N c_i \varphi(\|\zeta - \zeta_i\|) = \sum_{j=0}^K \omega_j u(\zeta_j),$$

$$\sum_{i=0}^N c_i \mathcal{L} \varphi(\|\zeta - \zeta_i\|) = \sum_{i=0}^N c_i \sum_{j=0}^K \omega_j \varphi(\|\zeta_j - \zeta_i\|)$$

$$\underbrace{\mathcal{L} \varphi(\|\zeta - \zeta_i\|)}_{rhs} = \sum_{j=0}^K \underbrace{\omega_j}_{\omega} \underbrace{\varphi(\|\zeta_j - \zeta_i\|)}_{\Phi}$$

$$\Phi \omega = rhs(\mathcal{L})$$

Operator assembly: Laplace-Beltrami

$$\partial_t \rho + \underbrace{\mathbf{v} \cdot \nabla \rho}_{\approx V(t, \Gamma(t)) \rho_h} + \underbrace{\rho \nabla_{\Gamma(t)} \cdot \mathbf{v}}_{\approx G(t, \Gamma(t)) \rho_h} + \underbrace{\nabla_{\Gamma(t)} \cdot (\mathbf{w} \rho)}_{\approx K(t, \mathbf{w}, \Gamma(t)) \rho_h} = \underbrace{D \Delta_{\Gamma(t)} \rho}_{\approx L(t, \Gamma(t)) \rho_h} + s(\cdot, \rho)$$

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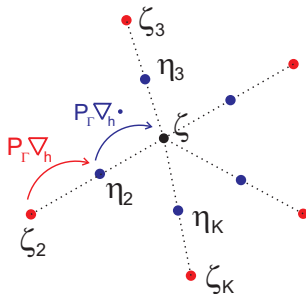
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$$L(t, \Gamma(t)) \rho_h = (P_\Gamma \nabla_h \cdot) (P_\Gamma \nabla_h) \rho_h$$

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where $\Xi = \{\zeta, \zeta_2, \zeta_3, \dots, \zeta_K\}$ and $\Sigma = \{\zeta, \eta_2, \eta_3, \dots, \eta_K\}$.

Operator assembly: convection

$$\partial_t \rho + \underbrace{\mathbf{v} \cdot \nabla \rho}_{\approx V(t, \Gamma(t)) \rho_h} + \underbrace{\rho \nabla_{\Gamma(t)} \cdot \mathbf{v}}_{\approx G(t, \Gamma(t)) \rho_h} + \underbrace{\nabla_{\Gamma(t)} \cdot (\mathbf{w} \rho)}_{\approx K(t, \mathbf{w}, \Gamma(t)) \rho_h} = \underbrace{D \Delta_{\Gamma(t)} \rho}_{\approx L(t, \Gamma(t)) \rho_h} + s(\cdot, \rho)$$

$$V(t, \Gamma(t)) \rho_h = (\mathbf{v} \cdot \nabla_h) \rho_h$$

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$$V(t, \Gamma(t)) \rho_h = (\mathbf{v} \cdot \nabla_h) \rho_h$$

$$(\mathbf{v} \cdot \nabla_h \rho_h)_i = \sum_{j \in \Xi} \sum_{p=1}^d v^p(\zeta_j) \omega_j^{\partial_p} \rho_h(\zeta_j).$$

Operator assembly: convection

$$\partial_t \rho + \underbrace{\mathbf{v} \cdot \nabla \rho}_{\approx V(t, \Gamma(t)) \rho_h} + \underbrace{\rho \nabla_{\Gamma(t)} \cdot \mathbf{v}}_{\approx G(t, \Gamma(t)) \rho_h} + \underbrace{\nabla_{\Gamma(t)} \cdot (\mathbf{w} \rho)}_{\approx K(t, \mathbf{w}, \Gamma(t)) \rho_h} = \underbrace{D \Delta_{\Gamma(t)} \rho}_{\approx L(t, \Gamma(t)) \rho_h} + s(\cdot, \rho)$$

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The operator $K(t, \mathbf{w}, \Gamma(t)) \approx \mathbf{w} \cdot \nabla_{\Gamma(t)} \rho$ is assembled in a similar way.

