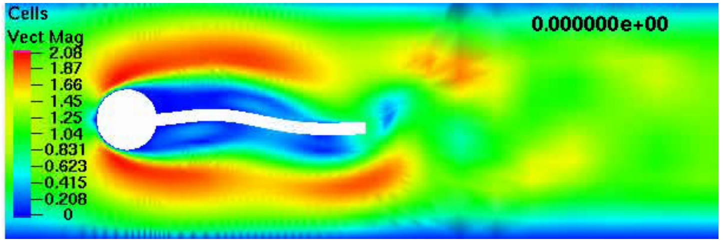
technische universität
dortmund

Fakultät für Mathematik
IAM

Fluid-Structure Interaction Problems:

FEM Multigrid Techniques and Benchmarking

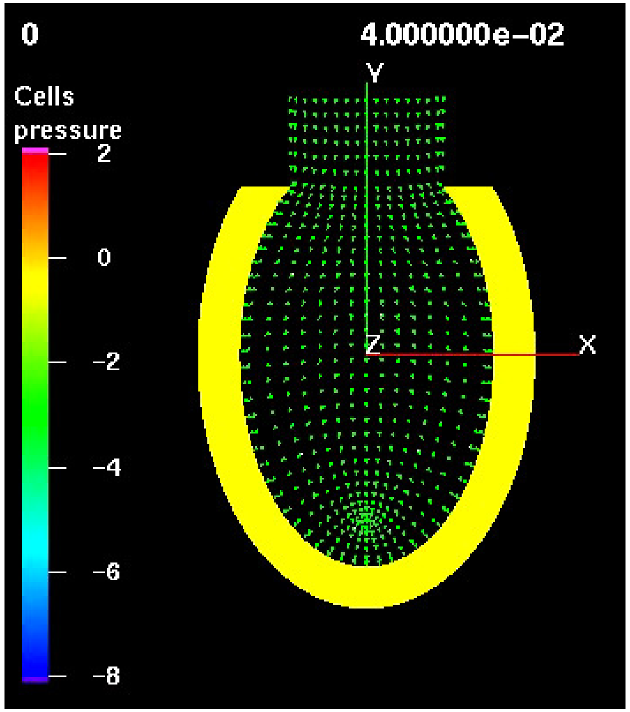


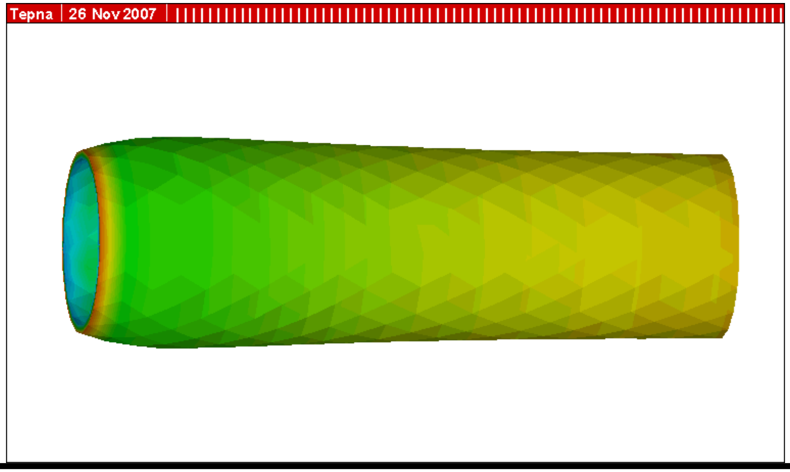
S. Turek
with support by the FEAST Group

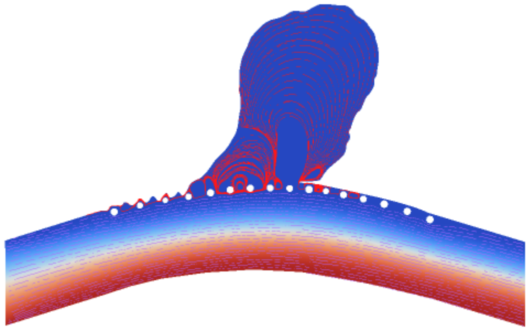
Institut für Angewandte Mathematik, TU Dortmund
http://www.mathematik.uni-dortmund.de/LS3
http://www.featflow.de

Multiphase FSI Problems: Elastic Solids

- ***Liquid – Rigid Solid***
  - Particulate Flow
  - Robofish
- ***Liquid – Elastic Solid***
  - Biomechanics
  - Medical applications
  - Aeroelasticity

