Computational Prediction

of Natural Convection Flows

in Enclosures

with FeatFlow

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http://www.featflow.de

Boussinesq Model:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu_{\mathbf{u}} \Delta \mathbf{u} + \mathbf{j}T \quad , \quad \nabla \cdot \mathbf{u} = 0$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \nu_T \Delta T$$

with $\quad \nu_{\mathbf{u}} = \sqrt{\frac{Pr}{Ra}} \quad$ and $\quad \nu_T = \sqrt{\frac{1}{Ra\,Pr}}$

FeatFlow Ingredients:

`http://www.featflow.de`

- Nonconforming **FEM** spaces (LBB-stable)
- Hybrid Upwind/Streamline-Diffusion stabilization
- **Adaptive** (fully/semi) **implicit** time stepping (2nd order)
- **Multilevel** Pressure Schur Complement methods
- Nonlinear Quasi-Newton Defect Correction
- Scalar **Multigrid** solvers with adaptive Grid Transfer

Time (k)/Space Discretization (h)

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(Nonlinear Coupled) Discrete Systems:

$$S_{\mathbf{u}}(\mathbf{u}^{n+1})\mathbf{u}^{n+1} + kBp^{n+1} + kM_T T^{n+1} = \mathbf{f}(n+1, n)$$

$$B^T \mathbf{u}^{n+1} = 0$$

$$S_T(\mathbf{u}^{n+1})T^{n+1} = \mathbf{g}(n+1, n)$$

$$\to \ S_{\mathbf{u}}(\mathbf{v}) := \alpha M + \nu_{\mathbf{u}} kL + kK_{\mathbf{u}}(\mathbf{v})$$
$$\to \ S_T(\mathbf{v}) := \alpha M + \nu_T kL + kK_T(\mathbf{v})$$

Matrix-Vector Notation:

$$\begin{bmatrix} S_{\mathbf{u}}(\mathbf{u}^{n+1}) & kM_T & kB \\ 0 & S_T(\mathbf{u}^{n+1}) & 0 \\ B^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ T^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(n+1, n) \\ \mathbf{g}(n+1, n) \\ 0 \end{bmatrix}$$

LOCAL MPSC (for "steady" configurations):

Outer **Newton-** or Fixpoint-like **nonlinear** iteration

Multigrid solver for resulting **linear** coupled subproblems

'*Solve* **"exactly"** *on "subsets/patches" and perform an outer* **Block–Gauß-Seidel/Vanka** *iteration'* as smoother

Algebraic Notation of the Newton-like Step:

$$\begin{bmatrix} \mathbf{u}^{l+1} \\ T^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^{l} \\ T^{l} \\ p^{l} \end{bmatrix} - \omega^{l+1} \begin{bmatrix} N_{\mathbf{u}}(\mathbf{u}^l) & kM_T & kB \\ \gamma N_{\mathbf{u}}(T^l) & S_T(\mathbf{u}^l) & 0 \\ B^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{def}_{\mathbf{u}}^l \\ \mathbf{def}_T^l \\ \mathbf{def}_p^l \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{def}_{\mathbf{u}}^l \\ \mathbf{def}_T^l \\ \mathbf{def}_p^l \end{bmatrix} := \begin{bmatrix} S_{\mathbf{u}}(\mathbf{u}^l) & kM_T & kB \\ 0 & S_T(\mathbf{u}^l) & 0 \\ B^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{l} \\ T^{l} \\ p^{l} \end{bmatrix} - \begin{bmatrix} \mathbf{f}(n+1,n) \\ \mathbf{g}(n+1,n) \\ 0 \end{bmatrix}$$

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Sequence of coupled linear problems

Preconditioner/Smoother on Patch Ω_i:

$$\begin{bmatrix} N_{\mathbf{u}|\Omega_i}(\mathbf{u}^l) & kM_{T|\Omega_i} & kB_{|\Omega_i} \\ \gamma N_{\mathbf{u}|\Omega_i}(T^l) & S_{T|\Omega_i}(\mathbf{u}^l) & 0 \\ B^T_{|\Omega_i} & 0 & 0 \end{bmatrix}^{-1}$$

