# Efficient numerical solution of incompressible flow problems

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## Towards adaptive error control

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# Demands for the numerical solution of:

**Incompressible Navier–Stokes equations**

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} \quad , \quad \nabla\cdot\mathbf{u} = 0$$

+ boundary and initial values

- stable and accurate discretizations in space and time

  $\rightarrow$ **FEM (UPW/SD) + CN/FS-$\theta$**
- adaptive (error) control strategies
- nonlinear solvers or linearization techniques in time
- **solvers for Stokes/Oseen–like equations**
- linear solvers for scalar subproblems

$\Downarrow$

# Multilevel Pressure Schur Complement

## Solution of discretized problems:

- **Nonlinearity** $\longrightarrow$ **'OK' (???)**
- **Saddle point character (!!!)**

$\longrightarrow$ **'large'** degrees of freedom $(10^6 - 10^8)$
$\longrightarrow$ **variable** time scales
$\longrightarrow$ **complex** geometries and locally **anisotropic** meshes
$\longrightarrow$ **varying** 'Reynolds numbers' in time

## Multilevel Pressure Schur Complement

1. **Separation** of discretization from **solver (!)** aspects
2. **Classification** and **generalization** of **algorithms**

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**Understanding of Advantages/Disadvantages**

3. **Improvements** and rigorous (!!!) **multigrid** embedding

$\Downarrow$

**'Local' and 'Global' MPSC techniques**

# Example for linear scalar problems:

**1.)** Numerical solution of **Poisson/Heat** equation

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Discretization in space and time

**2.)** Solution of discretized problems with **basic iterations**

$\Downarrow$

Jacobi, Gauß-Seidel, SOR, ILU, etc.

**3.)** **Characterization** of iterative schemes

$\Downarrow$

- Poisson problem ???    $k \to 0$ ???
- complex spatial meshes/variable time steps ???

**4.)** Acceleration as **preconditioner** in Krylov-methods

**5.)** Acceleration as **smoother** in **multigrid (!)**

**'Pressure Schur Complement' formulation:**

$$\begin{bmatrix} S & kB \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$B^T S^{-1} B p = \frac{1}{k} B^T S^{-1} \mathbf{g} \quad , \quad \mathbf{u} = S^{-1}(\mathbf{g} - kBp)$$

**Preconditioned Richardson schemes** for scalar **PSC**:

$$\begin{aligned} p^l &= p^{l-1} - C^{-1}(Ap^{l-1} - b) \\ &= p^{l-1} - C^{-1}(B^T S^{-1} B p^{l-1} - \frac{1}{k} B^T S^{-1} \mathbf{g}) \end{aligned}$$

$$\Downarrow$$

**Choice of PSC preconditioner $C^{-1}$:**

- $C^{-1} = \sum \alpha_i \tilde{C}_i^{-1}$
- $S := \alpha M + \theta_1 \nu k L + \theta_2 k N(\tilde{\mathbf{U}})$

$$\Downarrow$$

Smoother in **Multigrid** for $B^T S^{-1} B$

# <u>'Global' MPSC schemes</u>

'Construct **globally** defined preconditioners for the partial operators by **discrete** and **continuous** arguments and use them in an **additive** way'

- Generalization of **'Projection methods', 'Pressure Correction', 'Fractional Step'** und **'Uzawa'** schemes
- New interpretation of 'classical' schemes (**Chorin, Van Kan, Gresho–2**) as special **'incomplete solvers' (!!!)**
- More <u>**'efficiency'**</u> through **Multigrid** (MPSC)

$\Rightarrow$ Very **efficient** + **high MFLOP/s** rates possible !!!

$\Rightarrow$ Potentially **unstable** + **accuracy** ???

## **'Local' MPSC schemes**

'Solve **"exactly"** on "patches" and perform an outer **Block–Gauß-Seidel/Jacobi** iteration'

- Generalization of **Block-Jacobi/Gauß-Seidel** methods for saddle point problems
- Modification of 'classical' schemes (**'Vanka'**)
- **'Stabilization'** via **adaptive 'Patching'**

⇒ **Rigorous** Mathematics: very **robust** + **accurate** !!!

⇒ NO **efficiency** for fully nonsteady problems ???

⇒ **Low MFLOP/s** rates ???

## Global PSC Preconditioner "$C^{-1}$" ???

**Mass** **Laplacian** **Convection**

$$B^T S^{-1} B = B^T [M + \theta \nu k L + \theta k N]^{-1} B$$

$$\sim B^T M^{-1} B + \frac{1}{\theta \nu k} B^T L^{-1} B + \frac{1}{\theta k} B^T N^{-1} B$$

$$[B^T S^{-1} B]^{-1} \sim [B^T M^{-1} B + \frac{1}{\theta \nu k} B^T L^{-1} B + \frac{1}{\theta k} B^T N^{-1} B]^{-1}$$

$$\sim (B^T M^{-1} B)^{-1} + \theta \nu k (B^T L^{-1} B)^{-1} + \theta k (B^T N^{-1} B)^{-1}$$

**Reaction** **Diffusion** **Convection**

## Reactive Preconditioner:

- **explicit** calculation of $\boxed{P := B^T M_l^{-1} B}$
- **P "exact"** discrete PSC preconditioner for $\boxed{k \to 0}$
- nonconforming $\tilde{Q}_1/Q_0$: **5 (2D)/7 (3D) stencil !!!**

**OPTIMAL**: "exact" for $k \to 0$, "minimal" numerical complexity

## Diffusive Preconditioner:

- explicit calculation of $\boxed{B^T L^{-1} B}$ impossible ($L^{-1}$ **full !!!**)
- **However:** $\nabla \cdot \Delta^{-1} \nabla \sim I \implies \boxed{B^T L^{-1} B \sim M_p}$

**"OPTIMAL"**: spectrally optimal for discrete **Stokes** problems

$\Longrightarrow$ (Modified) **Uzawa** scheme !!!

## Convective Preconditioner: (???)

- discrete construction $\longrightarrow$ **ILU ???**
- continuous construction $\longrightarrow$ $\boxed{\nabla \cdot (\beta \cdot \nabla)^{-1} \nabla \sim \text{???}}$

## Complete "Additive" PSC Preconditioner:

$$\alpha_R P^{-1} + \alpha_D M_p^{-1}$$

**with:** $\alpha_R = 1$ (or 0), $\alpha_D = \theta\nu k$ + Damping if necessary

## Complete PSC Basic Iteration:

$$p^l = p^{l-1} - [\alpha_R P^{-1} + \alpha_D M_p^{-1}] \quad (B^T S^{-1} B p^{l-1} - \frac{1}{k} B^T S^{-1} \mathbf{g})$$

## Realization of 1 complete PSC Step:

$S\tilde{\mathbf{u}} = \mathbf{g} - kBp^{l-1}$ (with $p^{l-1}$ given)

$f_p := \frac{1}{k} B^T \tilde{\mathbf{u}}$ $[= \frac{1}{k} B^T S^{-1} \mathbf{g} - B^T S^{-1} B p^{l-1} =$ **Residual** $(p^{l-1})]$

