Multilevel Pressure Schur Complement

methods for solving the

incompressible Navier–Stokes equations

The case of GLOBAL MPSC

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Demands for the numerical solution of:

Incompressible Navier–Stokes equations

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} \quad , \quad \nabla\cdot\mathbf{u} = 0$$

$+$ boundary and initial values

- stable and accurate discretizations in space and time
- adaptive (error) control strategies
- nonlinear solvers or linearization techniques in time
- **solvers for Stokes/Oseen–like equations**
- linear solvers for scalar subproblems

Robustness **Efficiency**

$\Uparrow$

Implementation/Hardware (???)

Solution of discrete Nav.-St. problems:

- **Nonlinearity** $\longrightarrow$ **'OK' (???)**
- **Saddle point character (!!!)**

$\longrightarrow$ **'large'** degrees of freedom $(10^6 - 10^8)$
$\longrightarrow$ **variable** time scales
$\longrightarrow$ **complex** geometries and locally **anisotropic** meshes
$\longrightarrow$ **changing** 'Reynolds numbers' in time

Multilevel Pressure Schur Complement

1. **Separation** of discretization from **solver (!)** aspects
2. **Classification** and **generalization** of **algorithms**

$\Downarrow$

Understanding of Advantages/Disadvantages

3. **Improvements** and rigorous (!!!) **multigrid** embedding

$\Downarrow$

'Local' and 'Global' MPSC techniques

Example from Numerical Linear Algebra:

1.) Numerical solution of **Poisson/Heat** equation

$\Downarrow$

Discretization in space and time

2.) Solution of discretized problems with **basic iterations**

$\Downarrow$

Jacobi, Gauß-Seidel, SOR, ILU, etc.

3.) **Characterization** of iterative schemes

$\Downarrow$

- Poisson problem ???
- $k \to 0$???
- complex spatial meshes/variable time steps ???
- (nonlinearity ???)

4.) Acceleration as **preconditioner** in Krylov-methods

5.) Acceleration as **smoother** in **multigrid (!)**

‘Pressure Schur Complement’ formulation:

$$\begin{bmatrix} S & kB \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$B^T S^{-1} B p = \frac{1}{k} B^T S^{-1} \mathbf{g} \quad , \quad \mathbf{u} = S^{-1}(\mathbf{g} - kBp)$$

Preconditioned Richardson schemes for scalar **PSC**:

$$\begin{aligned} p^l &= p^{l-1} - C^{-1}(Ap^{l-1} - b) \\ &= p^{l-1} - C^{-1}(B^T S^{-1} B p^{l-1} - \frac{1}{k} B^T S^{-1} \mathbf{g}) \end{aligned}$$

$$\Downarrow$$

Choice of PSC preconditioner C^{-1}:

- $C^{-1} = \sum \alpha_i \tilde{C}_i^{-1}$
- $S := \alpha M + \theta_1 \nu k L + \theta_2 k N(\tilde{\mathbf{U}})$

$$\Downarrow$$

Smoother in **Multigrid** for $B^T S^{-1} B$

<u>‘Global’ MPSC schemes</u>

‘Construct **globally** defined preconditioners for the partial operators by **discrete** and **continuous** arguments and use them in an **additive** way’

- Generalization of **‘Projection methods’, ‘Pressure Correction’, ‘Fractional Step’** und **‘Uzawa’** schemes
- New interpretation of ‘classical’ schemes (**Chorin, Van Kan, Gresho–2**) as special **‘incomplete solvers’ (!!!)**
- More <u>**’efficiency’**</u> through **Multigrid** (MPSC)

<u>‘Local’ MPSC schemes</u>

‘Solve **”exactly”** on ”patches” and perform an outer **Block–Gauß-Seidel/Jacobi** iteration’

- Generalization of **Block-Jacobi/Gauß-Seidel** methods for saddle point problems
- Modification of ‘classical’ schemes (**‘Vanka’**)
- <u>**‘Stabilization’**</u> via **adaptive ‘Patching’**

Global PSC Preconditioner "C^{-1}" ???