$\rightarrow$ **Patch Ω_i:** Collection of $1-\mathrm{n}$ **neighboured elements**

- clustered due to "given" (blocked) subdomains
- **adaptively** clustered due to (mesh) anisotropies

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Sequence of "small" local problems

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- **direct steady** approaches possible
- **fully implicit** scheme ($\rightarrow$ nonsteady)
- **Problem:** Full Newton (= Linearization) !!!

  $\rightarrow$ quadr. convergence vs. stabilization ?

  $\rightarrow$ iterative solvers ?

Stationary Cases: $Ra = 1.7 \cdot 10^5$ / $Ra = 3.0 \cdot 10^5$

	$Ra = 1.7 \cdot 10^5$		$Ra = 3.0 \cdot 10^5$	
	Streamline-Diff./GRID0		Streamline-Diff./GRID0	
	#MG	#NL	#MG	#NL
Fixed Point	div.(80)	**div.(40)**	div.(324)	**div.(162)**
Newton	19	**6**	20	**7**

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Very Efficient Nonlinear Solver
(direct stationary !!!)

GLOBAL MPSC (fully "nonsteady" config.):

Outer **decoupling** of Navier-Stokes vs. energy equation

Newton-like **nonlinear** iteration for momentum equations

Multigrid solver for all **scalar** subproblems

Formulation of 1 Projection-like Step:

$$S_{\mathbf{u}}(\tilde{\mathbf{u}}^{n+1})\tilde{\mathbf{u}}^{n+1} = \mathbf{f}(n+1,n) - kBp^n + kM_T T^n \quad (\text{‘\textbf{Burgers}’})$$

$$\text{”}\Delta_h\text{”} q = f_p \quad , \quad f_p := \frac{1}{k} B^T \tilde{\mathbf{u}}^{n+1} \quad (\text{‘\textbf{Pressure Poisson}’})$$

$$p^{n+1} = p^n + q + \nu_{\mathbf{u}} k M_p^{-1} f_p \quad , \quad \mathbf{u}^{n+1} = \tilde{\mathbf{u}}^{n+1} - k M_{\mathbf{u}}^{-1} Bq$$

$$S_T(\mathbf{u}^{n+1}) T^{n+1} = \mathbf{g}(n+1,n) \quad (\text{‘\textbf{Temperature}’})$$

Sequence of "large" scalar problems

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- 'fast' Numerical Linear Algebra possible
- 'optimal' for fully **nonsteady** problems
  $\rightarrow$ **'exact'** preconditioner "Δ_h" $= P = B^T M_{\mathbf{u}}^{-1} B$
  $\rightarrow$ optimized multigrid for **5/7-point FEM** matrix
- **semi-implicit** (nonlinear) scheme

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Time step control???

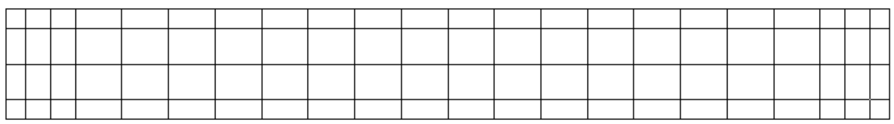
$$|f(\mathbf{u}_h^k(t_n), p_h^k(t_n)) - f(\mathbf{u}_h(t_n), p_h(t_n))| \leq TOL$$

(Error Indicator based on $\Delta T / 3\Delta T$)