$Pq = f_p$ ($\sim$ discrete **"Pressure-Poisson"** problem)

$p^l = p^{l-1} + \alpha_R q + \alpha_D M_p^{-1} f_p$

## MPSC Multigrid for $A := B^T S^{-1} B$:

Standard (scalar) Multigrid with **PSC smoother**

Additional "internal" multigrid for $S^{-1}$ and $P^{-1}$ **!!!**

# Stationary Navier–Stokes Equations I:

'Channel flow around square' with different **aspect ratios (AR)**

$\rightarrow$ **GLOBAL MPSC** (MG–Uzawa), **0/2** Smoothing Steps

- $\mathbf{1/\nu = 5}$

| LEV | $AR = 0.5 \cdot 10^1$ | $AR = 0.5 \cdot 10^3$ | $AR = 0.5 \cdot 10^5$ |
|---|---|---|---|
| 3 | **4**/ 6 /**1.5** | **4**/ 6 /**1.5** | **4**/ 6 /**1.5** |
| 4 | **4**/ 6 /**1.5** | **4**/ 6 /**1.5** | **4**/ 6 /**1.5** |
| 5 | **5**/ 7 /**1.5** | **4**/ 7 /**1.5** | **5**/ 8 /**1.5** |

- $\mathbf{1/\nu = 50}$

| LEV | $AR = 0.5 \cdot 10^1$ | $AR = 0.5 \cdot 10^3$ | $AR = 0.5 \cdot 10^5$ |
|---|---|---|---|
| 3 | **9**/ 25 /**3.0** | **6**/ 18 /**3.0** | **6**/ 18 /**3.0** |
| 4 | **6**/ 10 /**1.5** | **6**/ 10 /**2.0** | **6**/ 10 /**2.0** |
| 5 | **6**/ 6 /**1.0** | **6**/ 9 /**1.5** | **7**/ 9 /**1.5** |

- $\mathbf{1/\nu = 500}$

| LEV | $AR = 0.5 \cdot 10^1$ | $AR = 0.5 \cdot 10^3$ | $AR = 0.5 \cdot 10^5$ |
|---|---|---|---|
| 3 | **10**/ 76 /**7.5** | **10**/ 74 /**7.5** | **10**/ 73 /**7.5** |
| 4 | **11**/ 51 /**4.5** | **11**/ 49 /**4.5** | **11**/ 54 /**5.0** |
| 5 | **12**/ 31 /**2.5** | **13**/ 35 /**3.0** | **12**/ 31 /**2.5** |

- **Linear + Nonlinear rates** "independent" of **AR**
- **Linear rates** getting "better" for $h \rightarrow 0$, "worse" for $Re \rightarrow \infty$
- **Nonlinear rates** "independent" of $h$, "worse" for $Re \rightarrow \infty$

# Stationary Navier–Stokes Equations II:

→ **LOCAL MPSC** (Adapt. Patching), **0/4** Smoothing Steps

- $\mathbf{1/\nu = 5}$

| LEV | $AR = 0.5 \cdot 10^1$ | $AR = 0.5 \cdot 10^3$ | $AR = 0.5 \cdot 10^5$ |
|---|---|---|---|
| 3 | **4**/ 11 /**3.0** | **4**/ 11 /**3.0** | **4**/ 14 /**3.5** |
| 4 | **5**/ 17 /**3.5** | **5**/ 16 /**3.0** | **5**/ 17 /**3.5** |
| 5 | **4**/ 16 /**4.0** | **4**/ 16 /**4.0** | **5**/ 18 /**3.5** |

- $\mathbf{1/\nu = 50}$

| LEV | $AR = 0.5 \cdot 10^1$ | $AR = 0.5 \cdot 10^3$ | $AR = 0.5 \cdot 10^5$ |
|---|---|---|---|
| 3 | **6**/ 10 /**1.5** | **6**/ 13 /**2.0** | **6**/ 18 /**3.0** |
| 4 | **5**/ 12 /**2.5** | **5**/ 12 /**2.5** | **5**/ 12 /**2.5** |
| 5 | **7**/ 21 /**3.0** | **6**/ 20 /**3.5** | **6**/ 20 /**3.5** |

- $\mathbf{1/\nu = 500}$

| LEV | $AR = 0.5 \cdot 10^1$ | $AR = 0.5 \cdot 10^3$ | $AR = 0.5 \cdot 10^5$ |
|---|---|---|---|
| 3 | **10**/ 19 /**2.0** | **10**/ 19 /**2.0** | **11**/ 28 /**2.5** |
| 4 | **11**/ 25 /**2.0** | **10**/ 19 /**2.0** | **11**/ 21 /**2.0** |
| 5 | **11**/ 22 /**2.0** | **12**/ 22 /**2.0** | **11**/ 22 /**2.0** |

- **Linear rates** "independent" of **AR**, $h \to 0$, $Re \to \infty$ (, $\Delta t$)
- **Nonlinear rates** "independent" of **AR**, $h$
- **Nonlinear rates** getting "worse" for $Re \to \infty$

# Convection-oriented GLOBAL MPSC:

**1) Direct ILU decomposition of $B^T S^{-1} B$**

$$\begin{bmatrix} S & B \\ B^T & 0 \end{bmatrix} = \begin{bmatrix} S & 0 \\ B^T & I \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & -B^T S^{-1} B \end{bmatrix} \begin{bmatrix} S & B \\ 0 & I \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

$\Rightarrow$ $\boxed{L_{22} U_{22} = -B^T S^{-1} B}$

$\rightarrow$ Which sparsity pattern ($B^T S^{-1} B$ full !!!) for $\text{ILU}_{22}$ ???

$\rightarrow$ MG with $\text{ILU}_{22}(B^T S^{-1} B)$ ???

**2) Modified *curl* formulation of convection** (Olshanskii)

$$\boxed{(\mathbf{u} \cdot \nabla)\mathbf{u} = (curl\, \mathbf{u}) \times \mathbf{u} + 1/2 \nabla(\mathbf{u}^2)}$$

$\Rightarrow$ Replace: $\alpha \mathbf{u} - \nu \Delta \mathbf{u} + (\tilde{\mathbf{U}} \cdot \nabla)\mathbf{u}$ by $\alpha \mathbf{u} - \nu \Delta \mathbf{u} + curl\, \tilde{\mathbf{U}} \times \mathbf{u}$

$\Rightarrow$ $\boxed{B^T [\alpha M + M(curl\, \tilde{\mathbf{U}})]^{-1} B}$ is computable !!!

$\rightarrow$ Stabilization of the discretization for higher Re numbers ???

$\rightarrow$ Full Newton matrix ???

$\rightarrow$ MG for $\tilde{P} := B^T M(curl\, \tilde{\mathbf{U}})^{-1} B$ ???

**3) Doubled Preconditioners** (Elman/Sylvester)

$$\boxed{(B^T XB)(B^T S^{-1}B)^{-1}(B^T YB) \approx B^T XBB^{-1}SB^{-T}B^T YB \approx B^T XSYB}$$

$\Rightarrow$ $(B^T S^{-1}B)^{-1} \approx (B^T XB)^{-1}\,(B^T XSYB)\,(B^T YB)^{-1}$

$\rightarrow$ 2 preconditioners ??? Choice of X,Y: Diagonal matrices !!!