Operator assembly: $\rho \nabla_{\Gamma(t)} \cdot \mathbf{v}$

$$\partial_t \rho + \underbrace{\mathbf{v} \cdot \nabla \rho}_{\approx V(t, \Gamma(t)) \rho_h} + \underbrace{\rho \nabla_{\Gamma(t)} \cdot \mathbf{v}}_{\approx G(t, \Gamma(t)) \rho_h} + \underbrace{\nabla_{\Gamma(t)} \cdot (\mathbf{w} \rho)}_{\approx K(t, \mathbf{w}, \Gamma(t)) \rho_h} = \underbrace{D \Delta_{\Gamma(t)} \rho}_{\approx L(t, \Gamma(t)) \rho_h} + s(\cdot, \rho)$$

$$G(t, \Gamma(t)) \rho_h = \rho_h (P_\Gamma \nabla_h \cdot \mathbf{v}_h)$$

Operator assembly: $\rho \nabla_{\Gamma(t)} \cdot \mathbf{v}$

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$$(\rho_h P_\Gamma \nabla_h \cdot \mathbf{v}_h)_i = (\rho_h)_i (P_\Gamma \nabla_h \cdot \mathbf{v}_h)_i.$$

Operator assembly: $\rho \nabla_{\Gamma(t)} \cdot \mathbf{v}$

$$\partial_t \rho + \underbrace{\mathbf{v} \cdot \nabla \rho}_{\approx V(t, \Gamma(t)) \rho_h} + \underbrace{\rho \nabla_{\Gamma(t)} \cdot \mathbf{v}}_{\approx G(t, \Gamma(t)) \rho_h} + \underbrace{\nabla_{\Gamma(t)} \cdot (\mathbf{w} \rho)}_{\approx K(t, \mathbf{w}, \Gamma(t)) \rho_h} = \underbrace{D \Delta_{\Gamma(t)} \rho}_{\approx L(t, \Gamma(t)) \rho_h} + s(\cdot, \rho)$$

$$G(t, \Gamma(t)) \rho_h = \rho_h (P_\Gamma \nabla_h \cdot \mathbf{v}_h)$$

$$(\rho_h P_\Gamma \nabla_h \cdot \mathbf{v}_h)_i = (\rho_h)_i (P_\Gamma \nabla_h \cdot \mathbf{v}_h)_i.$$

This discrete operator leads to a diagonal matrix.

Surface PDE (evolving Γ): scheme

$$\partial_t \rho + \underbrace{\mathbf{v} \cdot \nabla \rho}_{\approx V(t, \Gamma(t)) \rho_h} + \underbrace{\rho \nabla_{\Gamma(t)} \cdot \mathbf{v}}_{\approx G(t, \Gamma(t)) \rho_h} + \underbrace{\nabla_{\Gamma(t)} \cdot (\mathbf{w} \rho)}_{\approx K(t, \mathbf{w}, \Gamma(t)) \rho_h} = \underbrace{D \Delta_{\Gamma(t)} \rho}_{\approx L(t, \Gamma(t)) \rho_h} + s(\cdot, \rho)$$

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Given ρ_h^n and $\Delta t = t_{n+1} - t_n$, solve for ρ_h^{n+1}

$$\begin{aligned} \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} &+ \theta (V^{n+1} + G^{n+1} + K^{n+1} - L^{n+1}) \rho_h^{n+1} \\ &= -(1 - \theta) (V^n + G^n + K^n - L^n) \rho_h^n \\ &+ \theta s^{n+1} + (1 - \theta) s^n. \end{aligned}$$

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$$\theta = 1 - \text{Implicit - Euler}$$

$$\theta = \frac{1}{2} - \text{Crank - Nicolson}$$

Numerical tests: example 1

Solve

$$\frac{\partial^* \rho(\mathbf{x}, t)}{\partial t} = D \Delta_{\Gamma(t)} \rho(\mathbf{x}, t) + f(\mathbf{x}, t) \quad \text{on } \Gamma(t),$$

where $\Gamma(t)$ is the zero level-set of

$$\phi(\mathbf{x}, t) = |\mathbf{x}| - 1.0 + \sin(4t)(|\mathbf{x}| - 0.5)(1.5 - |\mathbf{x}|).$$

Analytical solution is

$$\rho(\mathbf{x}, t) = e^{-t/|\mathbf{x}|^2} \frac{x_1}{|\mathbf{x}|}.$$

Numerical tests: example 1

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$$\partial_t \rho + \underbrace{\mathbf{v}_S \cdot \nabla \rho + V \frac{\partial \rho}{\partial \mathbf{n}}}_{=\mathbf{v} \cdot \nabla \rho} \overbrace{-V H \rho + \rho \nabla_{\Gamma} \cdot \mathbf{v}_S}^{=\rho \nabla_{\Gamma(t)} \cdot \mathbf{v}} - D \Delta_{\Gamma} \rho = f$$

Numerical tests: example 1

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Choose the time interval $[0, T = 0.1]$,
 $\Delta t \approx h^2$ (for IE) and $\Delta t \approx h$ (for CN).