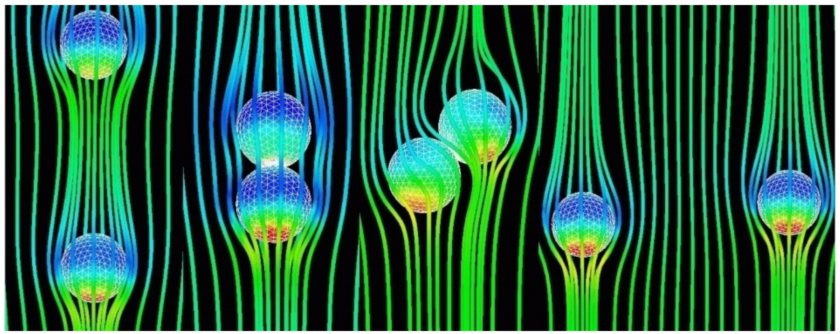


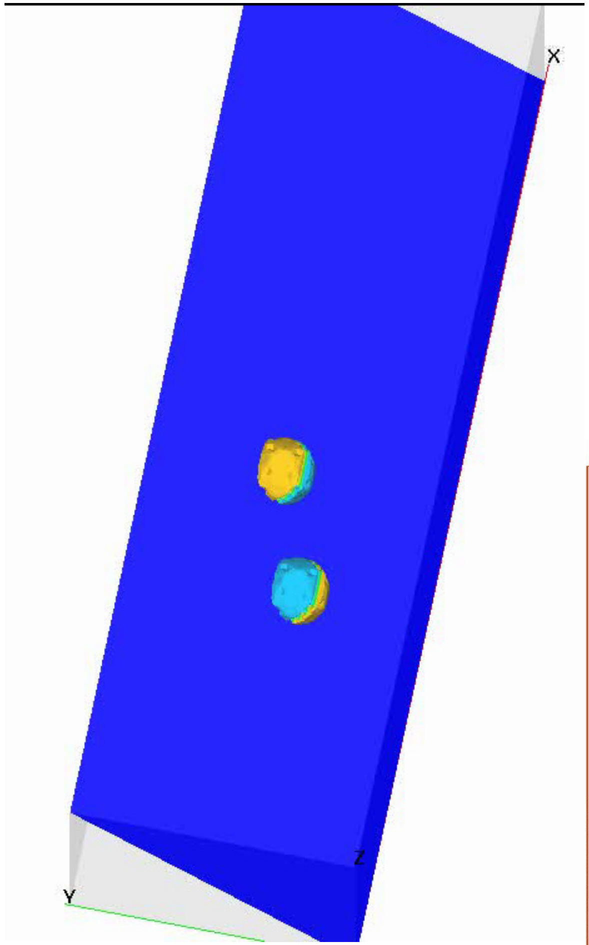


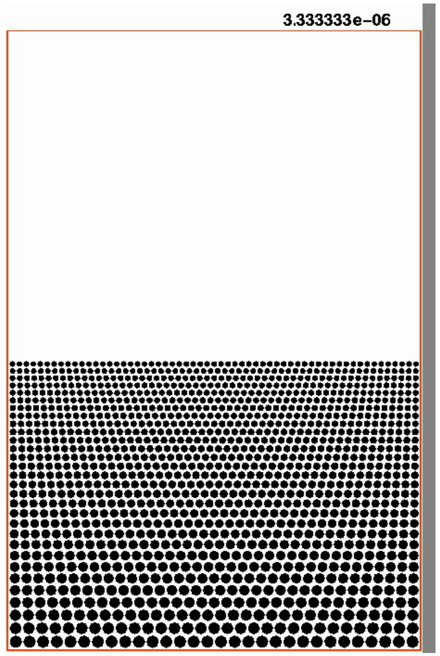


Multiphase FSI Problems: Rigid Solids

- *Liquid – Rigid Solid*
  - Particulate Flow
  - Robofish
- *Liquid – Elastic Solid*
  - Biomechanics
  - Medical applications
  - Aeroelasticity