$\rightarrow$ X=Y=I: Which PDE ??? Problems with anisotropic meshes !!!

$\rightarrow$ Stokes ??? $\Delta t \rightarrow 0$ ???

**4) Convective pressure formulation** (Elman/Sylv./Wathen)

'*Momentum equation discretized with pressure functions*'

$$\boxed{S_p = M_p + \nu k\,\text{"}\Delta_p\text{"} + k\,\text{"}N_p\text{"}} \quad \leftarrow \quad Q_0\ ???$$

$\Rightarrow$ Commute: $B^T S_u^{-1} B \approx (B^T M_u^{-1} B)\, S_p^{-1}\, M_p$

$\Rightarrow$ PSC: $M_p^{-1}[M_p + k\,\text{"}N_p\text{"}]P^{-1} \,+\, \nu k M_p^{-1}$

$$\Downarrow \ \text{"}N_p\text{"}\ ???$$

$$\boxed{C^{-1} = [I + kM_p^{-1}\,(T_u^p N_u T_p^u)]P^{-1} \,+\, \nu k M_p^{-1}}$$

$\Rightarrow$ **Natural** extension of MPSC approach !!!

$\Rightarrow$ Very **efficient** w.r.t. computational complexity !!!

## "2nd order accurate" Navier-Stokes solvers:

- **n-Gal** **Galerkin** approach with
  . **fully nonlinear** solver in each time step and
  . **global MPSC** steps for each Oseen problem
- **l-Gal** **global MPSC** with **2nd order linearization**
- **c-Gal** **global MPSC** with $\mathbf{u}^n \cdot \nabla \mathbf{u}^{n+1}$ **linearization**

*"fully coupled in* **u** *and p"*

- **n-Pro** *nonlinear projection* with **global PSC** ('Van Kan')
- **l-Pro** *semi-implicit projection* with **linear** extrapolation
- **c-Pro** *semi-implicit projection* with **constant** treatment

*"semi-implicit decoupled approach"*

- **xl-Pro** *semi-explicit projection* with **linear** extrapolation
- **xc-Pro** *semi-explicit projection* with **constant** treatment

*"semi-explicit approach with SPD's only"*

# Error indicator for adaptive time steps:

**Aim:** Solution at $t_{n+1} = t_n + k$, starting from $t_n$

$\rightarrow \quad v_k = v_k(t_{n+1}) = \{\mathbf{u}_k(t_{n+1}), p_k(t_{n+1})\}$ via time step $k$

$\rightarrow \quad v = v(t_{n+1})$ exact solution at $t_{n+1}$

**with:** $|||J(v) - J(v_k)||| \sim TOL$ for given (linear) functional $J(\cdot)$

$\rightarrow \quad J(\cdot) \approx$ velocity/pressure values, $|||J(v) - J(v_k)||| = \|v - v_k\|_{l_2}$

$\rightarrow \quad J(\cdot) \approx$ drag/lift/flux, $||| \cdot ||| = | \cdot |$

**Assumption:** $J(v) - J(v_k) \sim k^2\, e(v) + O(k^4)$

$\Rightarrow \quad$ 3 steps $\times\ \tilde{k}$, 1 step $\times\ 3\tilde{k}$, $\quad REL_{\tilde{k}} := J(v_{3\tilde{k}}) - J(v_{\tilde{k}})$

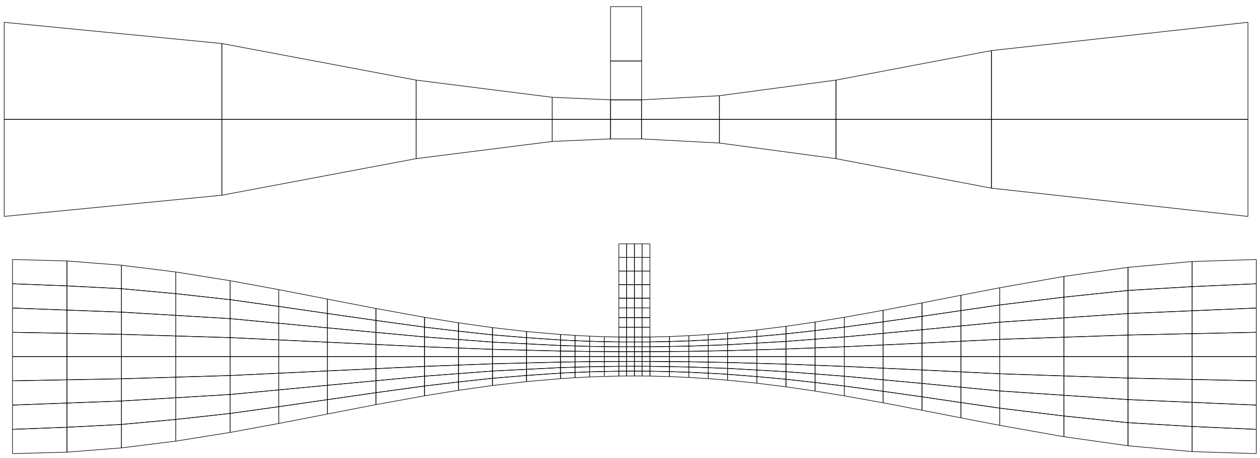
$$\Rightarrow \quad e(v) \sim \frac{J(v_{3\tilde{k}}) - J(v_{\tilde{k}})}{8\tilde{k}^2}, \quad J(v) - J(v_k) \sim \frac{k^2}{8\tilde{k}^2} REL_{\tilde{k}}$$

$\Rightarrow \quad$ Get new estimation: $k^2 \sim TOL \dfrac{8\,\tilde{k}^2}{|||\, REL_{\tilde{k}}|||}$