Mass **Laplacian** **Convection**

$$B^T S^{-1} B \;=\; B^T\,[M \quad + \quad \theta\nu k L \quad + \quad \theta k N]^{-1}\,B$$

$$\sim\; B^T M^{-1} B + \frac{1}{\theta\nu k} B^T L^{-1} B + \frac{1}{\theta k} B^T N^{-1} B$$

$$[B^T S^{-1} B]^{-1} \;\sim\; [B^T M^{-1} B + \frac{1}{\theta\nu k} B^T L^{-1} B + \frac{1}{\theta k} B^T N^{-1} B]^{-1}$$

$$\sim\; (B^T M^{-1} B)^{-1} + \theta\nu k (B^T L^{-1} B)^{-1} + \theta k (B^T N^{-1} B)^{-1}$$

Reaction **Diffusion** **Convection**

Reactive Preconditioner:

- **explicit** calculation of $\boxed{P := B^T M_l^{-1} B}$
- **P "exact"** discrete PSC preconditioner for $\boxed{k \to 0}$
- nonconforming $\tilde{Q}_1/Q_0$: **5 (2D)/7 (3D) stencil !!!**

OPTIMAL: "exact" for $k \to 0$, "minimal" numerical complexity

Diffusive Preconditioner:

- explicit calculation of $\boxed{B^T L^{-1} B}$ impossible (L^{-1} **full !!!**)
- **However:** $\nabla \cdot \Delta^{-1} \nabla \sim I \implies \boxed{B^T L^{-1} B \sim M_p}$

"OPTIMAL": spectrally optimal for discrete **Stokes** problems

$\Longrightarrow$ (Modified) **Uzawa** scheme !!!

Convective Preconditioner: (???)

- discrete construction $\longrightarrow$ **ILU ???**
- continuous construction $\longrightarrow$ $\boxed{\nabla \cdot (\beta \cdot \nabla)^{-1} \nabla \sim \text{???}}$

Complete "Additive" PSC Preconditioner:

$$\alpha_R P^{-1} + \alpha_D M_p^{-1}$$

with: $\alpha_R = 1$ (or 0), $\alpha_D = \theta\nu k$ + Damping if necessary

Complete PSC Basic Iteration:

$$p^l = p^{l-1} - [\alpha_R P^{-1} + \alpha_D M_p^{-1}] \quad (B^T S^{-1} B p^{l-1} - \frac{1}{k} B^T S^{-1} \mathbf{g})$$

Realization of 1 complete PSC Step:

$S\tilde{\mathbf{u}} = \mathbf{g} - kBp^{l-1}$ (with p^{l-1} given)

$f_p := \frac{1}{k} B^T \tilde{\mathbf{u}}$ $[= \frac{1}{k} B^T S^{-1} \mathbf{g} - B^T S^{-1} B p^{l-1} = \text{Residual}\,(p^{l-1})]$

$Pq = f_p$ ($\sim$ discrete **"Pressure-Poisson"** problem)

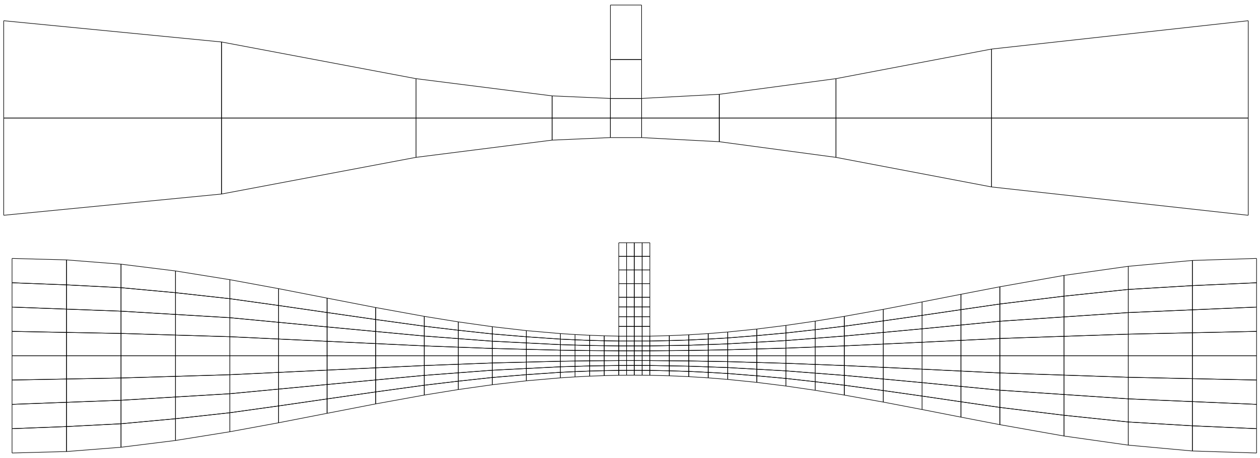
$p^l = p^{l-1} + \alpha_R q + \alpha_D M_p^{-1} f_p$