Numerical tests: example 1

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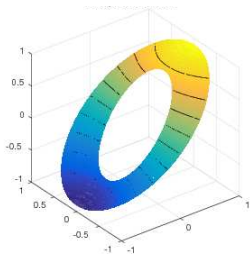
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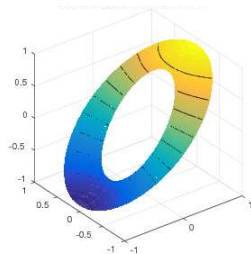
Choose the time interval $[0, T = 0.1]$,
 $\Delta t \approx h^2$ (for IE) and $\Delta t \approx h$ (for CN).

$$l_2(\Omega)\text{-error} = \left(\frac{1}{N} \sum_{i=1}^N |u_{\text{analyt}}(\mathbf{x}_i, T) - u_{\text{num}}(\mathbf{x}_i, T)|^2 \right)^{\frac{1}{2}}$$

Numerical tests: example 1



(a) analyt. solution at level lev 4



(b) num. solution at $T = 0.1$, level 4

Numerical tests: example 1

lev.	d.o.f	num. of time steps	$l_2(\Omega)$ -error	order
Implicit-Euler scheme				
1	30	3	0.035854	–
2	100	10	0.009567	1.905
3	360	40	0.002602	1.878
4	1360	160	0.000748	1.798
5	5280	640	0.000213	1.812
Crank-Nicolson				
1	30	5	0.040218	–
2	100	10	0.09203	2.127
3	360	20	0.002367	1.959
4	1360	40	0.000673	1.814
5	5280	80	0.000192	1.809

Table: Convergence of the Implicit-Euler and Crank-Nicolson schemes.

Numerical tests: example 2

Solve

$$\frac{\partial^* \rho(\mathbf{x}, t)}{\partial t} = D \Delta_{\Gamma(t)} \rho(\mathbf{x}, t) + f(\mathbf{x}, t) \quad \text{on } \Gamma(t),$$

where $\Gamma(t)$ is a zero level-set of

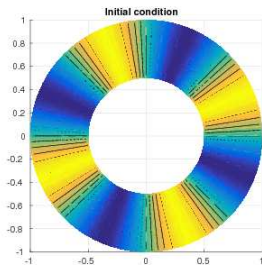
$$\phi(\mathbf{x}, t) = |\mathbf{x}| - 1.0.$$

Initial condition is

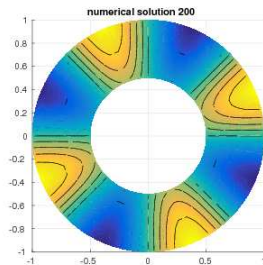
$$\rho_0(\mathbf{x}, t) = \sin(4\gamma), \quad \gamma \in [0, 2\pi].$$

$$\text{atan2}(x_2, x_1) = \begin{cases} \arctan(\frac{x_2}{x_1}) & , x_1 > 0 \\ \arctan(\frac{x_2}{x_1}) + \pi & , x_2 \geq 0, x_1 < 0 \\ \arctan(\frac{x_2}{x_1}) - \pi & , x_2 < 0, x_1 < 0 \\ +\frac{\pi}{2} & , x_2 > 0, x_1 = 0 \end{cases}$$

Numerical tests: example 2



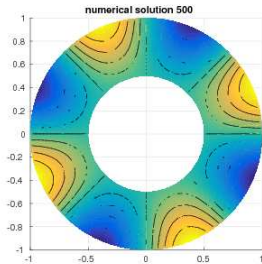
(c) at $t = 0.0$



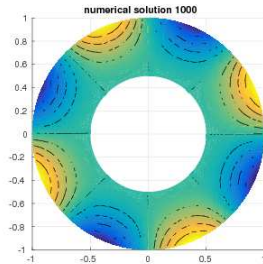
(d) at $t = 0.002$

Figure: Solution at various time instances, $\Delta t = 0.0001$

Numerical tests: example 2



(a) at $t = 0.05$



(b) at $t = 0.1$

Figure: Solution at various time instances, $\Delta t = 0.0001$

Vanishing of ρ_0 occurs at a rate
which depends on the radius of the circle.