$$\Downarrow$$

**Compare $k$ with $\tilde{k}$ !!!**

# Test I: 'Flow through a Venturi Pipe'

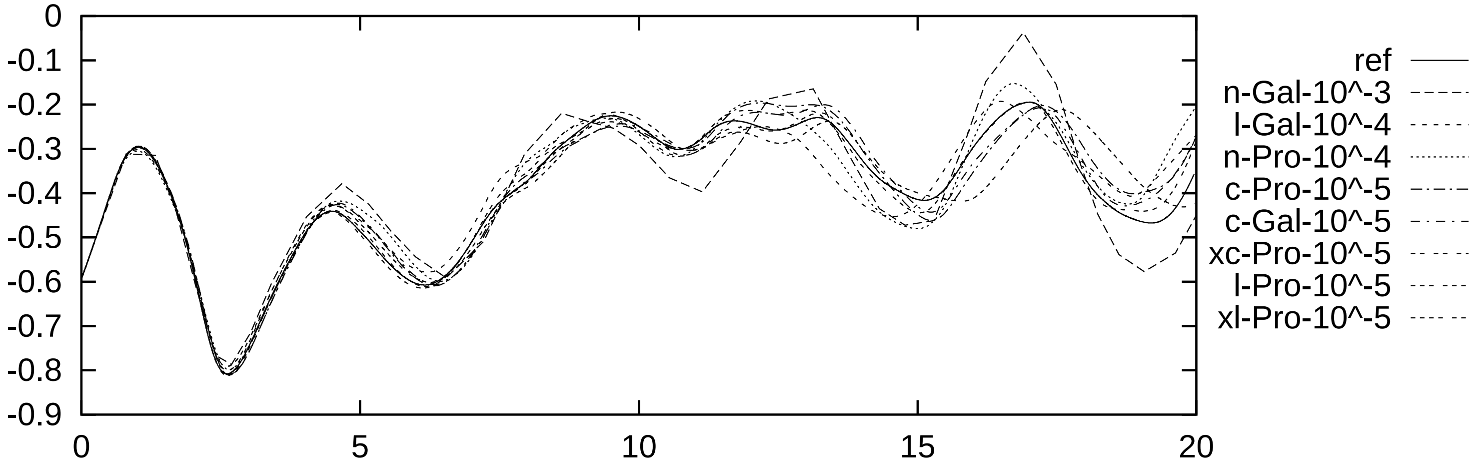


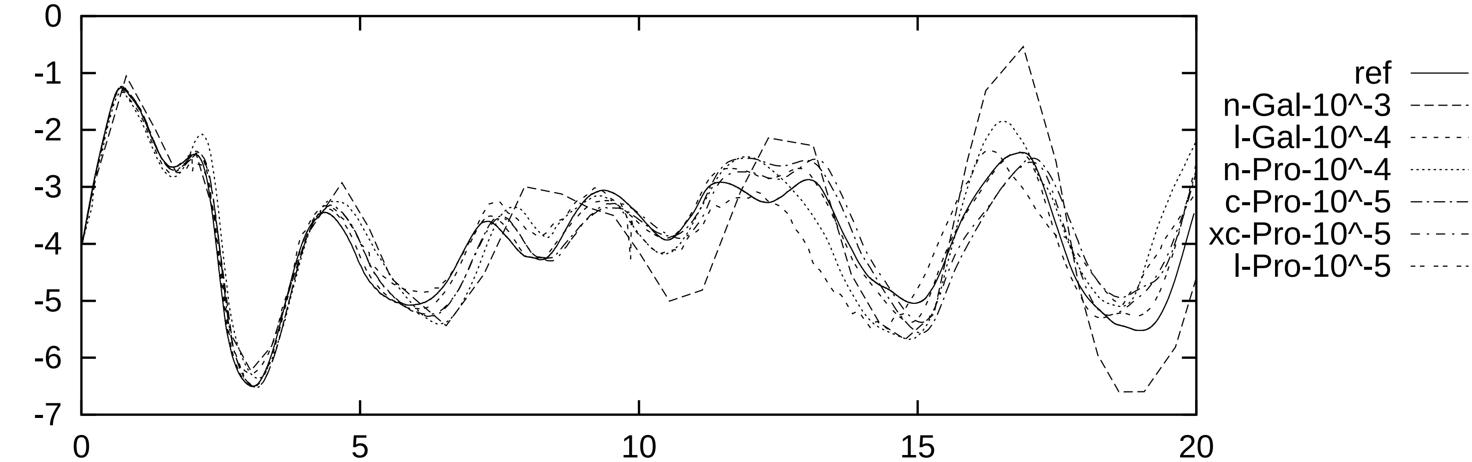
| level | vertices | elements | midpoints | total unknowns |
|---|---|---|---|---|
| 1 | 34 | 20 | 53 | 126 |
| 3 | 373 | 320 | 692 | 1,704 |
| **6** | **20,897** | **20,480** | **41,376** | **103,232** |

## Test configuration:

- $\nu = 10^{-3}$, resulting Reynolds number $Re \approx 5,000$
- (*adaptive*) upwind discretization of the convective terms
- Fractional–step–$\theta$-scheme as time discretization
- adaptive time step control for **velocity** (global, $l_2$) and **flux** (through upper device **!!!**)

