# Accurate and robust finite element solvers for chemotaxis-dominated partial differential equations

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# Outline

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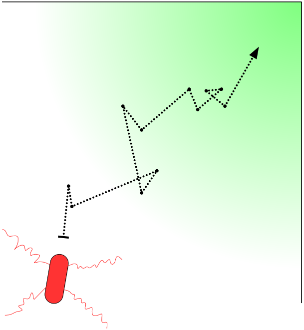
1. Introduction
2. Numerical Treatment
3. Conclusion

# Concept

**Chemotaxis** describes an oriented movement towards or away from regions of higher concentrations of chemical agents and plays a vitally important role in the evolution of many living organisms.

(a) Slime mold, http://dictybase.org



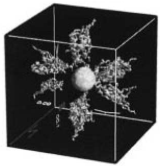
(b) Bacterial chemotaxis

# Where to find chemotaxis?

Certainly, applied mathematicians look for practical benefits of their work. Since chemotaxis plays a key-role for many organisms, plenty applications come into mind.

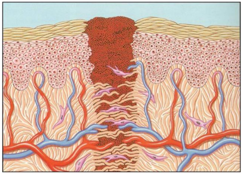
- proliferation of bacteria (not only in petri dishes)
- tumour growth/angiogenesis/haptotaxis
- breeding concerns (insemination of sea urchins)
- immunology/wound healing (production of chemokines at infection sites)

E. Ben-Jacob, http://star.tau.ac.il/∼eshel/image-flow.html



M.A.J. Chaplain, Journal of Neuro-Oncology

C. Pietschmann, MPI



www.surgical-blog.com/wound-healing-what-are-the-phases-of-wound-healing/

# Basic model

It is common to use continuous models → system of partial differential equations (PDE)

A general Keller-Segel model for chemotaxis:

equation for motile species $u$:

$$\frac{\partial u}{\partial t} = \nabla \cdot \big( \underbrace{D \nabla u}_{\text{diffusion}} - \underbrace{u \chi(v) \nabla v}_{\text{chemotaxis}} \big) + \underbrace{u\, g(u)}_{\textit{kinetics}}$$

equation for the chemical agent $v$:

$$\frac{\partial v}{\partial t} = \underbrace{\Delta v}_{\text{diffusion}} - \underbrace{\beta v + u\, s(u)}_{\text{reaction}}$$

(nonlinear) coefficients modeling saturation effects: e.g. $D$, $\chi(v)$, $s(u) \overset{u \to \infty}{\to} 0$

introducing kinetics: e.g. $g(u) = \nu(1-u)$ (logistic)

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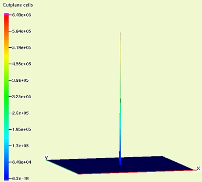
## Blow-up

$$
\begin{aligned}
\partial_t u &= \Delta u - \nabla \cdot \left(u \chi \nabla v\right) \\
\partial_t v &= \Delta v - v + u
\end{aligned}
$$

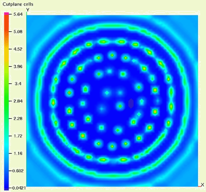



- solution might form singularities
- num. motivated by e.g. [Filbet '06, Chertock & Kurganov '08]
- theor. motivated by e.g.
  [Horstmann & Winkler '04, Tao & Winkler '11]
- theoretical results

| | | |
|---|---|---|
| $\mathbb{R}^1$ | : | all solutions are bounded |
| $\mathbb{R}^2$ | : | blow-up iff $\|\|u_0\|\|_1 > 8\pi/\chi$ |
| $\mathbb{R}^{\geq 3}$ | : | no explicit threshold is known |

## Pattern formation

$$\begin{aligned} \partial_t u &= \Delta u - \nabla \cdot \big(u\chi\nabla v\big) + \nu u(1-u) \\ \partial_t v &= \Delta v - \alpha v + \beta u \end{aligned}$$




- well documented patterns arise (experimental and math.)
- existence of non-trivial steady states
- num. motivated by e.g. [Mimura et al. '93, Chertock & Kurganov '08]
- theor. motivated by e.g. [Myerscough et al. '98, Tyson et al. '99]
- theoretical results $\mathbb{R}^{1,2}$ : unique global weak solution (at least for $\nu \gg 1$)
  $\mathbb{R}^{\geq 3}$ : far less is known

