Efficient Simulation of

Nonsteady Incompressible Flow

Towards adaptive error control

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Incompressible Navier–Stokes equations

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} \quad , \quad \nabla \cdot \mathbf{u} = 0$$

+ boundary and initial values

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Discretization in space (**FEM: UPW/SD**) and time (**CN/FS-** θ)

Treatment of **discrete** systems:

$$\begin{bmatrix} S & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}$$

- Nonlinearity !!!
- Saddle point character !!!
- **Huge** problems $(10^3 10^4 \times 10^4 10^{10}) !!!$

'Classical' Projection Methods:

 \rightarrow engineerical/commercial codes

→ **very "fast"** for fully **nonstationary** flows

Examplary Description as DPM:

Given: $\mathbf{u}(t_n), p(t_n)$ at time t_n Perform: 3 substeps1) $S\tilde{\mathbf{u}}(t_{n+1}) = \tilde{\mathbf{f}} - k\nabla p_{old}$ (Burgers or linearized)2) " Δ_h " $q = \frac{1}{k} \nabla \cdot \tilde{\mathbf{u}}(t_{n+1})$ ('Pressure Poisson Problem')3) $p(t_{n+1}) = p_{old} + \alpha q$, $\mathbf{u}(t_{n+1}) = \tilde{\mathbf{u}}(t_{n+1}) - k\nabla q$ With: $p_{old} = 0 \approx$ Chorin $p_{old} = p(t_n) \approx$ Van Kan

- pressure at **boundaries** ???
- low Re number/large viscosity ν ???
- small viscosity ν + 'large' timesteps ???
- rigorous time step **control** ???

Family of "2nd order" INSE solvers:

- **n-Pro** nonlinear Discrete Projection Method ('Van Kan')
- **l-Pro** DPM with **2nd order linear** extrapolation

 $\longrightarrow \mathbf{u}^{n+1} \approx 2\mathbf{u}^n - \mathbf{u}^{n-1} \Rightarrow (2\mathbf{u}^n - \mathbf{u}^{n-1}) \cdot \nabla \mathbf{u}^{n+1}$

• **xl-Pro** semi-explicit DPM with **linear** extrapolation

"Discrete Projection Methods"

- n-Gal Galerkin approach with
 fully nonlinear solver in each time step and
 "exact" multigrid Oseen solver
- **l-Gal** Oseen with **2nd order linearization**

"fully coupled in \mathbf{u} and p"

Error Indicator for adaptive time steps !!! based on local truncation error $("\Delta t, \Delta t/2")$

lest I: Flow through venturi Pipe



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- "x-variants" restricted due to **CFL condition**
- "l-variants" bad for **large ASPECT RATIOS**

Results I: 'OK' Ratings



			Flux		Pressure	
	Meth.	#NT	l_2	mean	l_2	mean
	n-Gal	38	15%	1%	19%	1%
Ο	l-Gal	222	11%	2%	13%	0%
Κ	n-Pro	263	11%	2%	14%	3%
	l-Pro	649	4%	0%	5%	0%
	xl-Pro	6941	11%	0%	14%	0%

'Smaller' time steps for DPM !!!

Results I: 'PERF' Ratings



			Flux		Pressure	
	Meth.	#NT	l_2	mean	l_2	mean
Р	n-Gal	117	5%	1%	7%	1%
Е	l-Gal	508	1%	0%	1%	0%
R	n-Pro	630	3%	0%	4%	0%
F	l-Pro	649	4%	0%	5%	0%

Different values for $TOL !!! \rightarrow$ Accumulation of local errors!





Control of:

Nonsteady behaviour + low RE???

Larger times steps??? Low Re numbers???

'Interpretation of Projection Methods as special **SOLVERS** for discretized NSE'

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Multilevel Pressure Schur Complement (MPSC)

$$\begin{bmatrix} S & kB \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ 0 \end{bmatrix}$$

1

$$B^T S^{-1} B p = \frac{1}{k} B^T S^{-1} \mathbf{g} \quad , \quad \mathbf{u} = S^{-1} (\mathbf{g} - k B p)$$

Scalar problem for Pressure only !!!