MPSC Multigrid for $A := B^T S^{-1} B$:

(Standard) Multigrid with **PSC smoother**

Additional "internal" multigrid for S^{-1} and P^{-1} **!!!**

Test: ‘Flow through a Venturi Pipe’



level	vertices	elements	midpoints	total unknowns
1	34	20	53	**126**
3	373	320	692	**1,704**
6	20,897	20,480	41,376	**103,232**

Test configuration:

- $\nu = 10^{-3}$, resulting Reynolds number $\boxed{Re \approx 5,000}$
- (*adaptive*) upwind discretization of the convective terms
- Fractional–step–θ-scheme as time discretization
- **adaptive time step control** for velocity (global, l_2) and **flux** (through upper device **!!!**)

Examined Navier-Stokes solvers:

- **n-Gal** **nonlinear full Galerkin** approach with **nonlinear** solver in each time step and **global MPSC** steps for each Oseen problem
- **l-Gal** **global MPSC** with **2nd order linearization**
- **c-Gal** **global MPSC** with $\mathbf{u}^n \cdot \nabla \mathbf{u}^{n+1}$ **linearization**

"fully coupled in **u** *and p"*

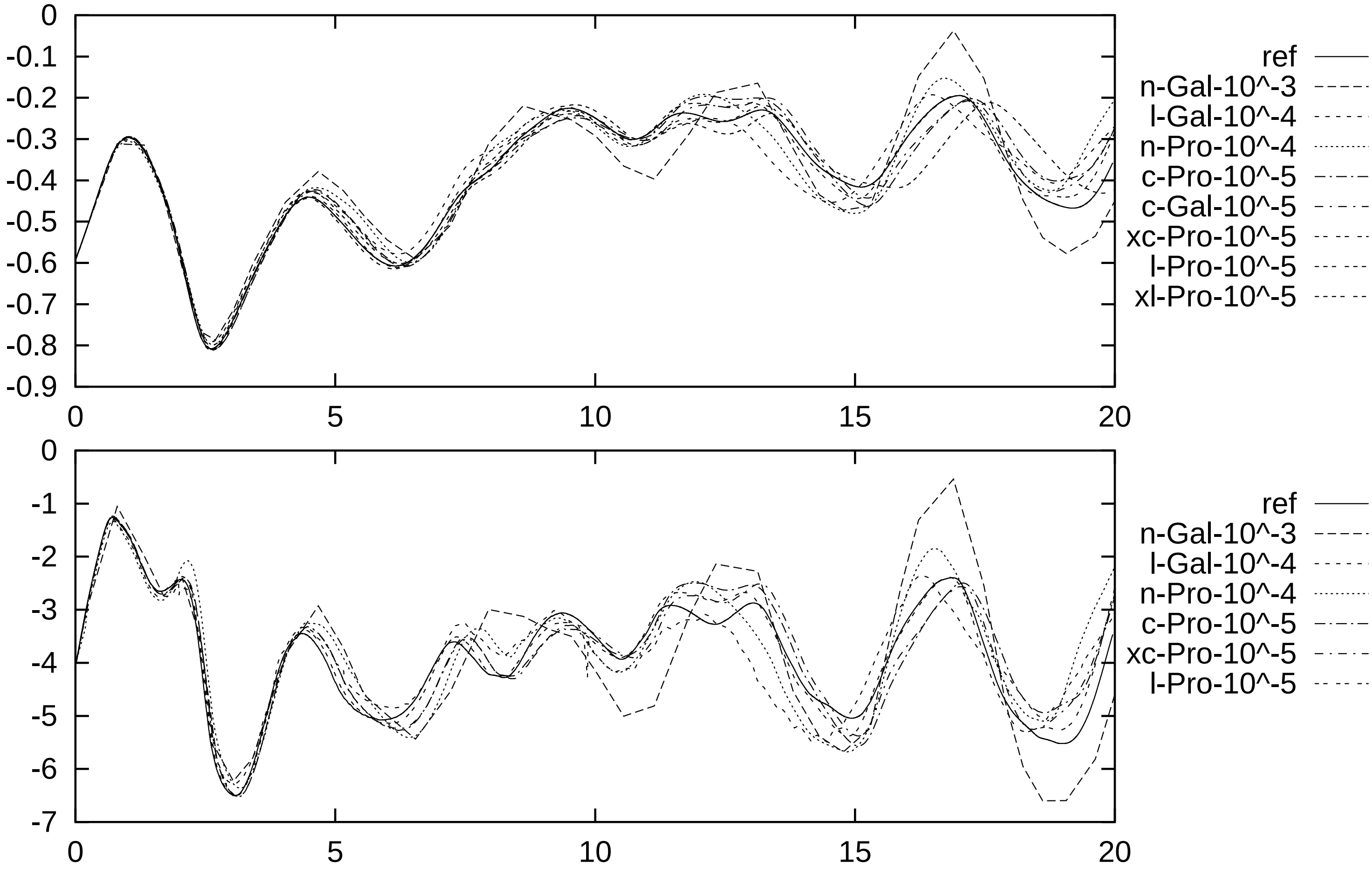
- **n-Pro** *nonlinear projection* with **global PSC** ('Van Kan')
- **l-Pro** *semi-implicit projection* with **linear** extrapolation
- **c-Pro** *semi-implicit projection* with **constant** treatment