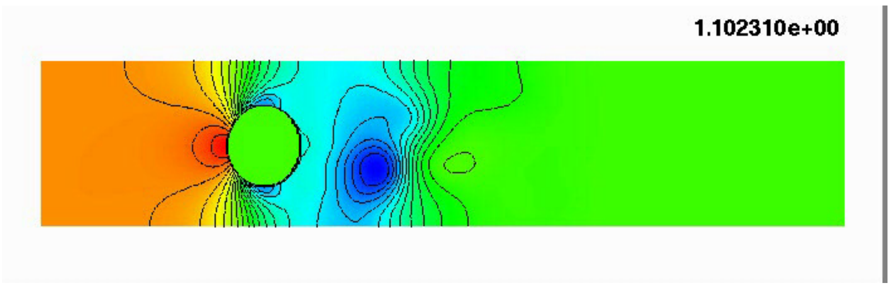






Required: Special Numerics for FSI

- Special FEM Techniques
- Multigrid Solvers
- Stabilization for high Re, Pe, We,… Numbers
- Implicit Approaches
- Space-Time Adaptivity
- Grid Deformation Methods
- Fictitious Boundary Methods



- GPU Computing

Computational mesh (can be) independent of 'internal objects'

Challenges for Numerics

- Special FEM discretization techniques to handle the following challenging points
  - Stable FE spaces for velocity and pressure fields, and velocity and extra-stress fields

    → Q2/P1/Q2, Q1(nc)/P0/Q1(nc) (new: Q2(nc)/P1/Q2(nc))
  - Special treatment of the „convective“ terms

    → edge-oriented/interior penalty EO-FEM, TVD/FCT
  - Special treatment of the „reactive" terms in viscoelastic problems

    → LCR + EO-FEM

- Special (nonlinear) solvers to deal with different sources of nonlinearity
  - nonlinear operators → Newton method via divided differences
  - stiff coupling of equations → monolithic/operator splitting multigrid
  - complex geometries and meshes

Nonlinear Solvers

Solve for the residual of the nonlinear system algebraic equations

$$R(\mathrm{x}) = 0, \quad \mathrm{x} = (\mathrm{u}, \Theta, \sigma, p)$$

Use Newton method with damping results in iterations of the form

$$\mathrm{x}^{n+1} = \mathrm{x}^{n} + \omega^{n} \left[\frac{\partial R(\mathrm{x}^{n})}{\partial \mathrm{x}} \right]^{-1} R(\mathrm{x}^{n})$$

- Continuous Newton: on variational level (before discretization)

→ The continuous Frechet operator can be **analytically calculated**

- Inexact Newton: on matrix level (after discretization)

→ The Jacobian matrix is **approximated** using finite differences as

$$\left[\frac{\partial R(\mathrm{x}^{n})}{\partial x} \right]_{ij} \approx \frac{R_i(\mathrm{x}^{n} + \varepsilon \mathrm{e}_j) - R_i(\mathrm{x}^{n} - \varepsilon \mathrm{e}_j)}{2\varepsilon}$$

Multigrid Solvers

- Standard geometric multigrid approach with full FEM grid transfer
- Smoother: Local/Global MPSC
  - Local MPSC via Vanka-like smoother

    → Monolithic multigrid solver

$$\begin{bmatrix} \mathrm{u}^{l+1} \\ \sigma^{l+1} \\ \Theta^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} \mathrm{u}^{l} \\ \sigma^{l} \\ \Theta^{l} \\ p^{l} \end{bmatrix} + \omega^{l} \, \Sigma_{T \in \tau_h} [K + J]_{|T}^{-1} \begin{bmatrix} \mathrm{Res}_{\mathrm{u}} \\ \mathrm{Res}_{\sigma} \\ \mathrm{Res}_{\Theta} \\ \mathrm{Res}_{p} \end{bmatrix}_{|T}$$

  - Global MPSC
    - solve for an intermediate **u** (generalized momentum equation)
    - solve for *p* (pressure Poisson equation)
    - update of **u** and *p*
    - solve for Θ (tracer equation)
    - solve for σ (constitutive equation)