# Numerical challenges



Highly localized solutions with steep gradients reveal particular numerical challenges

- CPU costs
- Memory concerns
- Convenient user interfaces
- Accuracy of discretization
- Robustness with respect to reasonable parameters (e.g. preservation of physical properties)

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"*The purpose of computing is insight, not numbers*"
*Hamming, 1971*

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# Numerical setting



Recapitulate the governing model

$$\begin{aligned} \partial_t u &= \nabla \cdot \big( D\, \nabla u - u\, \chi(v)\, \nabla v \big) + u\, g(u) \\ \partial_t v &= \Delta\, v - \beta\, v + u\, s(u) \end{aligned} \qquad (1)$$

## Discretisation techniques

We (currently) use

- a method of lines approach,
- a canonical, uniform refinement of the spatial grid,
- conform quadrilateral bilinear finite elements (Ritz-Galerkin),
- the standard $\theta-$scheme for temporal discretisation.

# Survey of num. schemes



**Nonlinear Richardson scheme**

$$\boldsymbol{A}(\boldsymbol{x}^n)\,\boldsymbol{x}^n = \boldsymbol{b}^{n-1}$$

$$\rightarrow \boldsymbol{x}^n_{m+1} = \boldsymbol{x}^n_m + \mathcal{P}^{n-1}(\boldsymbol{x}_m)(\boldsymbol{b}^n - \boldsymbol{A}(\boldsymbol{x}_m))$$

**Linearisation via Extrapolation LIN**

$$\boldsymbol{A}(2\boldsymbol{x}^{n-1} - \boldsymbol{x}^{n-2})\,\boldsymbol{x}^n = \boldsymbol{b}^{n-1}$$

**Picard linearisation PIC**

Choose $\mathcal{P}^n(\boldsymbol{x}_m) = \boldsymbol{A}(\boldsymbol{x}_m)$

**(Ordinary) Newtons method NEWT**

Choose $\mathcal{P}^n(\boldsymbol{x}_m) = jac(\boldsymbol{A}(\boldsymbol{x}_m))$

**Decoupled Picard linearisation DEC**

Reduction yields two $n \times n$ systems

$$\boldsymbol{A}_{11}(\boldsymbol{x}_m)\,\boldsymbol{y}_1 = \boldsymbol{res}_1$$

$$\boldsymbol{A}_{22}\,\boldsymbol{y}_2 = -\boldsymbol{A}_{21}\,\boldsymbol{y}_1 + \boldsymbol{res}_2$$

(Strong decoupled Picard linearisation)

**Decoupled Newtons method**

Reduction yields three $n \times n$ systems

$$\boldsymbol{A}_{22}\,\boldsymbol{z}_1 = \boldsymbol{res}_2$$

$$\boldsymbol{C}(\boldsymbol{x}_m)\,\boldsymbol{y}_1 = \boldsymbol{res}_1 - \boldsymbol{A}_{12}\,\boldsymbol{z}_1$$

$$\boldsymbol{A}_{22}\,\boldsymbol{z}_2 = \boldsymbol{A}_{21}\,\boldsymbol{y}_1$$

# Comparison of iteration schemes

Model under consideration: 2D Pattern model on a square
Plots show convergence to num. reference solution

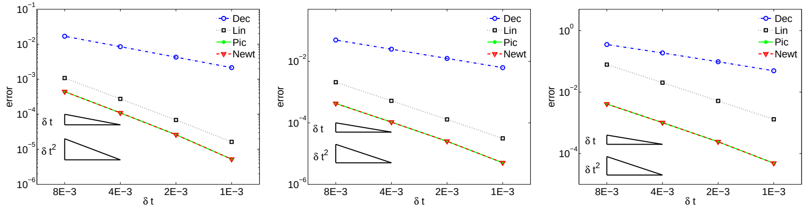


Figure: Convergence with varying chemosensitivities, $\chi = 10, 20, 50$.