"semi-implicit decoupled approach"

- **xl-Pro** *semi-explicit projection* with **linear** extrapolation
- **xc-Pro** *semi-explicit projection* with **constant** treatment

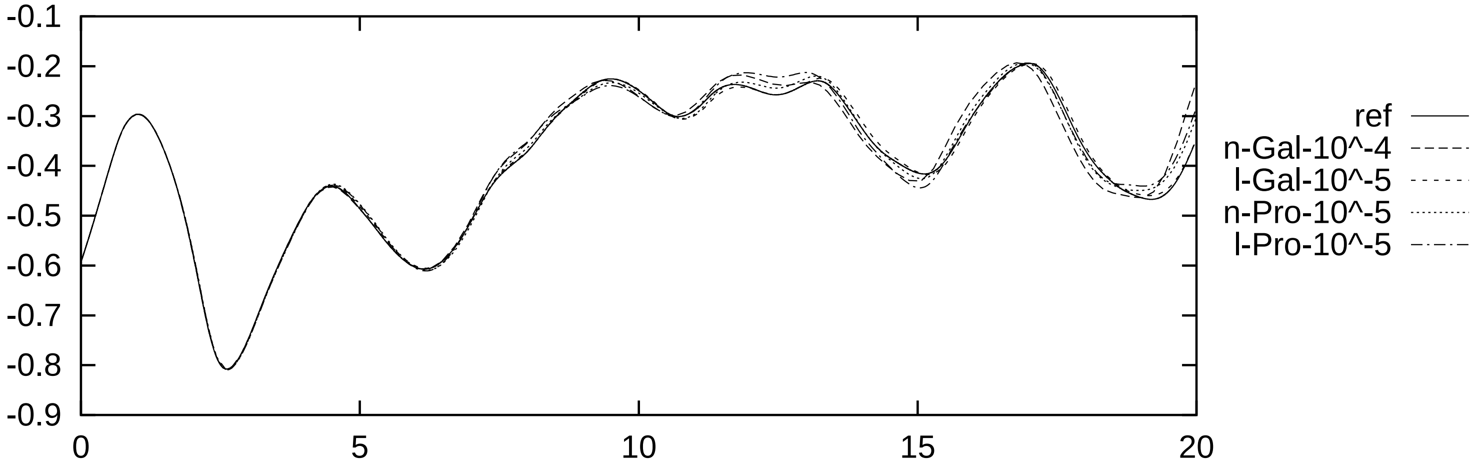
"semi-explicit approach with SPD's only"

