Some remarks on pressure separation and edge-oriented FEM stabilization for incompressible flow

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Pressure separation: Motivation

- A priori error estimate for Navier-Stokes problem

$$h|\boldsymbol{u}-\boldsymbol{u}_h|_{1,\Omega}+\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}\leq \quad Ch^{k+1}\left\{|\boldsymbol{u}|_{k+1,\Omega}+\tfrac{1}{\nu}|p|_{k,\Omega}\right\}$$

- Modified problem with pressure separation **(Ganesan & John)**

$$\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\nu\Delta\boldsymbol{u}+\nabla\tilde{p}=\boldsymbol{f}-\nabla p_{sep}\quad(\tilde{p}=p-p_{sep})$$

- New a priori error estimate

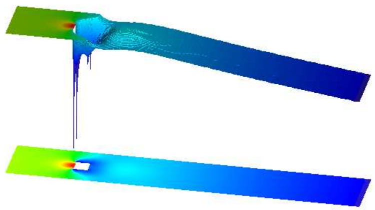
$$h|\boldsymbol{u}-\boldsymbol{u}_h|_{1,\Omega}+\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}\leq \quad Ch^{k+1}\left\{|\boldsymbol{u}|_{k+1,\Omega}+\tfrac{1}{\nu}|p-p_{sep}|_{k,\Omega}\right\}$$

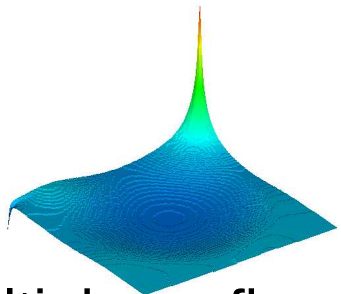
- Improvement if $\frac{1}{\nu}|p|_{k,\Omega}$ dominant and $|p-p_{sep}|_{k,\Omega}\ll|p|_{k,\Omega}$

! How relevant is this modification in real CFD simulations ?

Prototypical flow applications

- Flow dominated by pressure gradient in the case of problems with
  1. External force of the type $\boldsymbol{f} = \nabla \phi$
  2. Small viscosity parameter ν and the increase of $\|\nabla p\|$ in the boundary layer
  3. Singularities due to geometry or boundary conditions

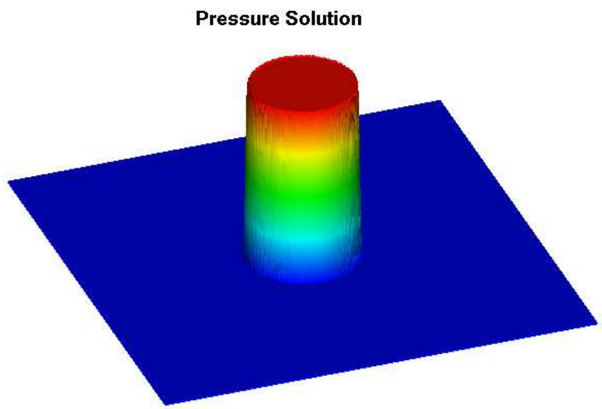




- Special local external force: multiphase flow with surface tension

$$\boldsymbol{f}_{st} = \sigma \int_{\partial\Omega_1} k\,\boldsymbol{n}\,\delta(\partial\Omega_1, x)\,ds$$

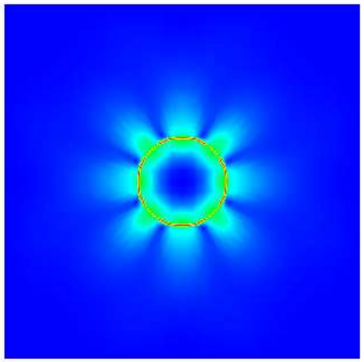
$\partial\Omega_1$	The interface
$\boldsymbol{n}$	The unit normal to the surface
k	The surface curvature
σ	The surface tension coefficient
$\delta(\partial\Omega_1, x)$	The Dirac delta function



! What about the resulting velocity ?