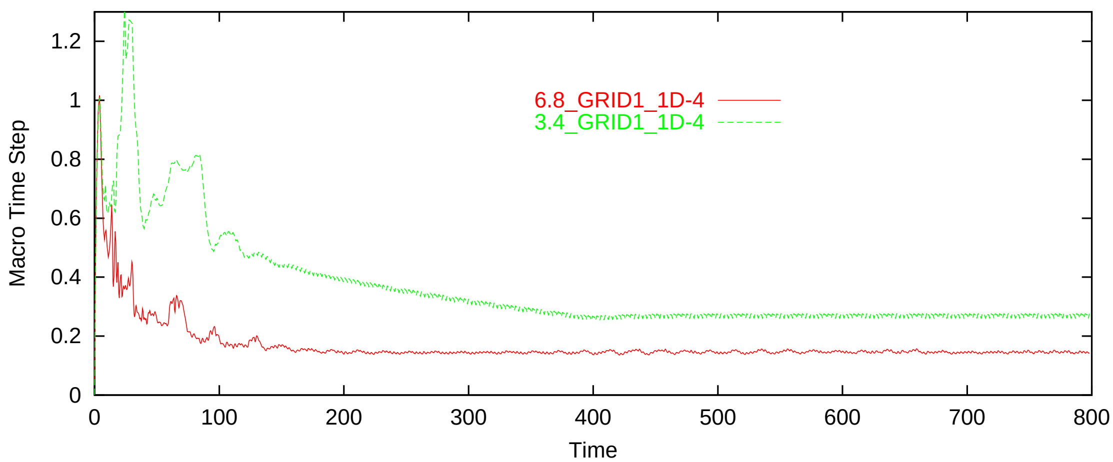
Details concerning spatial/temporal Meshes

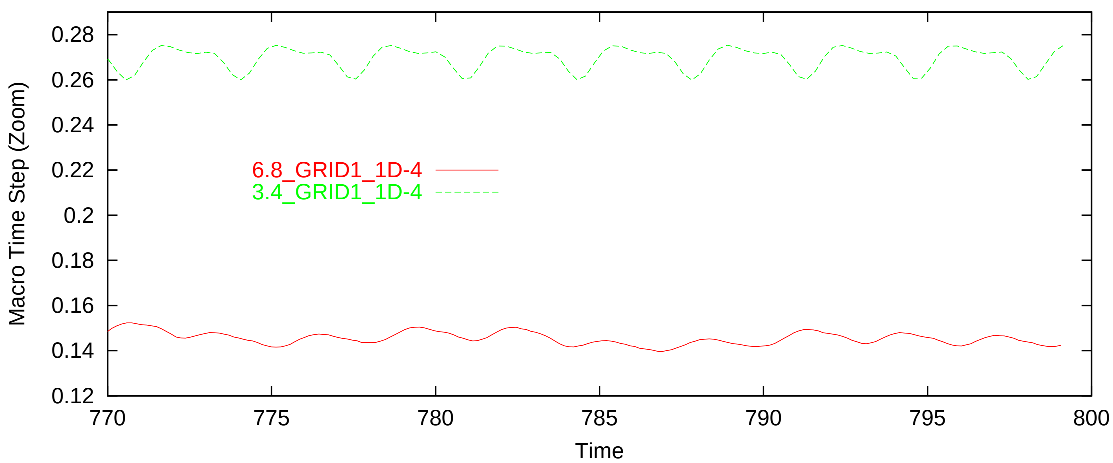
Coarse Mesh/Geometrical Details

level	vertices	elements	midpoints	total unknowns
1	115	88	202	**492**
GRID0	22,945	22,528	45,472	**113,472**
GRID1	90,945	90,112	181,056	**453,224**
GRID2	362,113	360,448	722,560	**1,805,568**