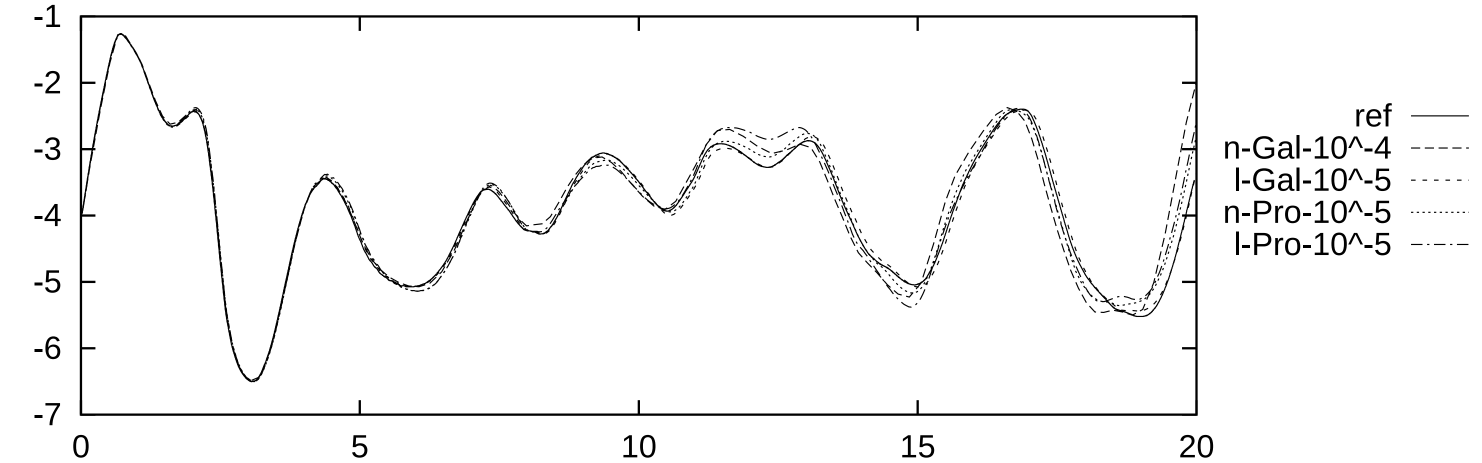
Results: ‘OK’ Ratings



			Flux value			**Pressure value**		
	Meth.	#NT	l_2	mean	peak	l_2	mean	peak
OK	n-Gal	38	15%	1%	3%	19%	1%	1%
	l-Gal	222	11%	2%	1%	13%	0%	3%
	n-Pro	263	11%	2%	1%	14%	3%	2%
	l-Pro	649	4%	0%	0%	5%	0%	0%
	c-Gal	768	6%	1%	0%	8%	1%	0%
	c-Pro	769	8%	2%	0%	10%	2%	0%
	xc-Pro	2730	7%	1%	1%	8%	0%	4%
	xl-Pro	6941	11%	0%	2%	14%	0%	22%