Preconditioned Richardson Scheme for scalar **PSC**:

$$p^{l} = p^{l-1} - C^{-1}(Ap^{l-1} - b)$$

= $p^{l-1} - C^{-1}(B^{T}S^{-1}Bp^{l-1} - \frac{1}{k}B^{T}S^{-1}\mathbf{g})$
 \downarrow

Choice of (global) PSC preconditioner C^{-1} :

•
$$C^{-1} = \sum \alpha_i \tilde{C}_i^{-1} \quad (\approx [B^T S^{-1} B]^{-1})$$

• $S := \alpha M + \theta_1 \nu k L + \theta_2 k N(\tilde{\mathbf{U}})$

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Smoother in **Multigrid** for $B^T S^{-1} B$

 \uparrow

Numerical Linear Algebra

<u>Global PSC Preconditioner</u> " C^{-1} "???



$$B^{T}S^{-1}B = B^{T}[M + \theta\nu kL + \theta kN]^{-1}B$$
$$\sim B^{T}M^{-1}B + \frac{1}{\theta\nu k}B^{T}L^{-1}B + \frac{1}{\theta k}B^{T}N^{-1}B$$

$$[B^{T}S^{-1}B]^{-1} \sim [B^{T}M^{-1}B + \frac{1}{\theta\nu k}B^{T}L^{-1}B + \frac{1}{\theta k}B^{T}N^{-1}B]^{-1}$$

 $\sim (B^{T}M^{-1}B)^{-1} + \theta\nu k(B^{T}L^{-1}B)^{-1} + \theta k(B^{T}N^{-1}B)^{-1}$

Reaction Diffusion Convection







Complete "Additive" PSC Preconditioner:

$$\alpha_R P^{-1} + \alpha_D M_p^{-1}$$

with: $\alpha_R = 1 \text{ (or 0)}, \alpha_D = \theta \nu k + \text{Damping if necessary}$

Complete PSC Basic Iteration:

$$p^{l} = p^{l-1} - \left[\alpha_{R}P^{-1} + \alpha_{D}M_{p}^{-1}\right] \quad \left(B^{T}S^{-1}Bp^{l-1} - \frac{1}{k}B^{T}S^{-1}\mathbf{g}\right)$$

$\begin{array}{l} \label{eq:spectrum} \hline \mathbf{Realization of 1 complete PSC Step:} \\ \hline S \tilde{\mathbf{u}} = \mathbf{g} - kBp^{l-1} & (\text{Burgers problem with } p^{l-1} \text{ given}) \\ \hline f_p := \frac{1}{k}B^T \tilde{\mathbf{u}} & [= \frac{1}{k}B^TS^{-1}\mathbf{g} - B^TS^{-1}Bp^{l-1} = \mathbf{Residual} \ (p^{l-1})] \\ \hline Pq = f_p & (\sim \text{discrete "Pressure-Poisson" problem}) \\ \hline p^l = p^{l-1} + \alpha_R q + \alpha_D M_p^{-1}f_p \end{array}$

$$DPM = 1 \times PSC(\mathbf{no \ conv.})$$
$$VanKan = 1 \times PSC(\alpha_D = 0, \mathbf{no \ conv.})$$

Convective PSC Preconditioners (I):

1) Direct ILU decomposition of $B^T S^{-1} B$

$$\begin{bmatrix} S & B \\ B^T & 0 \end{bmatrix} = \begin{bmatrix} S & 0 \\ B^T & I \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & -B^T S^{-1} B \end{bmatrix} \begin{bmatrix} S & B \\ 0 & I \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$
$$\Rightarrow \quad L_{22}U_{22} = -B^T S^{-1} B$$

- \rightarrow Which sparsity pattern $(B^T S^{-1} B \text{ full } !!!)$ for ILU₂₂ ???
- \rightarrow MG with ILU₂₂ $(B^T S^{-1} B)$???

2) Modified *curl* formulation of convection (Olshanskii)

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = (curl \,\mathbf{u}) \times \mathbf{u} + 1/2\nabla(\mathbf{u}^2)$$

$$\Rightarrow \text{ Replace: } \alpha \mathbf{u} - \nu \Delta \mathbf{u} + (\tilde{\mathbf{U}} \cdot \nabla) \mathbf{u} \text{ by } \alpha \mathbf{u} - \nu \Delta \mathbf{u} + curl \, \tilde{\mathbf{U}} \times \mathbf{u}$$
$$\Rightarrow B^T [\alpha M + M(curl \, \tilde{\mathbf{U}})]^{-1} B \text{ is computable } !!!$$

- \rightarrow Stabilization of the discretization for higher Re numbers ???
- \rightarrow Full Newton matrix ???
- \rightarrow MG for $\tilde{P} := B^T M(\operatorname{curl} \tilde{\mathbf{U}})^{-1} B$???