- efficiency scales remarkably with $\chi$
- Dec not comparable in terms of $\#\mathrm{IT}$
- Dec and Lin reveal inconsistencies
- Pic vs. Newt strongly emphasized for higher nonlinearity

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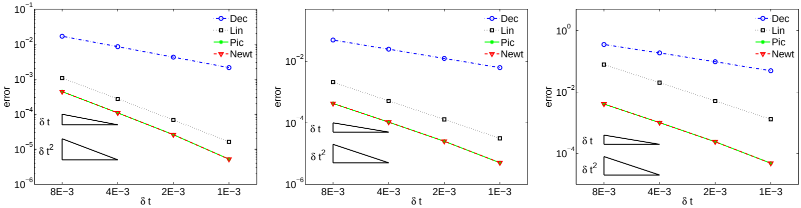


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## Motivation

- standard FEM fail for chemotaxis dominated PDEs
- upwinding aims at 'smoothing-out' instabilities and preserve physical entities ...
- ... at costs of (first order) accuracy

# Stabilisation via AFC

## Motivation

- standard FEM fail for chemotaxis dominated PDEs
- upwinding aims at 'smoothing-out' instabilities and preserve physical entities ...
- ... at costs of (first order) accuracy

REMEDY: merging of the two approaches is the motivation of Algebraic Flux Correction (AFC), [Kuzmin '09]

**Standard Galerkin**
+ second order
− num. artifacts

→ **AFC** ←
+ mixed order
+ failsafe

**Discrete Upwinding**
+ failsafe
− first order

# AFC sketch

**Standard Galerkin**

+ second order
— num. artifacts

**Discrete Upwinding**

+ failsafe
— first order

**AFC**

+ mixed order
+ failsafe

convenient semi-discretized formulation

$$\boldsymbol{M}\partial_t \boldsymbol{u} = \boldsymbol{B}(\boldsymbol{u})\boldsymbol{u}$$

# AFC sketch

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**Discrete Upwinding**

- + failsafe
- − first order

modification with discrete upwinding

$$\boldsymbol{M}^L\partial_t \boldsymbol{u} = (\boldsymbol{B}+\boldsymbol{D})(\boldsymbol{u})\boldsymbol{u} = \widetilde{\boldsymbol{B}}(\boldsymbol{u})\boldsymbol{u}$$

**AFC**

- + mixed order
- + failsafe

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**AFC**

+ mixed order
+ failsafe

correction of over-diffusive fluxes

$$\underbrace{\boldsymbol{M}^L\partial_t \boldsymbol{u} = \widetilde{\boldsymbol{B}}(\boldsymbol{u})\,\boldsymbol{u}}_{\text{low order scheme}} + \underbrace{\bar{\boldsymbol{f}}(\boldsymbol{u})}_{\text{antidiff. flux}}, \quad \bar{\boldsymbol{f}}_i = \sum_{j\neq i} \underbrace{\alpha_{ij}}_{\text{lim. factors}} \boldsymbol{f}_{ij}$$

# Roadmap of explicit AFC

*timestep iteration*

*nonlinear iteration*

**High-order Galerkin scheme**

$$\mathcal{P}^n(\boldsymbol{x}_m)\,\boldsymbol{y} = \boldsymbol{res}^n(\boldsymbol{x}_m)$$

↓

**Low-order scheme**
(pos.-/mass preserving scheme)

$$\widetilde{\mathcal{P}^n}(\boldsymbol{x}_m)\,\boldsymbol{y} = \widetilde{\boldsymbol{res}}^n(\boldsymbol{x}_m)$$

↓

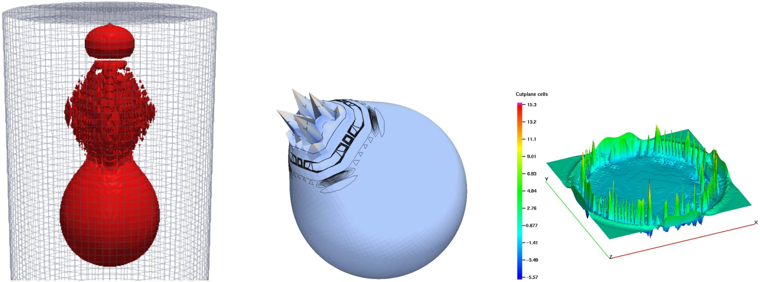
**Explicit flux correction**

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^L + \delta t\,\bar{\boldsymbol{f}}(\boldsymbol{x}^L)$$