    → Decoupled multigrid solver

1) Aspects of (Elastic) FSI Problems

- incompressible Newtonian fluid (with nonlinear extensions)

$$\sigma^f = -p\mathrm{I} + 2\nu\mathrm{D}$$

- hyperelastic material, incompressible

$$\sigma^f = -p\mathrm{I} + 2\mathrm{F}\frac{\partial\Psi}{\partial\mathrm{F}}\mathrm{F}^T, \qquad \det F = 1$$

$$\Psi(\mathrm{F}) = \alpha(\mathrm{I}_\mathrm{C} - 3)\,\mathrm{Neo\text{-}Hook}$$

$$\Psi(\mathrm{F}) = \alpha_1(\mathrm{I}_\mathrm{C} - 3) + \alpha_2(\mathrm{I}_\mathrm{C} - 3) + \alpha_3(|\mathrm{Fe}| - 1)^2\ \mathrm{Mooney\text{-}Rivlin + anisotropic}$$

where $\mathrm{C} = \mathrm{F}\mathrm{F}^T$ and $\mathrm{I}_\mathrm{C} = \mathrm{trC}$, $\mathrm{I}_\mathrm{C} = \frac{1}{2}\left(\mathrm{trC}^2 - (\mathrm{trC})^2\right)$

- or **St. Venant-Kirchhoff** material, compressible

$$\sigma^s = \frac{1}{J}\mathrm{F}\left(\lambda^s(\mathrm{trE})\mathrm{I} + \mu^s\mathrm{E}\right)\mathrm{F}^T$$

where $\mathrm{E} = \frac{1}{2}\left(\mathrm{F}^T\mathrm{F} - \mathrm{I}\right)$

Monolithic ALE-FEM Approach

$$R(\mathrm{x}) = 0 \qquad \mathrm{x} = (u_h, v_h, p_h) \in U_h \times V_h \times P_h$$

$$Mu_h - \frac{k}{2}\left(M^s v_h + L^f u_h\right) = \mathrm{rhs}\left(u_h^n, v_h^n\right)$$

$$\left(M^f + \beta M^s\right)v_h + \frac{k}{2}N_1(v_h, u_h) + \frac{1}{2}N_2(v_h, u_h) + \frac{k}{2}\left(S^s(u_h) + S^f(v_h)\right) - kBp_h = \mathrm{rhs}\left(u_h^n, v_h^n, p_h^n\right)$$

$$C(u_h) + B^{f^T} v_h = 1$$

$$\Downarrow$$

$$\frac{\partial R}{\partial \mathrm{x}}(\mathrm{x}) = \begin{pmatrix} M - \frac{k}{2}L^f & \frac{k}{2}M^s & 0 \\ \frac{1}{2}\frac{\partial\left(N_1 + S^s + S^f\right)}{\partial u_h} + k\frac{\partial B}{\partial u_h}p_h & M^s + \beta M^f + \frac{1}{2}\frac{\partial N_2}{\partial v_h} + \frac{k}{2}\frac{\partial\left(N_1 + S_f^2\right)}{\partial v_h} & kB \\ B^{s^T} + \frac{\partial B^{f^T}}{\partial u_h}v_h & B^{f^T} & 0 \end{pmatrix}$$

Monolithic ALE-FEM Approach

$$R(\mathrm{x}) = 0 \qquad \mathrm{x} = (u_h,\, v_h,\, p_h) \in U_h \times V_h \times P_h$$

$$Mu_h - \frac{k}{2}\left(M^s v_h + L^f u_h\right) = \mathrm{rhs}\left(u_h^n, v_h^n\right)$$

$$\left(M^f + \beta M^s\right)v_h + \frac{k}{2} N_1(v_h, u_h) + \frac{1}{2} N_2(v_h, u_h) + \frac{k}{2}\left(S^s(u_h) + S^f(v_h)\right) - kBp_h = \mathrm{rhs}\left(u_h^n, v_h^n, p_h^n\right)$$

$$C(u_h) + B^{f^T} v_h = 1$$

$$\Downarrow$$

$$\begin{bmatrix} S_{uu} & S_{uv} & 0 \\ S_{vu} & S_{vv} & kB \\ c_u B_s^T & c_v B_f^T & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ p \end{bmatrix} = \begin{bmatrix} \mathrm{f}_{\mathrm{u}} \\ \mathrm{f}_{\mathrm{u}} \\ f_p \end{bmatrix}$$

Typical discrete saddle-point problem

Multigrid Solver for Q2/Q2/P1

- standard geometric multigrid approach
- smoother by local MPSC-Ansatz (Vanka-like smoother

$$\begin{bmatrix} u^{l+1} \\ v^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} u^{l} \\ v^{l} \\ p^{l} \end{bmatrix} - \omega \sum_{\mathrm{Patch}\Omega_i} \begin{bmatrix} S_{uu|\Omega_i} & S_{uv|\Omega_i} & 0 \\ S_{vu|\Omega_i} & S_{vv|\Omega_i} & kB_{|\Omega_i} \\ c_u B^T_{s|\Omega_i} & c_v B^T_{f|\Omega_i} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{def}^l_u \\ \mathrm{def}^l_v \\ def^l_p \end{bmatrix}$$