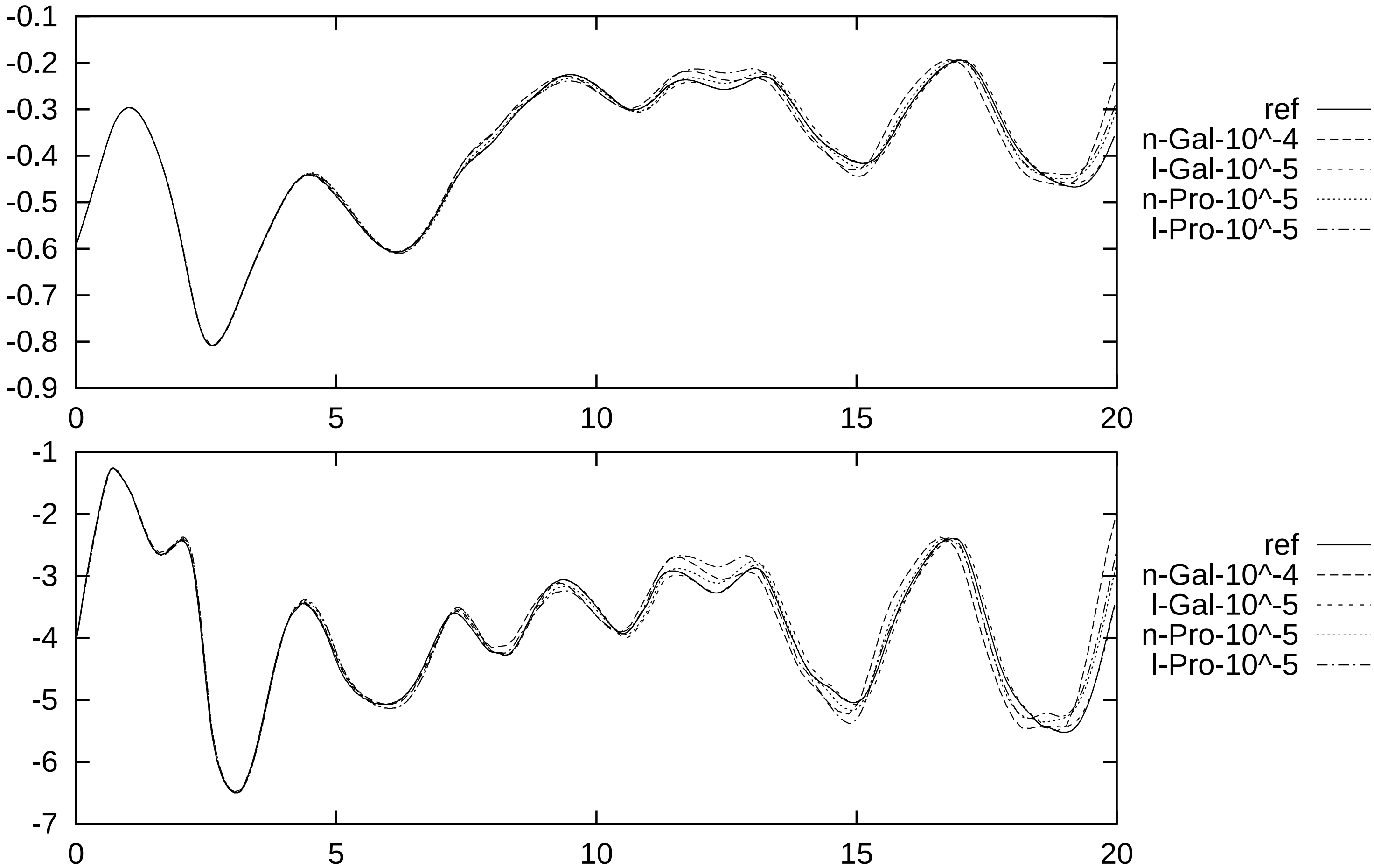
# Results I: 'OK' Ratings




| | Meth. | #NT | Flux $l_2$ | Flux mean | Pressure $l_2$ | Pressure mean |
|---|---|---|---|---|---|---|
| | n-Gal | **38** | 15% | 1% | 19% | 1% |
| | l-Gal | **222** | 11% | 2% | 13% | 0% |
| | n-Pro | **263** | 11% | 2% | 14% | 3% |
| O | l-Pro | **649** | 4% | 0% | 5% | 0% |
| K | c-Gal | **768** | 6% | 1% | 8% | 1% |
| | c-Pro | **769** | 8% | 2% | 10% | 2% |
| | xc-Pro | **2730** | 7% | 1% | 8% | 0% |
| | xl-Pro | **6941** | 11% | 0% | 14% | 0% |

# Results I: 'PERF' Ratings




| | Meth. | #NT | Flux $l_2$ | Flux mean | Pressure $l_2$ | Pressure mean |
|---|---|---|---|---|---|---|
| P | n-Gal | **117** | 5% | 1% | 7% | 1% |
| E | l-Gal | **508** | 1% | 0% | 1% | 0% |
| R | n-Pro | **630** | 3% | 0% | 4% | 0% |
| F | l-Pro | **649** | 4% | 0% | 5% | 0% |

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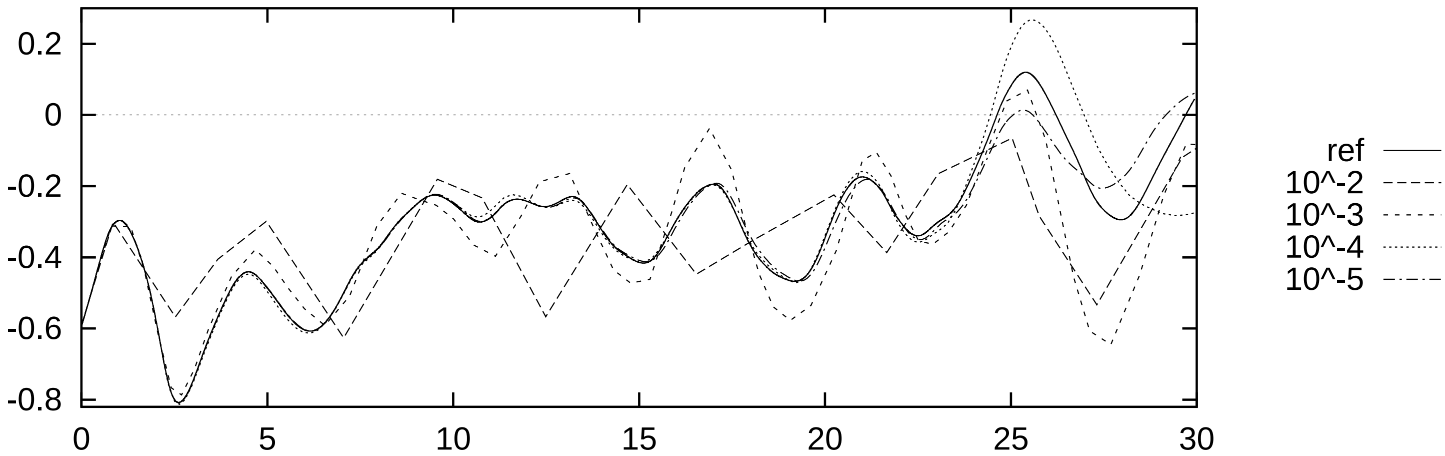
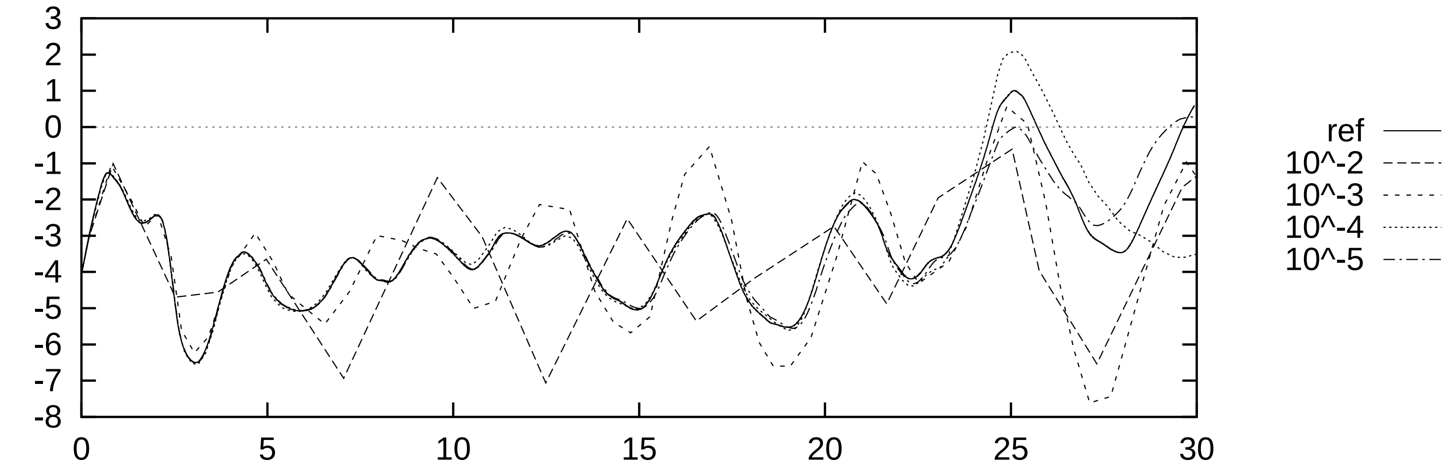
Different values for $TOL$ !!!