# The need of AFC



In generic situations, classical Galerkin schemes provide unphysical results, e.g. severe oscillations, negative densities, loss of characteristic profiles → possibly solver-breakdown



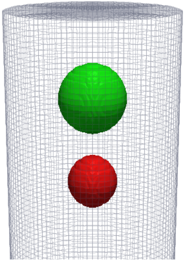

(a) blowup (b) blowup (c) pattern

Figure: Challenges, blowup: Steep gradients, pattern: Maintenance of travelling waves/trailing spots

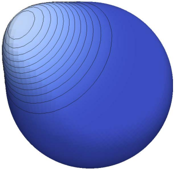
# The need of AFC



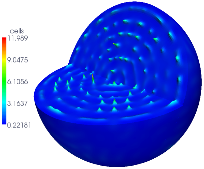
AFC stabilised schemes resolve the problem at costs linear in $\#DOF$ (per $IT_{NL}$)



(a) blowup



(b) blowup




(c) pattern

Figure: No oscillations, no negative values, patterns are recaptured.

# Outline

# Take home messages

- Numerical studies offer validation and reshaping of underlying models and provide quantitative insights into complex dynamics
- Identification of proper num. scheme is a challenging task (user customisation), focus: accuracy, number of iterations, complexity of iterations, stability
- A first glimpse revealed the potential of elaborate solver strategies, particularly in case of large chemotaxis factors $\chi$ or poor (temporal) discretisations
- An AFC-like stabilisation counters chemotaxis-dominated num. artifacts and is highly flexible and inexpensive

# Future interests

## Numerical improvements

- jacobian-free Newton methods
- "global" AFC techniques
- adaptive schemes

## Model considerations

- modeling aspects (variety and comparison of derivations, model assumptions, combination of models)
- possible patterns and steady states ($\rightarrow$ Winkler)
- multi species interactions ($\rightarrow$ Horstmann)
- follow signal transduction pathways up to the cell membranes
  $\rightarrow$ chemotaxis on surfaces (preliminary work by Sokolov exists)

# Appendix: Gallery

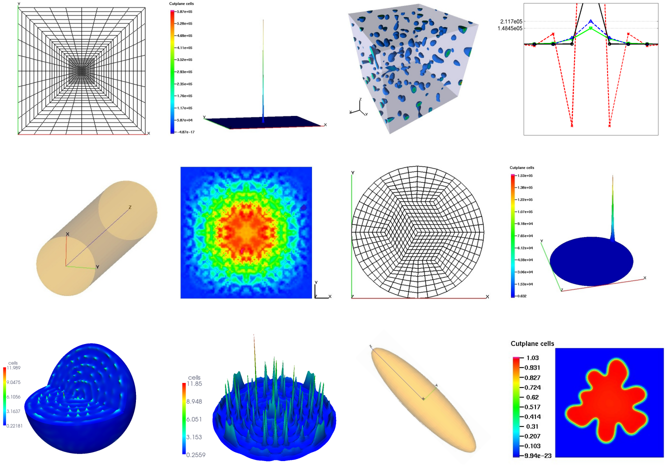



Figure: Some impressions.

- E.E.E. Arenas, A. Stevens, J.J.L. Vel'azquez , Simultaneous finite time blow-up in a two-species model for chemotaxis, Analysis **29**, no.3, pp. 317–338 (2009).
- R.E. Bank, M. Benbourenane, The Hierarchical Basis Multigrid Method For Convection-Diffusion Equations, Method For Convection-Diffusion Equations (1992).
- M. Benzi, G.H. Golub, J. Liesen , *Numerical solution of saddle point problems*, Acta Numerica **14**, No. 1, pp. 1–137 (2005).
- A. Chertock, A. Kurganov, *A second-order positivity preserving central-upwind scheme for chemotaxis and haptotaxis models*, Numer. Math. **111** (2008), pp. 169–205.
- R. Erban, H.G. Othmer , *From individual to collective behavior in bacterial chemotaxis* , SIAM J Appl Math **65**, no.2, pp. 361–391 (2004).