- full inverse of the local problems by LAPACK (39 x39 systems)
- alternatives: simplified local problems (3x3 systems) or ILU(k)
- combination with GMRES/BiCGStab methods possible
- full (canonical) FEM prolongation, restriction by $\mathrm{R} = \mathrm{P}^T$

Very accurate, flexible and highly efficient FSI solver

(→ FSI Benchmarks)

2) Aspects of Particulate Flow

Fluid flow is modelled by the **Navier-Stokes equations**:

$$\rho\left(\frac{\partial u}{\partial t}+u\cdot\nabla u\right)-\nabla\cdot\sigma = f\,,\ \nabla\cdot u=0\ \ ,\ \ \sigma(X,t)=-pI+\mu[\nabla u+(\nabla u)^T]$$

Motion of particles is described by the **Newton-Euler equations**, i.e., the **translational velocities** and **angular velocities** of the p-th particle satisfy:

$$M_p\frac{dU_p}{dt}=F_p+F_p^{'}+(\Delta M_p)g,\quad I_p\frac{d\omega_p}{dt}+\omega_p\times(I_p\omega_p)=T_p.$$

F_p and T_p are the **hydrodynamical forces** and the **torque** at mass center acting on the p-th particle and $F_p^{'}$ are the **collision forces**

$$F_p=-\int_{\Gamma_p}\sigma\cdot n_p\,d\Gamma_p,\qquad T_p=-\int_{\Gamma_p}(X-X_p)\times(\sigma\cdot n_p)\,d\Gamma_p$$

Particle-Fluid Interaction

No slip boundary conditions at interface Γ_p between particles and fluid i.e., for any $X \in \Gamma_p$, the velocity **u**(X) is defined by:

$$u(X) = U_p + \omega_p \times (X - X_p)$$

The **position** X_p of the p-th particle and its **angle** θ_p are obtained by integration of the kinematic equations:

$$\frac{dX_p}{dt} = U_p, \qquad \frac{d\theta_p}{dt} = \omega_p$$

How to Calculate the Forces?

Hydrodynamic forces and torque acting on the i-th particle

$$F_i = -\int_{\partial P_i} \sigma \cdot n_i d\Gamma_i, \qquad T_i = -\int_{\partial P_i} (X - X_i) \times (\sigma \cdot n_i) d\Gamma_i$$

Idea: 'Replace the surface integral by a volume integral' and use indicator functions ($n_p \approx \nabla \alpha_p$)

$$F_p = -\int_{\Gamma_p} \sigma \cdot n_p d\Gamma_p = -\int_{\Omega_T} \sigma \cdot \nabla \alpha_p d\Omega_T$$

+

Fictitious Boundary Method on Generalized Tensorproduct Meshes

Grid Deformation Methods

Idea : construct transformation ϕ, $x = \phi(\xi, t)$ with $\det \nabla \phi = f$