Results: ‘PERF’ Ratings

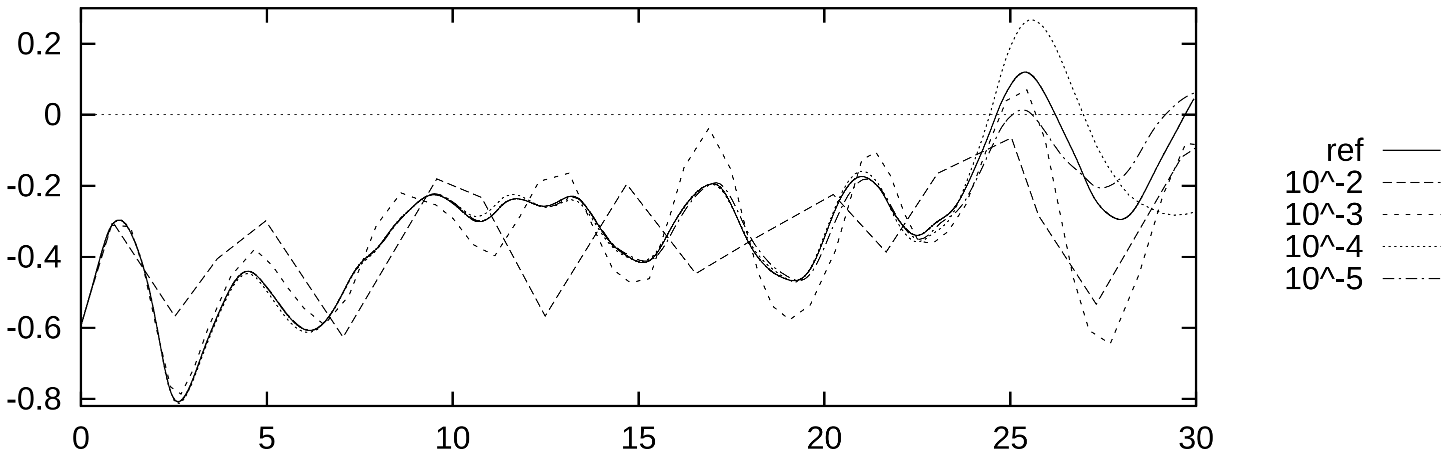


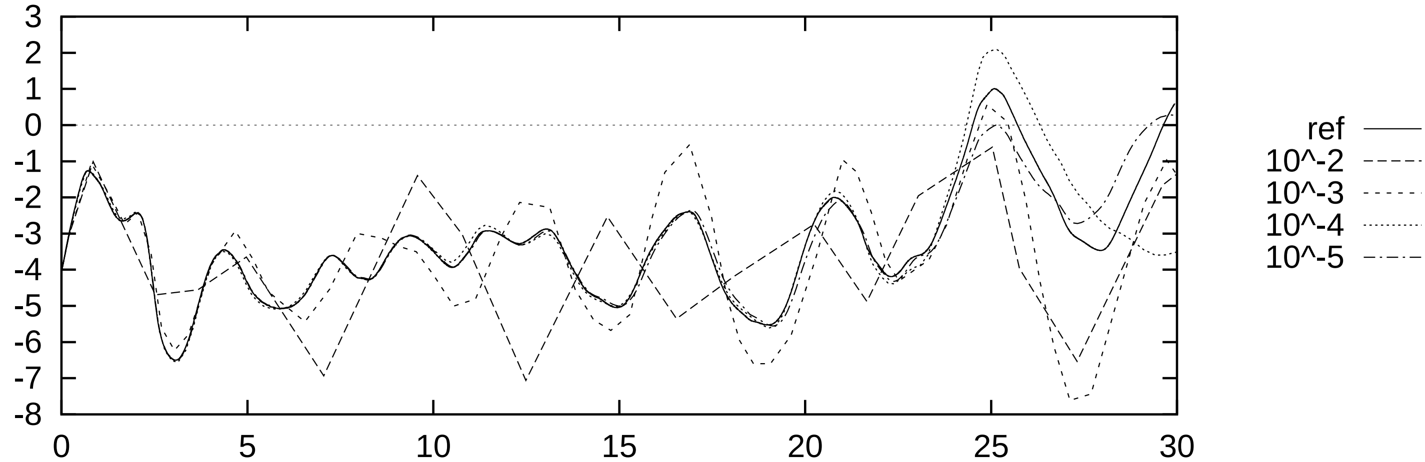


			Flux value			**Pressure value**		
	Meth.	#NT	l_2	mean	peak	l_2	mean	peak
P	n-Gal	117	5%	1%	0%	7%	1%	0%
E	l-Gal	508	1%	0%	0%	1%	0%	0%
R	n-Pro	630	3%	0%	0%	4%	0%	0%
F	l-Pro	649	4%	0%	0%	5%	0%	0%

Results: Comparison of n-Gal

Different tolerances for adaptive control





Conclusions:

- **"fully coupled in u and p"**

  ⇒ **Several PSC** projection steps in time step

  ⇒ Accelerated via **multigrid**

- **"fully nonlinear treatment"**

  ⇒ **Quasi-Newton** fixed point iteration

  ⇒ 2nd order linearization for **anisotropic meshes ???**

- **"Adaptive mixing of** GLOBAL + LOCAL MPSC"

  ⇒ LOCAL MPSC: **"optimal"** for **low** – **medium** Re

  ⇒ GLOBAL MPSC: **"optimal"** for **medium** – **high** Re

$$\Downarrow$$

Combination of higher accuracy of **Galerkin-type** *methods with high speed of* **Projection-type** *solvers is possible !!!*