Numerical tests: example 3

Solve

$$\frac{\partial^* \rho(\mathbf{x}, t)}{\partial t} = D \Delta_{\Gamma(t)} \rho(\mathbf{x}, t) + f(\mathbf{x}, t) \quad \text{on } \Gamma(t),$$

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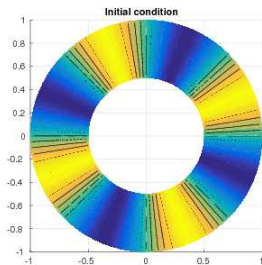
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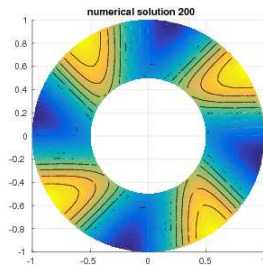
and

$$\mathbf{v} := \mathbf{v}_S = \frac{(-\phi_{x_2}, \phi_{x_1})^T}{|\nabla \phi|}.$$

Numerical tests: example 3



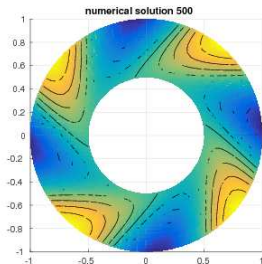
(a) at $t = 0.0$



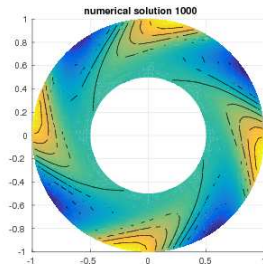
(b) at $t = 0.002$

Figure: Solution at various time instances, $\Delta t = 0.0001$

Numerical tests: example 3



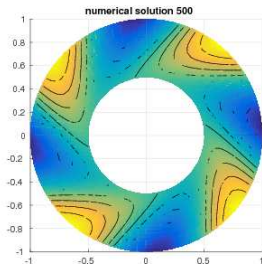
(a) at $t = 0.05$



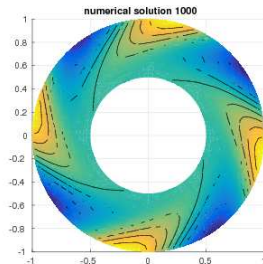
(b) at $t = 0.1$

Figure: Solution at various time instances, $\Delta t = 0.0001$

Numerical tests: example 3



(a) at $t = 0.05$



(b) at $t = 0.1$

Figure: Solution at various time instances, $\Delta t = 0.0001$

PDE on a surface which 'evolves' in the tangential direction.

Numerical tests: anisotropic diffusion

Anisotropic diffusion:

$$-\nabla (\mathbf{A} \nabla u(\mathbf{x})) = f(\mathbf{x}) \quad \text{in} \quad \Omega = [0, 1]^2,$$

where

$$\mathbf{A} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.$$

Special treatment of the RBF-FD stencil is required!!

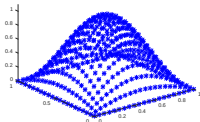


Figure: Numerical solution, $h=1/20$.

Numerical tests: anisotropic diffusion

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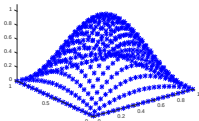


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Bengt Fornberg, Elisabeth Larsson, and Natasha Flyer, "Stable Computations with Gaussian Radial Basis Functions", *SIAM J. Sci. Comput.*, **33**(2), pp. 869-892, (2011).

Elisabeth Larsson, Erik Lehto, Alfa Heryudono, and Bengt Fornberg, "Stable Computation of Differentiation Matrices and Scattered Node Stencils Based on Gaussian Radial Basis Functions", *SIAM J. Sci. Comput.*, **35**(4), pp. A2096-A2119, (2013).

Natasha Flyer, Gregory A. Barnett, Louis J. Wicker, "Enhancing finite differences with radial basis functions: Experiments on the Navier-Stokes equations", *Journal of Computational Physics*, **316**, pp. 39-62, (2016).