⇨ local mesh area $\approx f$

1. Compute monitor function $f(x,t) > 0$, $f \in C^1$ and

$$\int_\Omega f^{-1}(x,t)\,dx = |\Omega|, \quad \forall t \in [0,1]$$

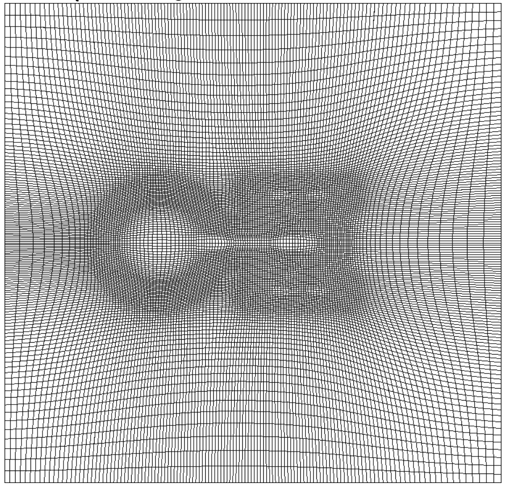
2. Solve $(t \in [0,1])$

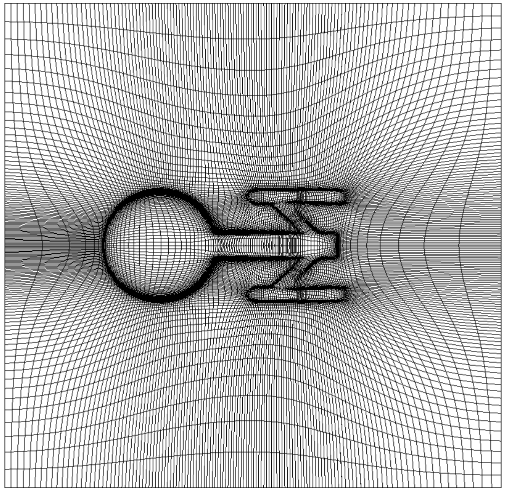
$$\Delta v(\xi, t) = -\frac{\partial}{\partial t}\left(\frac{1}{f(\xi, t)}\right), \quad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0$$

3. Solve the ODE system

$$\frac{\partial}{\partial t}\phi(\xi, t) = f(\phi(\xi, t), t)\nabla v(\phi(\xi, t), t)$$

new grid points: $x_i = \phi(\xi_i, 1)$





Grid deformation preserves the (local) logical structure of the grid

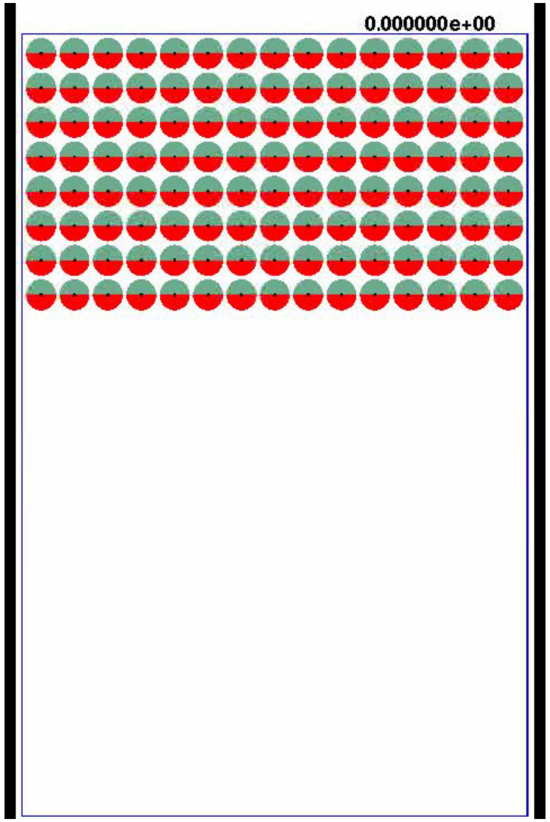
(Semi-explicit) Operator-Splitting Approach

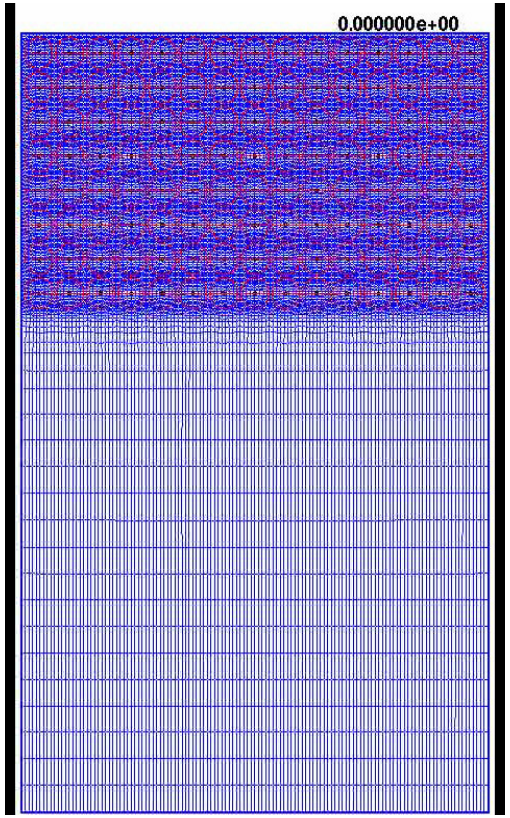
The algorithm for $t^n \rightarrow t^{n+1}$ consists of the following 5 substeps