Convective PSC Preconditioners (Π) :

3) Convective Pressure Form (Elman/Kay/Sylv./Wathen)

Green's tensor for Oseen equations:

$$\nabla\!\cdot\!S^{-1}\nabla\approx\left(\nabla\!\cdot\!\nabla\right)S^{-1}$$

$$\Rightarrow \quad B^T S_u^{-1} B \approx (B^T M_u^{-1} B) \, S_p^{-1} \, M_p = P \, S_p^{-1} \, M_p$$

Problem: $S_p = M_p + \nu k "\Delta_p" + k "N_p" \leftarrow Q_0 ???$

 $(B^T S_u^{-1} B)^{-1} \approx M_p^{-1} S_p P^{-1} \approx M_p^{-1} [M_p + k "N_p"] P^{-1} + \nu k M_p^{-1}$

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$$C^{-1} = [I + kM_p^{-1}(T_u^p N_u T_p^u)]P^{-1} + \nu kM_p^{-1}$$

Preliminary Tests I: Cylinder



Observations:

- small improvements w.r.t. amplitudes of lift/drag
- 'significant' improvements w.r.t. **frequencies** of **lift/drag**

However:

DPM cannot use <u>much larger</u> time steps since even the use of a (almost) spectrally equivalent preconditioner gets useless when only 1 (!!!) incomplete solution step is performed!

Compare with numerical solution of discretized Heat equation resp. Poisson Problem:

'1 preconditoned Richardson step is not sufficient if overall condition number behaves like $O(h^{-\alpha})$ '!!!





<u>Further Observations:</u>

- Significant improvements for approximating the **dynamics** !
- **But:** Which boundary conditions for " N_p " (\rightarrow Oscillations)???
- **But:** Which velocity **u** for " $N_p(\mathbf{u})$ "???
- **But:** Working with complete matrix S_u ???
- But: Application to nonlinear fluids (with $\nu = \nu(\mathbf{u})$ and/or pressure dependent viscosity) or linearized NSE ???

Knowledge of:

Strong relation between Projection Methods as Navier-Stokes schemes and iterative solvers for discretized NSE's in the PSC formulation !

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Rigorous multigrid embeddding towards strongly coupled multigrid solver MPSC is required!

Design of future Flow Solvers:

• Full Galerkin discretization in space/time

- \Rightarrow FEM in space with Upwind/Streamline-Diffusion
- \Rightarrow (Modified) Crank-Nicolson (~ Petrov-Galerkin with cG(1))

• Strongly coupled solvers in u and p

 \Rightarrow PSC preconditioners with (standard) multigrid embedding \Rightarrow fully nonlinear treatment for complex configurations

• Error Estimator instead of Error Indicator !!!

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Combination of higher accuracy of Galerkin-type methods with high speed of Projection-type solvers is possible !!!

Residual-based a posteriori error control in space/time with dual solutions based on **meanvalue** functionals in **space/time** is aimed !?!

Robust solvers for linearized NS problems with large time steps !?!



Control of the **mean value** for the 'wall pressure' in space/time???

Error indicator for adaptive time steps:

Aim: Solution at $t_{n+1} = t_n + k$, starting from t_n $\rightarrow v_k = v_k(t_{n+1}) = \{ \mathbf{u}_k(t_{n+1}), p_k(t_{n+1}) \}$ via time step k $\rightarrow v = v(t_{n+1})$ exact solution at t_{n+1} with: $|||J(v) - J(v_k)||| \sim TOL$ for given (linear) functional $J(\cdot)$ $\rightarrow J(\cdot) \approx \text{velocity/pressure values}, |||J(v) - J(v_k)||| = ||v - v_k||_{l_2}$ $\rightarrow J(\cdot) \approx \text{drag/lift/flux}, ||| \cdot ||| = |\cdot|$ Assumption: $J(v) - J(v_k) \sim k^2 e(v) + O(k^4)$ $\Rightarrow 3 \text{ steps} \times \tilde{k}, 1 \text{ step} \times 3\tilde{k}, \quad REL_{\tilde{k}} := J(v_{3\tilde{k}}) - J(v_{\tilde{k}})$ $\Rightarrow e(v) \sim \frac{J(v_{3\tilde{k}}) - J(v_{\tilde{k}})}{8\tilde{k}^2}, \quad J(v) - J(v_k) \sim \frac{k^2}{8\tilde{k}^2}REL_{\tilde{k}}$ $\Rightarrow \text{ Get new estimation: } k^2 \sim TOL \frac{8 \tilde{k}^2}{||| REL_{\tilde{k}} |||}$ \downarrow