Spurious velocities

- Example of spurious velocity with $\tilde{Q}1$ for flow with interface



- These spurious currents
  1. are shared by large class of numerical methods: FEM, FVM, conf., nonconf.,....
  2. have different causes: external force, jump in the pressure, approximation of the interface,...

- The spurious velocity is not restricted to flow with interface

! **Can pressure separation deal with spurious velocity ?**

! **What about edge-oriented FEM ?**

Pressure Separation algorithm (I)

- **Stationary case**
  1. Compute $(\boldsymbol{u}_h, p_h)$, as finite element solution for the original Navier-stokes equations
  2. Compute $p_{sep,h} = I(p_h)$, interpolation of p_h, such that $|p_h - p_{sep,h}|_k \ll |p_h|_k$
  3. Compute $(\boldsymbol{u}_{sep,h}, \tilde{p}_h)$, the finite element solution of the modified Navier-Stokes equations with $\nabla p_{sep,h}$ as right hand side
  4. Set $\boldsymbol{u}_h = \boldsymbol{u}_{sep,h}$ and $p_h = p_h + \tilde{p}_h$

- **Remarks**
  - ◊ This algorithm requires almost double CPU times w.r.t solving the original problem.
  - ◊ $p_{sep,h}$ can be deduced from p_{2h} in multigrid.
  - ◊ If p_h was approximated by piecewise constant function, $p_{sep,h}$ in the second step can be taken as its linear interpolation.
  - ◊ Further variants for computing $p_{sep,h}$ (see Ganesan & John (2005)).

Pressure Separation algorithm (II)

- **Nonstationary case**
  1. Compute $p^n_{sep,h} := I(p^{n-1}_h)$, interpolation of p^{n-1}_h
  2. Compute $(\boldsymbol{u}^n_{sep,h}, \tilde{p}^n_h)$, the finite element solution of the modified Navier-Stokes equations with $\nabla p^n_{sep,h}$ as right hand side
  3. Set $\boldsymbol{u}^n_h = \boldsymbol{u}^n_{sep,h}$ and $p^n_h = p^{n-1}_h + \tilde{p}^n_h$

- **Remarks**
  - ◊ This algorithm is simple.
  - ◊ $p^n_{sep,h}$ in the first step can be taken as high order exrapolation, as for instance $p^n_{sep,h} = I(2p^{n-1}_h - p^{n-2}_h)$.

Edge-oriented Stabilization FEM

- **Based only on the "smoothness" of the discrete solution** we have proposed the following jump term

$$\sum_{\text{edge E}} \max(\gamma\nu h_E, \gamma^* h_E^2) \int_E [\nabla \boldsymbol{u}][\nabla \boldsymbol{v}] d\sigma \quad \text{with}\ \gamma, \gamma^* \in [0.0001, 0.1]$$

- only one generic stabilization takes care of all instabilities
  1. insatisfaction of Korn's inequality ($\gamma\nu h_E$)
  2. convection dominated flow for medium and high Reynolds number, even for pure transport ($\gamma^* h_E^2$)
- independent of the local Reynolds number and finite element space

! Can EO-FEM solve the spurious velocity problem ?

! If so, how to generalize the mesh-dependent penalty parameter ?