1. Fluid velocity and pressure : $NSE\left(u_f^{n+1}, p^{n+1}\right) = BC\left(\Omega_p^n, u_p^n\right)$
2. Calculate hydrodynamic forces: F_p^{n+1}
3. Calculate velocity of particles: $u_p^{n+1} = g\left(F_p^{n+1}\right)$
4. Update position of particles: $\Omega_p^{n+1} = f\left(u_p^{n+1}\right)$
5. Align new mesh

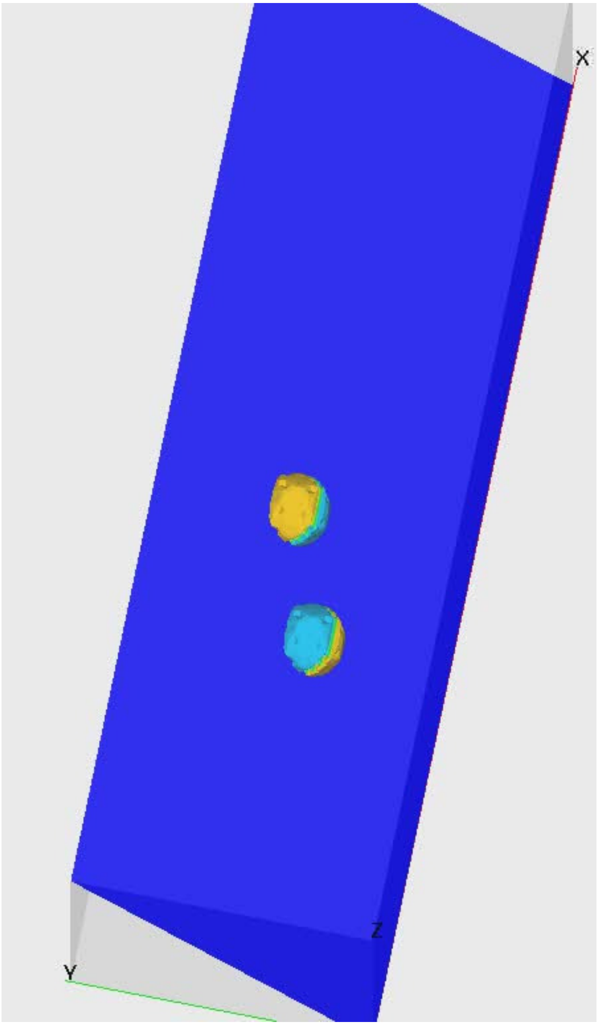
→ Required: efficient calculation of hydrodynamic forces
→ Required: efficient treatment of (many) particle interaction
→ Required: efficient (dynamic) grid alignment
→ Required: fast (nonstationary) Navier-Stokes solver FEASTFLOW