Adaptively chosen Macro Time Steps





Symmetric Periodical Case: $Ra = 3.4 \cdot 10^5$

TOL	Grid0		Grid1		Grid2	
(adaptiv)	mean	[Min:Max]	**mean**	**[Min:Max]**	mean	[Min:Max]
- Nusselt Number						
10^{-3}	4.5824	[4.5778:4.5881]	**4.5773**	**[4.5718:4.5826]**	4.5766	[4.5708:4.5826]
$\mathbf{10^{-4}}$	**4.5799**	**[4.5780:4.5816]**	**4.5802**	**[4.5758:4.5843]**	**4.5803**	**[4.5752:4.5851]**
10^{-5}	4.5797	[4.5796:4.5799]	**4.5795**	**[4.5766:4.5823]**	4.5791	[4.5743:4.5850]
Temperature						
10^{-3}	0.2563	[0.2143:0.2976]	**0.2595**	**[0.2199:0.3025]**	0.2604	[0.2197:0.3058]
$\mathbf{10^{-4}}$	**0.2589**	**[0.2488:0.2691]**	**0.2637**	**[0.2393:0.2911]**	**0.2651**	**[0.2369:0.2981]**
10^{-5}	0.2590	[0.2579:0.2601]	**0.2638**	**[0.2476:0.2812]**	0.2647	[0.2442:0.2884]
U1 Component						
10^{-3}	0.0668	[0.0346:0.1241]	**0.0628**	**[0.0260:0.1213]**	0.0631	[0.0249:0.1253]
$\mathbf{10^{-4}}$	**0.0549**	**[0.0432:0.0682]**	**0.0585**	**[0.0329:0.0984]**	**0.0606**	**[0.0318:0.1107]**
10^{-5}	0.0542	[0.0528:0.0555]	**0.0552**	**[0.0373:0.0796]**	0.0572	[0.0344:0.0909]
U2 Component						
10^{-3}	0.4841	[0.4307:0.5404]	**0.4647**	**[0.4134:0.5161]**	0.4600	[0.4078:0.5132]
$\mathbf{10^{-4}}$	**0.4836**	**[0.4654:0.5016]**	**0.4676**	**[0.4240:0.5130]**	**0.4634**	**[0.4148:0.5163]**
10^{-5}	0.4837	[0.4818:0.4856]	**0.4675**	**[0.4373:0.4988]**	0.4642	[0.4253:0.5049]
Pressure Difference 14						
10^{-3}	-0.0010	[-0.0142:0.0096]	**-0.0021**	**[-0.0168:0.0108]**	-0.0023	[-0.0176:0.0111]
$\mathbf{10^{-4}}$	**-0.0013**	**[-0.0060:0.0032]**	**-0.0016**	**[-0.0142:0.0095]**	**-0.0016**	**[-0.0162:0.0110]**
10^{-5}	-0.0013	[-0.0018:0.0001]	**-0.0017**	**[-0.0102:0.0060]**	-0.0020	[-0.0128:0.0081]
Pressure Difference 15						
10^{-3}	0.5085	[0.4894:0.5251]	**0.5322**	**[0.5206:0.5434]**	0.5371	[0.5270:0.5491]
$\mathbf{10^{-4}}$	**0.5096**	**[0.5044:0.5146]**	**0.5280**	**[0.5139:0.5397]**	**0.5332**	**[0.5171:0.5458]**
10^{-5}	0.5095	[0.5088:0.5100]	**0.5276**	**[0.5178:0.5361]**	0.5322	[0.5195:0.5431]

Symmetric Periodical Case: $Ra = 3.4 \cdot 10^5$

time control TOL	Grid0	**Grid1**	Grid2
Averaged Time Step (adatively chosen)			
10^{-3}	0.2525	**0.2164**	0.2064
$\mathbf{10^{-4}}$	**0.1572**	**0.0898**	**0.0803**
10^{-5}	0.1518	**0.0484**	0.0297
(Approximative) Oscillation Period (evaluated graphically)			
10^{-3}	3.977	**3.475**	3.582
$\mathbf{10^{-4}}$	**3.566**	**3.447**	**3.416**
10^{-5}	3.564	**3.438**	3.422
Averaged CPU per Macro Timestep ($\approx$ 3 small (ΔT) + 1 large ($3\Delta T$) substeps) (COMPAQ ES40/667MHz, sequentiell, Standard-FeatFlow)			
10^{-3}	6	**31**	159
$\mathbf{10^{-4}}$	**6**	**27**	**129**
10^{-5}	5	**24**	108

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Accuracy of 'graphical' evaluation + 'mean values' ???