Phase-field method

The phase-field method:

$$u_t - \nabla \cdot (\nabla_{\Gamma} u(\boldsymbol{x})) = u(\boldsymbol{x}) + f \quad \text{on } \Gamma \subset \mathbb{R}^d,$$

Phase-field method

The phase-field method:

$$B(\phi) u_t - \nabla \cdot (B(\phi) \nabla u(\mathbf{x})) = B(\phi) (u(\mathbf{x}) + f) \quad \text{in } \Omega_\epsilon \subset \mathbb{R}^d,$$

where

$$\phi(\mathbf{x}) = \frac{1}{2} \left(1.0 - \tanh \left(\frac{2}{5h} (|\mathbf{x}| - 0.3) \right) \right),$$

and

$$B(\phi) = 36 \phi^2 (1 - \phi^2).$$

$$\Gamma = \{\mathbf{x} : |\mathbf{x} - (0.5, 0.5)^T| = 0.3\}$$

Numerical tests: transport equation

Transport equation (the solid body rotation):

$$u_t + \mathbf{v} \cdot \nabla u = 0 \quad \text{in } \Omega = [0, 1]^2.$$

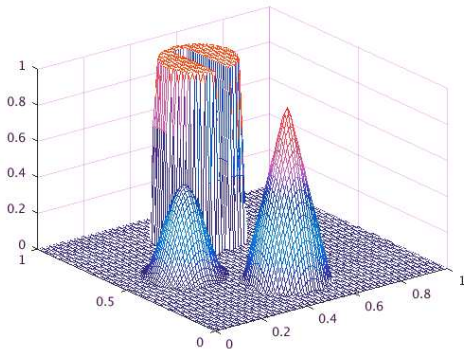


Figure: Initial condition.

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Transport equation (the solid body rotation):

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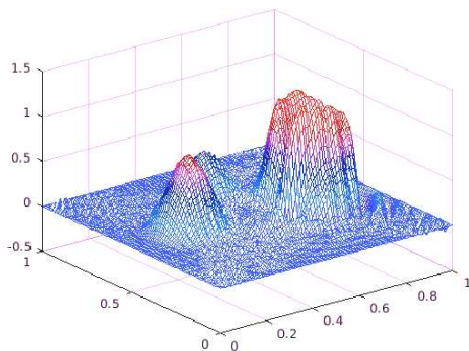


Figure: Non-stabilized solution, rotation angle $\alpha = \frac{2\pi}{3}$, $N = 100 \times 100$.

Numerical tests: transport equation

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... or

$$u_t + \mathbf{v} \cdot \nabla u - \alpha \Delta u = 0.$$

Numerical tests: transport equation

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$$u_t + \mathbf{v} \cdot \nabla u - \alpha \Delta u = 0.$$

... or???

FCT technique

Consider the general transport problem:

$$\underbrace{u_t - \nabla_{\mathcal{P}} \cdot (D \nabla_{\mathcal{P}} u)}_{\approx L} + \overbrace{\mathbf{v} \cdot \nabla_{\mathcal{P}} u}^{\approx K} = g, \quad (1)$$

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$$u_t - \underbrace{\nabla_{\mathcal{P}} \cdot (D \nabla_{\mathcal{P}} u)}_{\approx L} + \underbrace{\mathbf{v} \cdot \nabla_{\mathcal{P}} u}_{\approx K} = g, \quad (1)$$

After the RBF-FD discretization one obtains the following semi-discrete problem:

$$\mathcal{M} \frac{d\mathbf{u}}{dt} = \mathcal{K} \mathbf{u} + g, \quad (2)$$

where $\mathcal{K} = -\mathcal{M} (\mathbf{L} + \mathbf{K})$, and $\mathcal{M} = \text{diag}(m_1, \dots, m_N)$ is a diagonal mass matrix

FCT technique (implicit Euler)

$$[\mathcal{M} - \Delta t \mathcal{K}] u^{n+1} = \mathcal{M} u^n + \Delta t g^n.$$

- 1 Calculate the low-order (overdiffusive) solution u^{Low} :

$$[\mathcal{M} - \Delta t \mathcal{K}^*] u^{\text{Low}} = \mathcal{M} u^n + \Delta t g^n,$$

where $\mathcal{K}^* = \mathcal{K} + \mathcal{D}$ is M-Matrix and $d_{ij} = \max\{-k_{ij}, 0, -k_{ji}\}$, $j \neq i$.