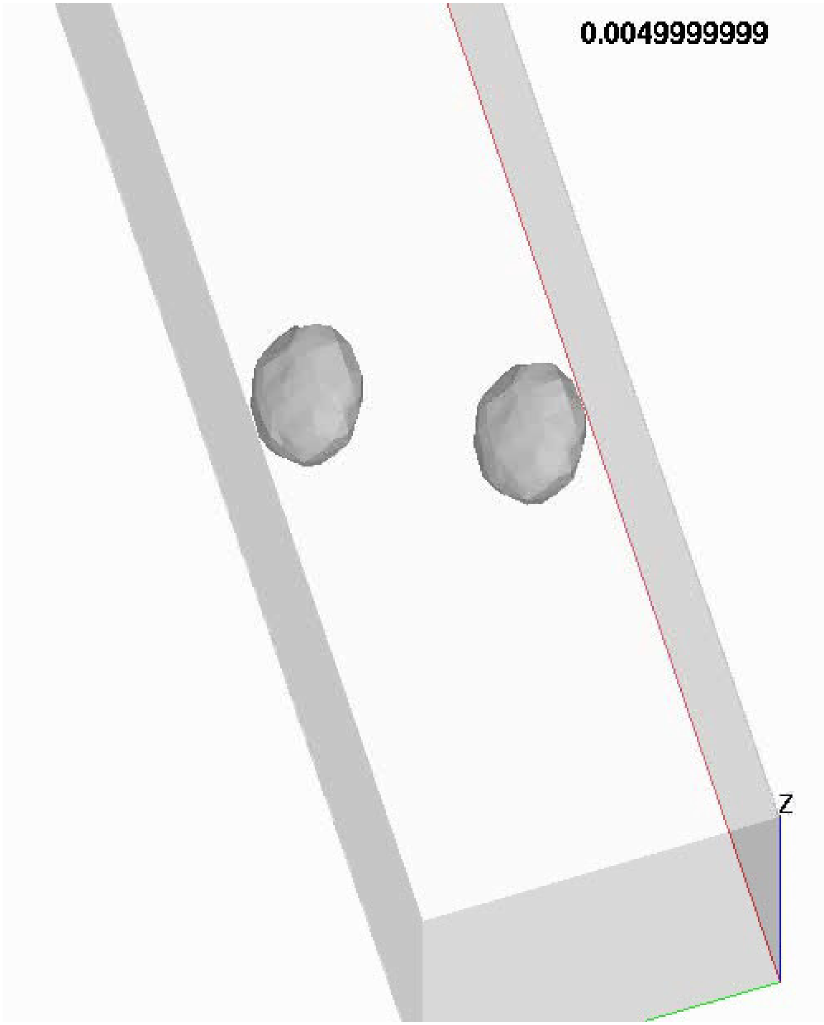
Dynamic Adaptation: 2D Sedimentation





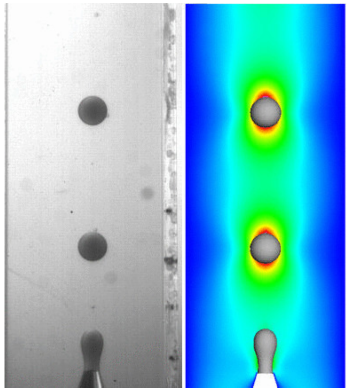
3D Examples

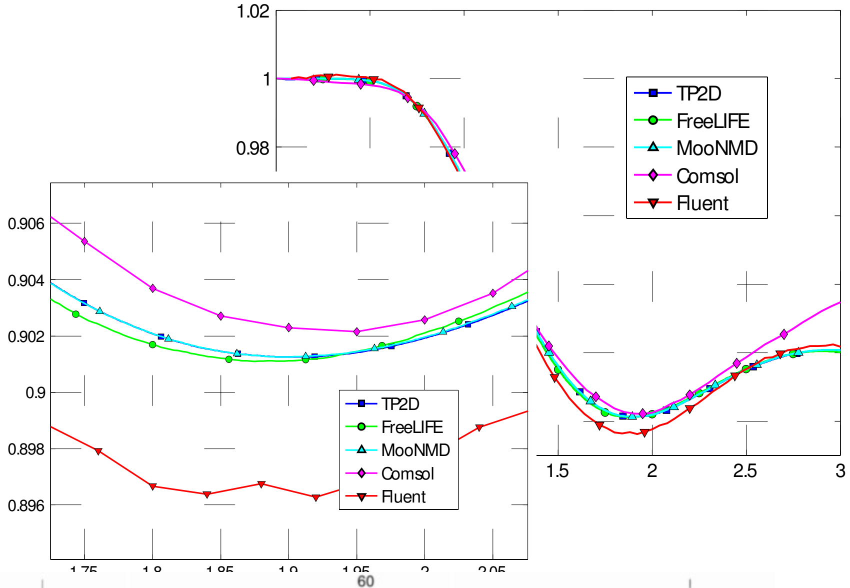




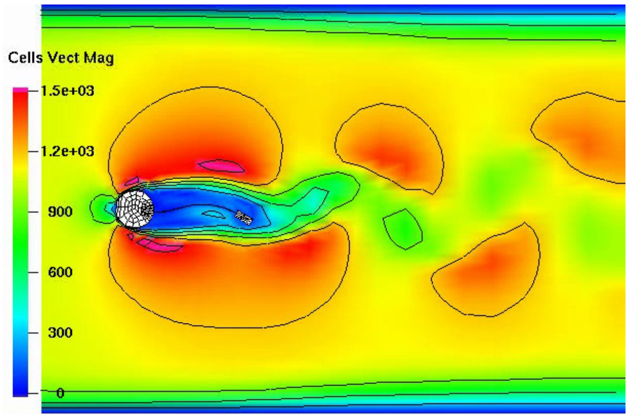
3) Benchmarking of Multiphase CFD

- **Initiative "Rising Bubble"**
  → *quantitative* validation and comparison of multiphase codes





- **Initiative "Elastic FSI"**
  → *quantitative* validation and comparison of monolithic vs. decoupled approaches



- **Initiative "Particulate Flow"**
  → *quantitative* validation and comparison with experimental configurations