Symmetric Periodical Case: $Ra = 3.4 \cdot 10^5$

How to choose the appropriate Time Steps ???

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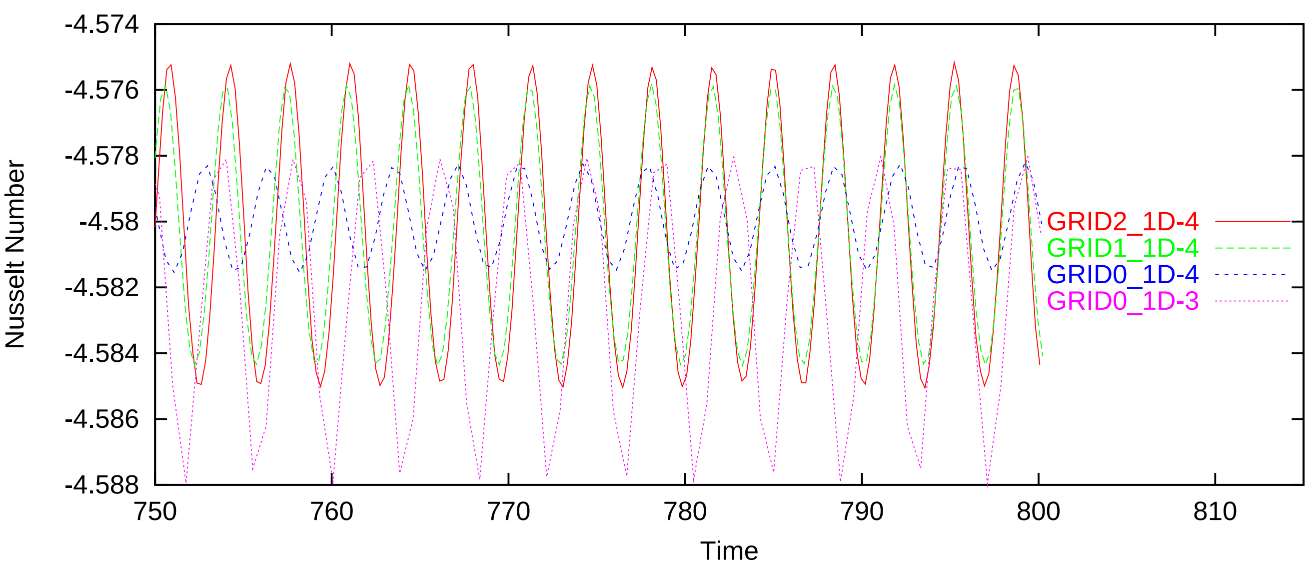
- **larger** amplitudes by **bigger** time steps in **Projection-like** schemes
- 'good' results for coarse meshes and large time steps !!!