# Results I: Comparison of n-Gal




## General Results:

- "c–variants" only **first order accurate** in time
- "x–variants" restricted due to **CFL condition**
- "l–variants" bad for **large ASPECT RATIOS**

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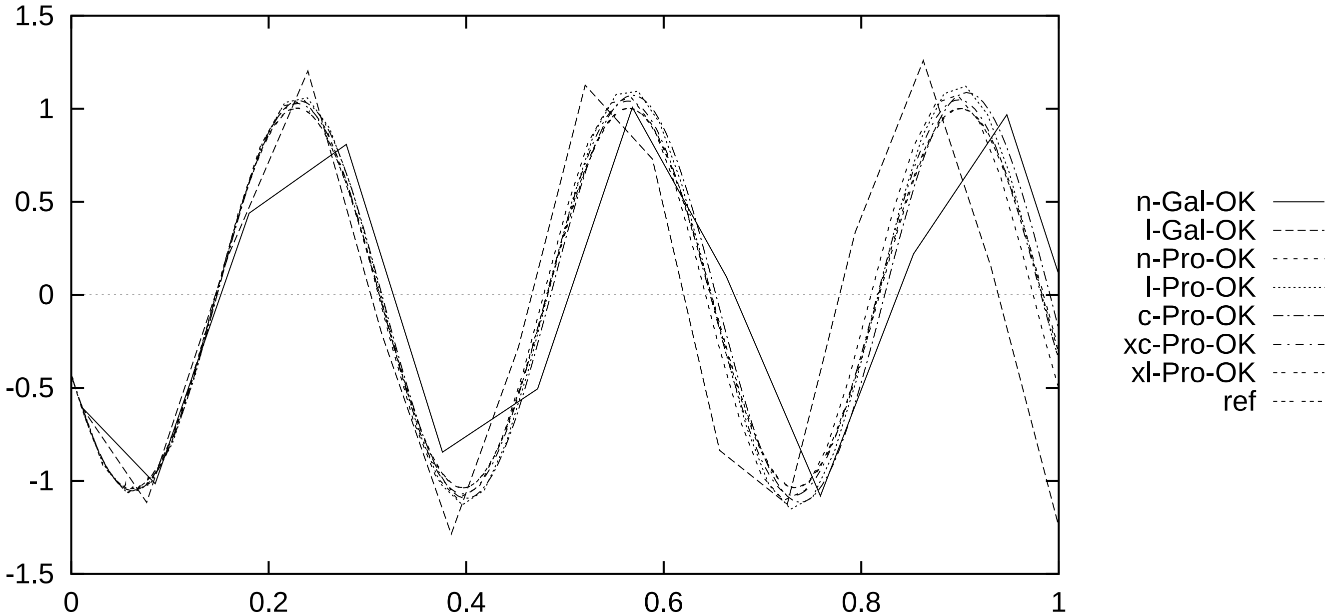
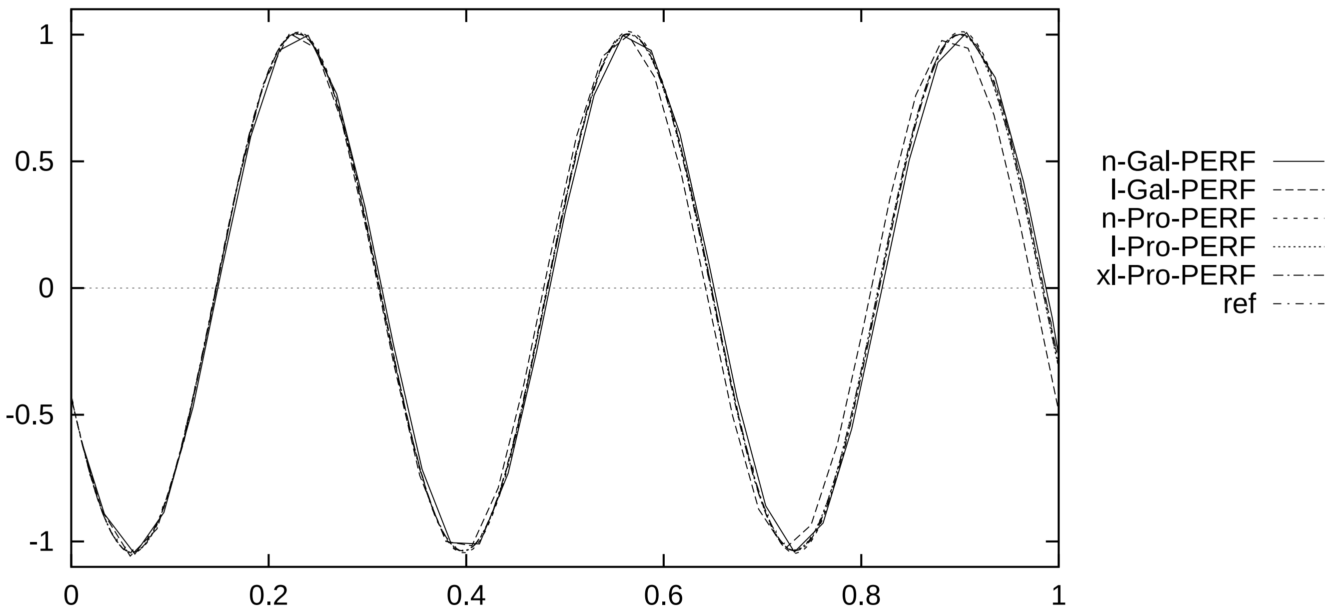
**n-GAL** or **n-PRO**!!!