Compare k with \tilde{k} !!!

Stationary Navier–Stokes Equations 1:

'Channel flow around square' with different aspect ratios (AR)

 \rightarrow DPM (~ GLOBAL MPSC) smoothing (0/2 steps)

• $1/\nu = 5$

LEV	$AR = 0.5 \cdot 10^1$	$AR = 0.5 \cdot 10^3$	$AR = 0.5 \cdot 10^5$
3	4 / 6 / 1.5	4/ 6 /1.5	4/ 6 /1.5
4	4 / 6 / 1.5	4/ 6 /1.5	4/ 6 /1.5
5	5/7/1.5	4/~~7~~/1.5	5 / 8 / 1.5

• $1/\nu = 50$

LEV	$AR = 0.5 \cdot 10^1$	$AR = 0.5 \cdot 10^3$	$AR = 0.5 \cdot 10^5$
3	9 / 25 / 3.0	6 / 18 / 3.0	6 / 18 / 3.0
4	6 / 10 / 1.5	6 / 10 / 2.0	6 / 10 / 2.0
5	6 / 6 / 1.0	6 / 9 / 1.5	7 / 9 / 1.5

• $1/\nu = 500$

LEV	$AR = 0.5 \cdot 10^1$	$AR = 0.5 \cdot 10^3$	$AR = 0.5 \cdot 10^5$
3	10 / 76 / 7.5	10 / 74 / 7.5	10 / 73 / 7.5
4	11 / 51 / 4.5	11/49/4.5	11 / 54 / 5.0
5	12 / 31 / 2.5	13 / 35 / 3.0	12 / 31 / 2.5

- Linear + Nonlinear rates "independent" of AR
- Linear rates getting "better" for $h \to 0$, "worse" for $Re \to \infty$
- Nonlinear rates "independent" of h, "worse" for $Re \to \infty$

Stationary Navier–Stokes Equations II:

- \rightarrow LOCAL MPSC (Adapt. Patching), 0/4 Smoothing Steps
 - $1/\nu = 5$

LEV	$AR = 0.5 \cdot 10^1$	$AR = 0.5 \cdot 10^3$	$AR = 0.5 \cdot 10^5$
3	4 / 11 / 3.0	4 / 11 / 3.0	4 / 14 / 3.5
4	5 / 17 / 3.5	5 / 16 / 3.0	5 / 17 / 3.5
5	4 / 16 / 4.0	4/ 16 /4.0	5 / 18 / 3.5

• $1/\nu = 50$

LEV	$AR = 0.5 \cdot 10^1$	$AR = 0.5 \cdot 10^3$	$AR = 0.5 \cdot 10^5$
3	6 / 10 / 1.5	6 / 13 / 2.0	6 / 18 / 3.0
4	5 / 12 / 2.5	5 / 12 / 2.5	5 / 12 / 2.5
5	7 / 21 / 3.0	6 / 20 / 3.5	6 / 20 / 3.5

• $1/\nu = 500$

LEV	$AR = 0.5 \cdot 10^1$	$AR = 0.5 \cdot 10^3$	$AR = 0.5 \cdot 10^5$
3	10 / 19 / 2.0	10 / 19 / 2.0	11/28/2.5
4	11 / 25 / 2.0	10 / 19 / 2.0	11 / 21 / 2.0
5	11 / 22 / 2.0	12 / 22 / 2.0	11 / 22 / 2.0

- Linear rates "independent" of AR, $h \to 0, Re \to \infty$ $(, \Delta t)$
- Nonlinear rates "independent" of AR, h
- Nonlinear rates getting "worse" for $Re \to \infty$

Preliminary Tests I: Cylinder







Project: Perpendicular inflow into pipe