- 2 reconstruct the high-order solution u^{m+1} (i.e. to remove the artificial diffusion where it is possible):

$$\mathcal{M} u^{n+1} = \mathcal{M} u^{\text{Low}} + \Delta t \bar{f}(u^{\text{Low}}, u^n),$$

where $\bar{f}_i = \sum_{j \neq i} \alpha_{ij} f_{ij}$ with flux limiters $\alpha_{ij} \in [0, 1]$ and

$f_{ij} = m_{ij}(\dot{u}_i^{\text{Low}} - \dot{u}_j^{\text{Low}}) + d_{ij}(u_i^{\text{Low}} - u_j^{\text{Low}})$ – antidiffusive fluxes.

$$\dot{u}^{\text{Low}} = \mathcal{M}^{-1} (\mathcal{K}^*) u^{\text{Low}}$$

FCT technique (implicit Euler)

$$[\mathcal{M} - \Delta t \mathcal{K}] u^{n+1} = \mathcal{M} u^n + \Delta t g^n.$$

Flux-corrected transport algorithm:

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FCT: flux limiters for node i

Elimination of the artificial diffusion, which is due to $\mathcal{D} = \{d_{ij}\}$:

- 1 Compute the sums of positive/negative antidiffusive fluxes into node i

$$P_i^+ = \sum_{j \neq i} \max\{0, f_{ij}\}, \quad P_i^- = \sum_{j \neq i} \min\{0, f_{ij}\}.$$

- 2 Compute the distance to a local extremum of the auxiliary solution u

$$Q_i^+ = \max\{0, \max_{j \neq i} (u_j^{n, \text{Low}} - u_i^{n, \text{Low}})\}, \quad Q_i^- = \min\{0, \min_{j \neq i} (u_j^{n, \text{Low}} - u_i^{n, \text{Low}})\}.$$

- 3 Compute the nodal correction factors for the net increment to node i

$$R_i^+ = \begin{cases} \min\left\{1, \frac{m_i Q_i^+}{\Delta t P_i^+}\right\}, & P_i^+ \neq 0 \\ 0, & P_i^+ = 0 \end{cases} \quad R_i^- = \begin{cases} \min\left\{1, \frac{m_i Q_i^-}{\Delta t P_i^-}\right\}, & P_i^- \neq 0 \\ 0, & P_i^- = 0 \end{cases}$$

- 4 Check the sign of the antidiffusive flux and apply the correction factor

$$\alpha_{ij} = \begin{cases} \min\{R_i^+, R_j^-\}, & \text{if } f_{ij} > 0, \\ \min\{R_i^-, R_j^+\}, & \text{otherwise.} \end{cases}$$

FCT technique (θ -scheme)

$$(\mathcal{M} - \theta \Delta t \mathcal{K}) \mathbf{u}^{n+1} = (\mathcal{M} + (1 - \theta) \Delta t \mathcal{K}) \mathbf{u}^n.$$

- 1 Compute the high-order solution $\bar{\mathbf{u}}^{n+1}$ from the algebraic system

$$(\mathcal{M} - \theta \Delta t \mathcal{K}) \bar{\mathbf{u}}^{n+1} = (\mathcal{M} + (1 - \theta) \Delta t \mathcal{K}) \mathbf{u}^n.$$

- 2 Compute the intermediate solution $\mathbf{u}^{n+\theta}$ by the low-order scheme

$$\mathcal{M} \mathbf{u}^{n+\theta} = \left(\mathcal{M} + (1 - \theta) \Delta t \mathcal{K}^L \right) \mathbf{u}^n.$$

- 3 Compute antidiffusive fluxes $f_{ij}(\mathbf{u}^n, \mathbf{u}^{n+\theta})$ and correction factors $\alpha_{ij}^{n+\theta}$ to find the intermediate solution $\mathbf{u}^*$:

$$\mathcal{M} \mathbf{u}^* = \mathcal{M} \mathbf{u}^{n+\theta} + \Delta t \sum_{j \neq i} \alpha_{ij}^{n+\theta} f_{ij}(\mathbf{u}^n, \mathbf{u}^{n+\theta}).$$

- 4 Compute the stabilized high-order solution $\mathbf{u}^{n+1}$:

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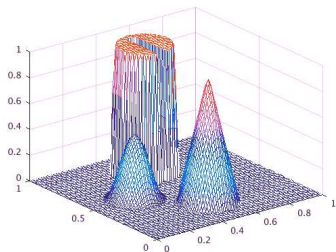
FCT technique: properties

- 1 Positivity preservation
- 2 Mass conservation
- 3 High-order (cf., upwind-like schemes)
- 4 General purpose wrt. dimension of a problem and mesh
- 5 Does not require any artificial parameters
- 6 Applicable for convection dominated problems in domains and on surfaces