# Results II: Cylinder(int) Benchmark

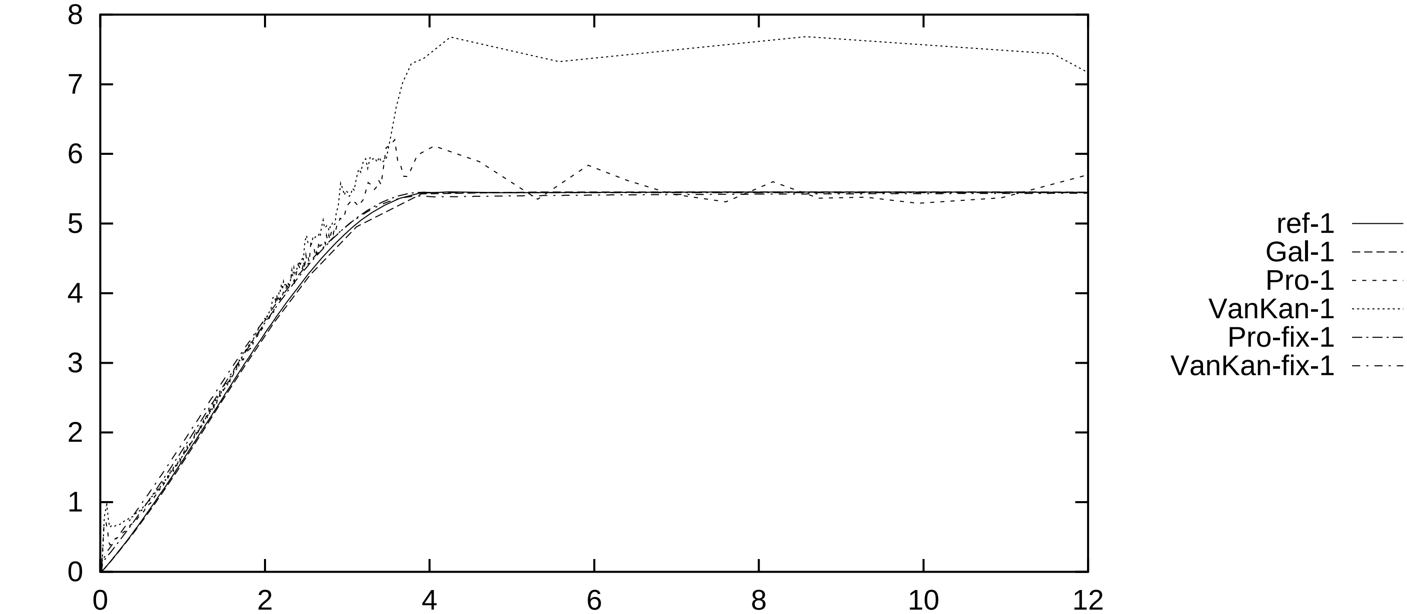


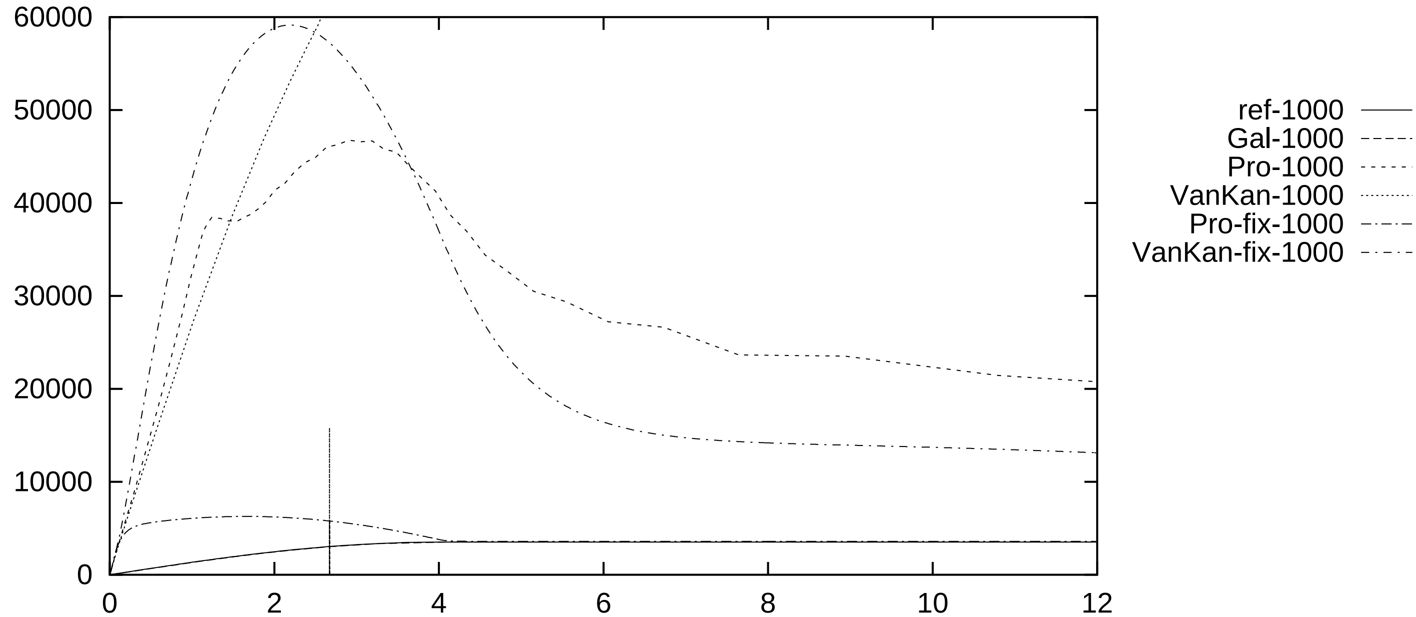




| L6 | CPU | Meth. | NT |
|---|---|---|---|
| | 14,876 (78/10%) | n-Gal | **12** |
| | 6,910 (82/ 9%) | l-Gal | **16** |
| O | 16,884 (56/30%) | n-Pro | **46** |
| K | 6,749 (67/23%) | l-Pro | **46** |
| | 64,485 (54/27%) | xl-Pro | **889** |
| P | 32,538 (73/13%) | n-Gal | **36** |
| E | 14,358 (81/ 9%) | l-Gal | **39** |
| R | 42,679 (51/34%) | n-Pro | **165** |
| F | 21,409 (63/26%) | l-Pro | **171** |

| L4 | CPU | Meth. | NT |
|---|---|---|---|
| | 760 (81/ 9%) | n-Gal | **9** |
| | 342 (84/ 8%) | l-Gal | **12** |
| O | 744 (56/30%) | n-Pro | **33** |
| K | 315 (68/22%) | l-Pro | **34** |
| | 1,515 (53/28%) | xl-Pro | **338** |
| P | 1,548 (76/12%) | n-Gal | **27** |
| E | 712 (82/ 9%) | l-Gal | **30** |
| R | 1,449 (47/36%) | n-Pro | **97** |
| F | 740 (63/26%) | l-Pro | **96** |

# Results III: Low Reynolds numbers




| $\nu = 1$ | NT | **Lift T=12** value | error |
|---|---|---|---|
| REF | | 5.448 | – |
| Gal | 13 | 5.450 | 0% |
| Pro | 329 | 5.702 | 5% |
| VanKan | 313 | 7.172 | **31**% |
| Pro-fix | 1200 | 5.448 | 0% |
| VanKan-fix | 1200 | 5.438 | 0% |

| $\nu = 1,000$ | NT | **Lift T=12** value | error |
|---|---|---|---|
| REF | | 3.513+3 | – |
| Gal | 20 | 3.513+3 | 0% |
| Pro | 53 | 2.077+4 | **277**% |
| VanKan | — | — | — |
| Pro-fix | 1200 | 3.592+3 | 4% |
| VanKan-fix | 1200 | 1.311+4 | **217**% |

# Conclusion: Adaptive time stepping ???

- **Error Estimator instead of Error Indicator !!!**

- **Galerkin approach in space/time**

⇒ FEM in space with Upwind/Streamline-Diffusion

⇒ (Modified) Crank-Nicolson ($\sim$ Petrov-Galerkin with cG(1))