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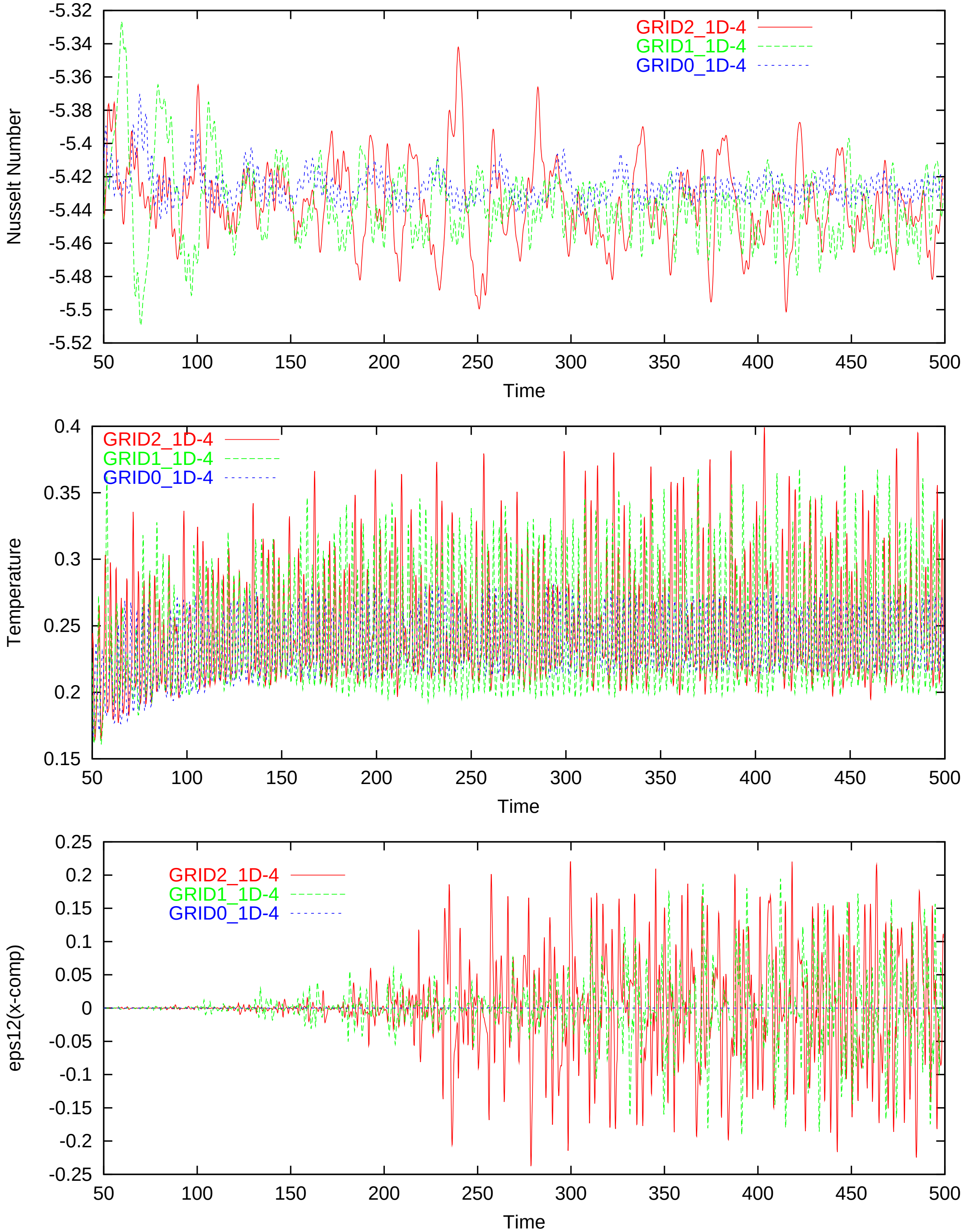
Time Step Setting of Participants ???

Nonsymmetric/-periodical Case: $Ra = 6.8 \cdot 10^5$



Nonsymmetric/-periodical Case: $Ra = 6.8 \cdot 10^5$

time control TOL	Grid0	**Grid1**	Grid2
Averaged Time Step (adatively chosen, T=[0:500])			
10^{-4}	0.0911	0.0546	0.0534
Averaged CPU per Macro Timestep ($\approx$ 3 small (ΔT) + 1 large ($3\Delta T$) substeps) (COMPAQ ES40/667MHz, sequentiell, Standard-FeatFlow)			
10^{-4}	5	27	108

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- no periodical oscillations visible
- larger amplitudes
- skewness is lost
- smaller time steps necessary (?)

$\Downarrow$

Adaptive Time step control!!!
Prediction of Point Values (in time)???
Prediction of Mean Values!?!

'My' conclusions:

'Scalability' via implicit components and multigrid solvers !!!

How did all participants choose their time step sizes if using implicit schemes ???