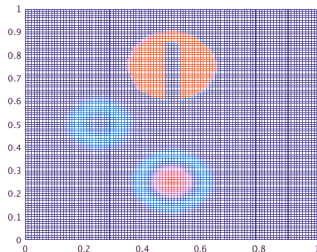
FCT technique: transport equation

The solid body rotation:

$$u_t + \mathbf{v} \cdot \nabla u = 0 \quad \text{in } \Omega = [0, 1]^2.$$



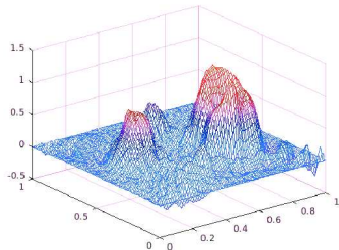
(a) front view



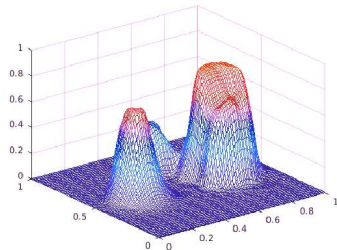
(b) top view

Figure: Initial condition.

FCT technique: transport equation



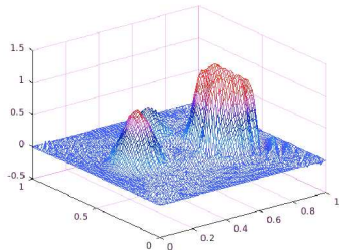
(a) non-stabilized, $N = 50 \times 50$



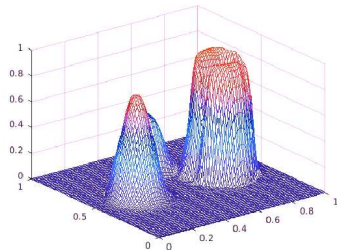
(b) FCT-stabilized, $N = 50 \times 50$

Figure: Numerical solution after rotation by the angle $\alpha = \frac{2\pi}{3}$.

FCT technique: transport equation



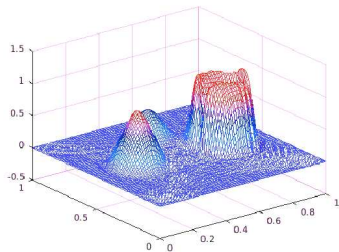
(a) non-stabilized, $N = 100 \times 100$



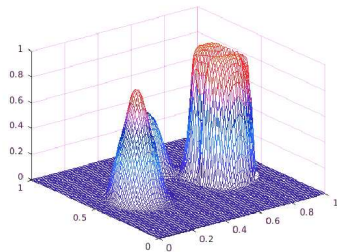
(b) FCT-stabilized, $N = 100 \times 100$

Figure: Numerical solution after rotation by the angle $\alpha = \frac{2\pi}{3}$.

FCT technique: transport equation



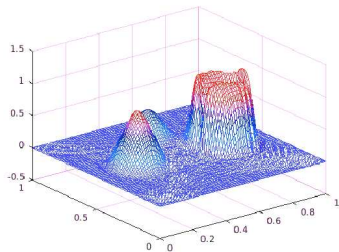
(a) non-stabilized, $N = 200 \times 200$



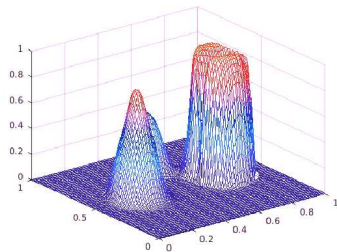
(b) FCT-stabilized, $N = 200 \times 200$

Figure: Numerical solution after rotation by the angle $\alpha = \frac{2\pi}{3}$.

FCT technique: transport equation



(a) non-stabilized, $N = 200 \times 200$



(b) FCT-stabilized, $N = 200 \times 200$

Figure: Numerical solution after rotation by the angle $\alpha = \frac{2\pi}{3}$.

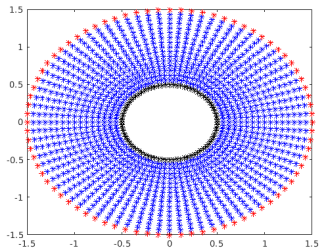
for details see Kuzmin, Sokolov, Davydov, Turek 2018

FCT technique: transport on manifold

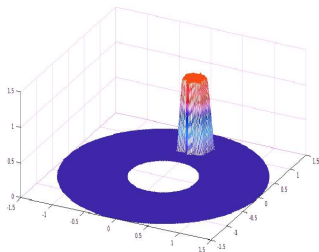
Configuration:

$$u_t + \mathbf{v} \cdot \nabla_{\Gamma} u = 0,$$

where $\phi(\mathbf{x}) = |\mathbf{x}| - 1.0$ and $\mathbf{v} = (-1, 0)^T$.