# Literature


- F. Filbet, *A finite volume scheme for the Patlak-Keller-Segel chemotaxis model*, Numer. Math. **104** (2006), no. 4, 457–488.
- S.K. Godunov, *Finite difference method for numerical computation of discontinuous solutions of the equations of fluid dynamics*, Mat. Sb. **47**, pp. 271–306 (1959).
- D. Horstmann, M. Winkler, *Boundedness vs. blow-up in a chemotaxis system*, Journal of Differential Equations **215** (2004), pp. 52–107.
- D. Horstmann , *Generalizing the Keller-Segel Model: Lyapunov Functionals, Steady State Analysis, and Blow-Up Results for Multi-species Chemotaxis Models in the Presence of Attraction and Repulsion Between Competitive Interacting Species*, J. Nonlinear Sci. **21**, pp. 231–270 (2011).
- E. F. Keller, L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol. **26** (1970), pp. 399–415.

# Literature


- D. Kuzmin, M. Möller, S. Turek, *High-resolution FEM-FCT schemes for multidimensional conservation laws*, Computer Meth. Appl. Mech. Engrg., **193**, pp. 4915–4946, (2004).
- D. Kuzmin , *Explicit and implicit FEM-FCT algorithms with flux linearization* , Journal of Computational Physics **228**, no.7, pp. 2517–2534 (2009).
- M. Mimura, T. Tsujikawa, R. Kobayashi, D. Ueyama, *Dynamics of Aggregating Patterns in a Chemotaxis-Diffusion-Growth Model*, Forma, **8** (1993), no. 2, pp. 179–195.
- M. R. Myerscough, P. K. Maini, K. J. Painter, *Pattern formation in a generalized chemotaxis model*, Bull. of Math. Biol. **60** (1998), pp. 1–26.
- H.G. Othmer, S. Dunbar, W. Alt , *Models of dispersal in biological systems* , J. Math. Biol. **26**, pp. 263–298 (1988).

# Literature


- H.G. OTHMER, A. STEVENS , *Aggregation, blowup, and collapse: the ABC of taxis in reinforced random walks* , SIAM J Appl Math **57**, pp. 1044–1081 (1997).
- R. STREHL, A. SOKOLOV, S. TUREK , *Efficient, accurate and flexible Finite Element solvers for Chemotaxis problems* , Comput Math Appl **64**, no.3, pp. 161–390 (2012).
- Y. TAO, M. WINKLER, *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity*, ArXiv e-prints (2011).
- R. TYSON, S. R. LUBKIN, J. D. MURRAY, *A minimal mechanism for bacterial pattern formation*, Proc. R. Soc. Lond. B **266** (1999), pp. 299–304.
- S. T. ZALESAK, *Fully multidimensional Flux-Corrected Transport algorithm for fluids*, Journal of Computational Physics **31** (1979), pp. 335–362.

# Appendix: Modeling aspects

- Macroscopic derivation (e.g. [Keller & Segel '70]) require $\delta t/\delta h^2 = const$. Does it make sense to study chemotaxis-dominating scenarios (from the modeling pov)?
- Understand the motiviations for different microscopic approaches, space vs. velocity jump processes, [Othmer et al. '88, Othmer & Stevens '97]. What are their differences numerically, [Erban & Othmer, '04]? Is it perhaps numerically favorable to consider microscopic models?

return

# Appendix: Multi-Species

- higher coupling requires even more carefully chosen discretisations. Does a segregated approach still provide reasonable/reliable results?
- stabilisation techniques may also be required for Diffusion-like terms (in the presence of conflicts)
- consider (free-boundaries) multiphase-like scenarios. What about single species space posession, e.g. at most one species lives in designated areas?
- conditions for a blow-up are even less analysed, [Horstmann '11, Arenas et al., '09]. The results in [Arenas et al., '09] show that in the radial symmetric setting the blow-up for multispecies has to be simultaneous, but what happens for other initial symmetries?
- What effect does a new approach have to the receptor-based chemotactic sensitivity on the time asymptotic behavior and pattern forming mechanism?

return

# Appendix: PDEs on surfaces

- consider chemotaxis on the individual cell level: the chemo-gradient induces a polarisation of the cell in terms of localisation of membrane receptors $\rightarrow$ chemotaxis on surfaces
- coupling with surface PDEs promote a level-set ansatz (different scaling of grids)
- numerical and mathematical analysis for gradient-based slope limiters for PDEs on surfaces is desired (in context of AFC)

return