- **Coupled solvers in u and p**

⇒ global/local PSC with (doubled) multigrid embedding

⇒ fully nonlinear treatment for complex configurations

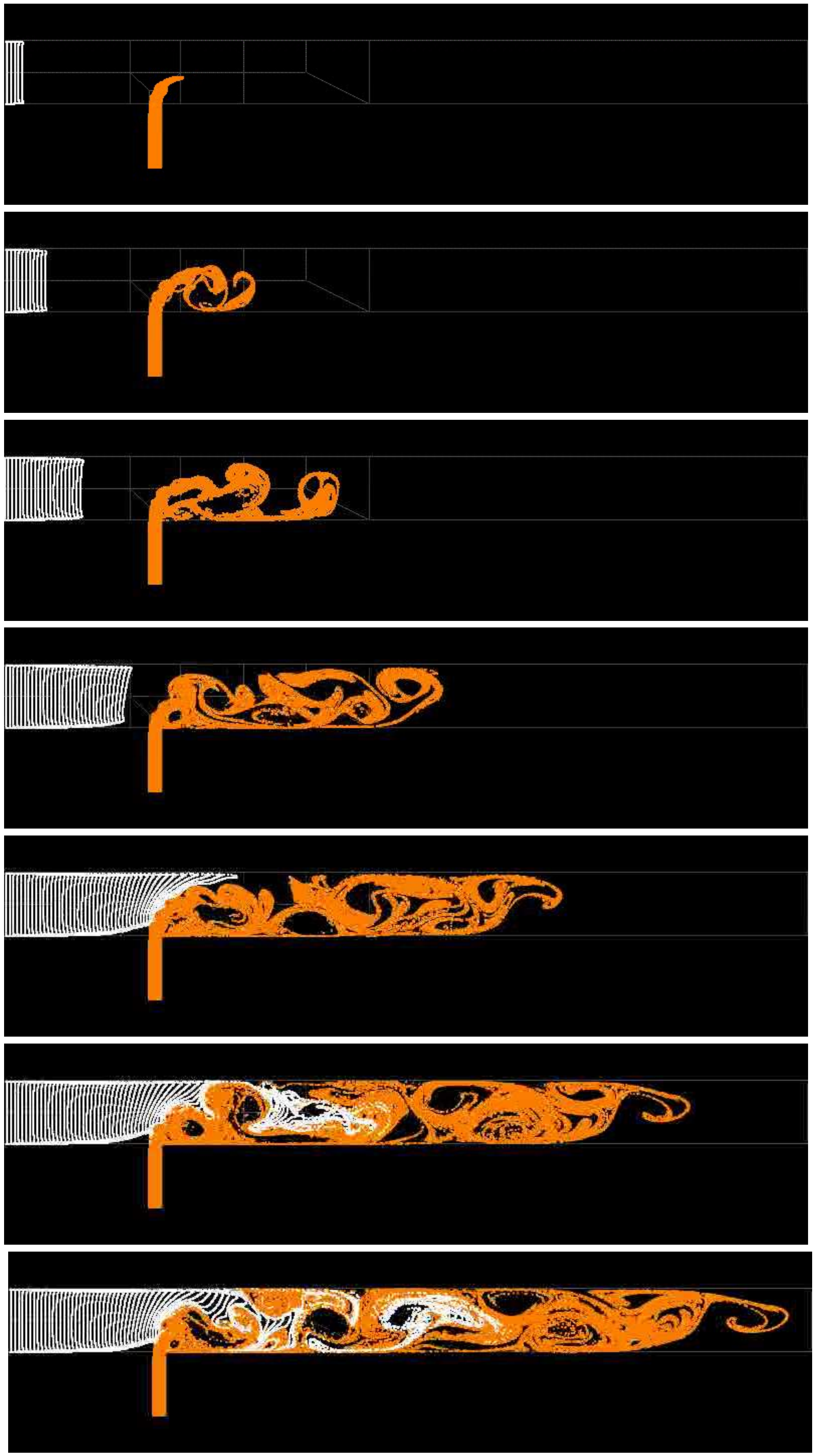
⇓

*Combination of higher accuracy of* **Galerkin-type** *methods with high speed of* **Projection-type** *solvers is possible !!!*

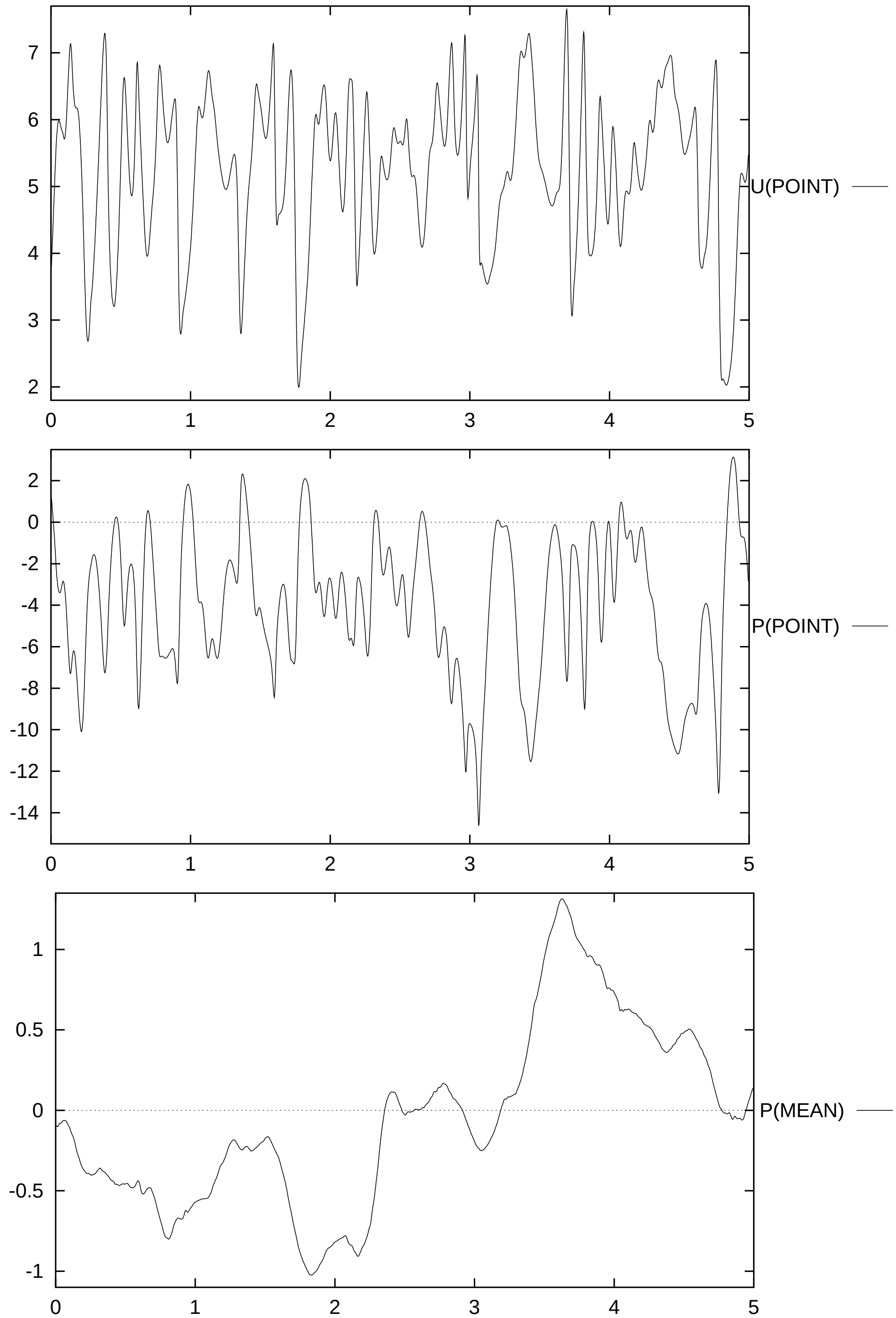
**Residual-based** *a posteriori error control in space/time with dual solutions based on* **meanvalue** *functionals in* **space/time** *is aimed !!!*

**Robust** *solvers for linearized NS problems with* **large** *time steps !?!*

# Project: Perpendicular inflow into pipe



## Snapshots:




$\Downarrow$

Control of the **mean value** for the 'wall pressure' in space/time???