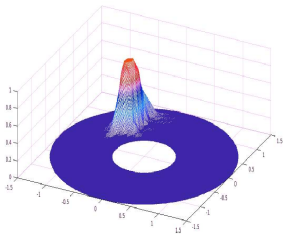
(a) placement of 1377 data points



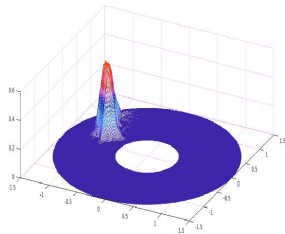
(b) initial condition

Figure: Initial setting.

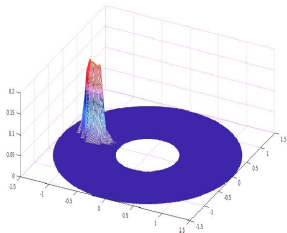
FCT technique: transport on manifold



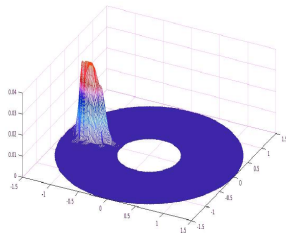
(a) $\alpha = \frac{\pi}{4}$



(b) $\alpha = \frac{\pi}{2}$

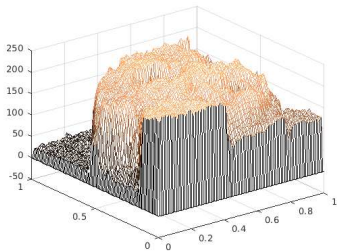


(c) $\alpha = \pi$

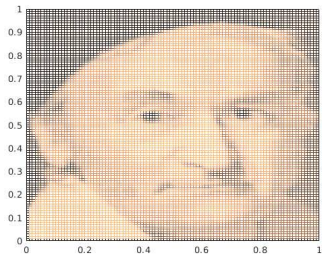


(d) $\alpha = \frac{3\pi}{2}$

Final example



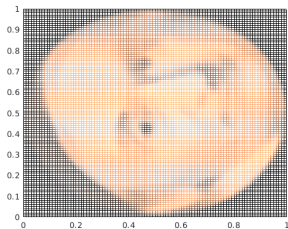
(e) front view



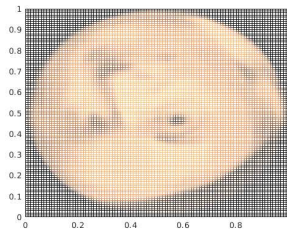
(f) top view

Figure: Initial condition, $N = 200$.

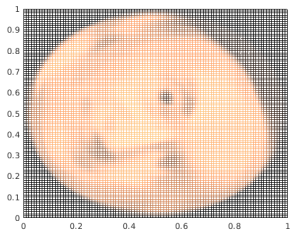
Final example



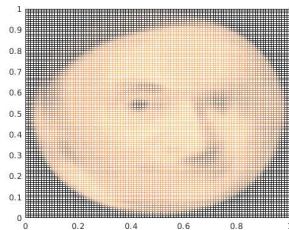
(a) $\alpha = \frac{\pi}{2}$



(b) $\alpha = \pi$



(c) $\alpha = \frac{3\pi}{2}$



(d) $\alpha = 2\pi$

Conclusions

- 1 It is possible to treat PDEs of time-dependent surfaces which evolve both in normal and in tangential directions.
- 2 The method is accurate and robust.
- 3 The RBF-FD nature of the method allows sufficient flexibility while working with meshes.
- 4 The incorporated FCT-stabilization technique make its possible to treat convection dominated problems in domains and on surfaces.

Acknowledgements

- Prof. Dr. Oleg Davydov, University of Giessen
- Prof. Dr. Dmitri Kuzmin, TU Dortmund
- Prof. Dr. Stefan Turek, TU Dortmund

**Thank you very much
for your attention!**

Backup 1

Weights are combined in a form

$$\text{trace}(A B) = \text{sum} \left(\text{sum} \left(A^T \cdot * B \right) \right) .$$