Numerical Example: Exact Solution

$$u_1(x,y) = 2x^2(1-x^2)(y(1-y)^2 - y^2(1-y)),$$

$$u_2(x,y) = 2y^2(1-y^2)(x(1-x)^2 - x^2(1-x)),$$ and

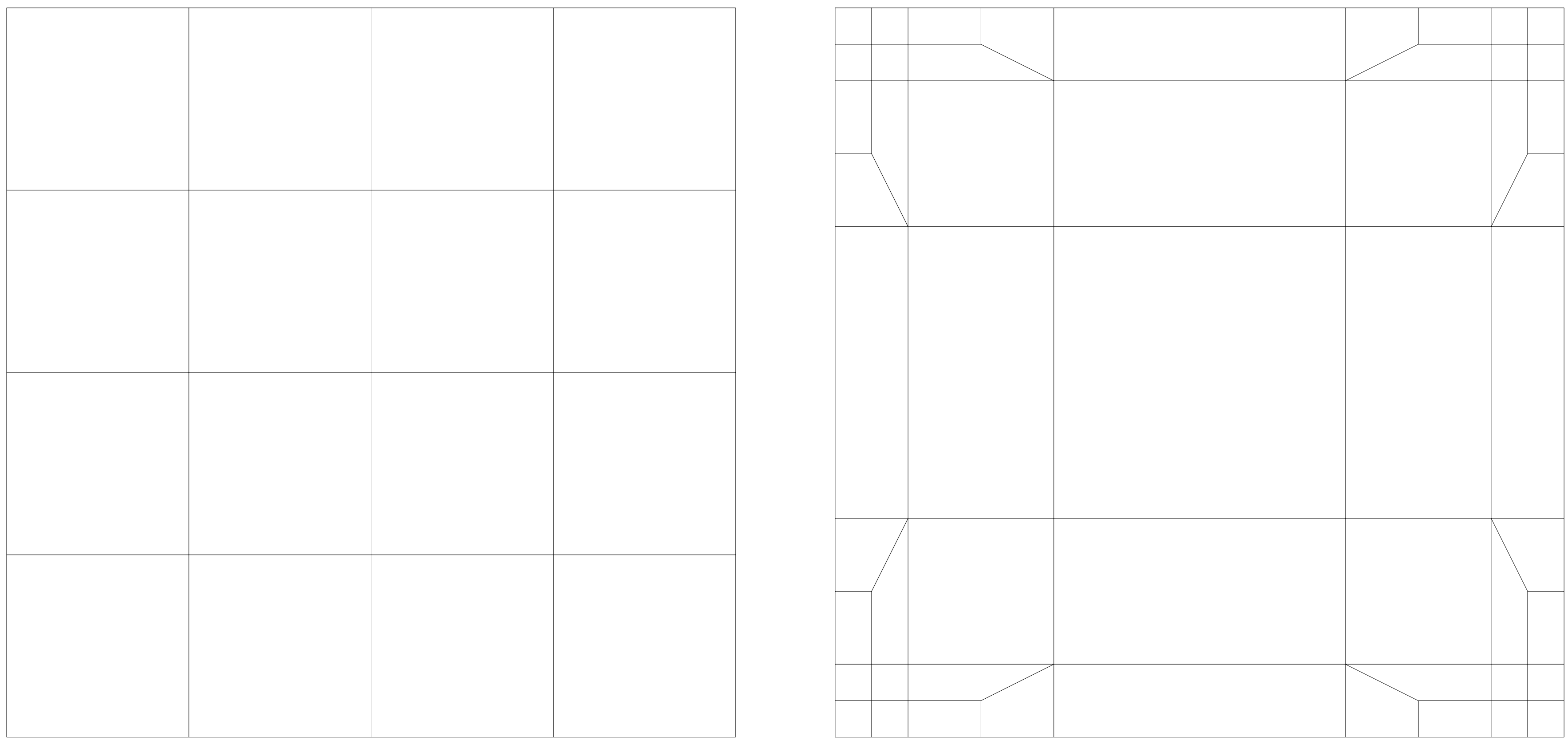
$$p(x,y) = c(x^3 - y^3 - 0.5)$$

Level	$\|\boldsymbol{u}-\boldsymbol{u}_h\|$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,h}$	$\|p-p_h\|$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,h}$	$\|p-p_h\|$
	without pressure separation			with pressure separation		
			Re=1d0, c=1d0			
8	5.2246E-05	0.01388741	0.00266865	1.3252E-05	0.00426955	0.0026781
9	1.3064E-05	0.00694537	0.00133426	3.2879E-06	0.00212224	0.0013366
			Re=1d0, c=1d3			
8	0.04997221	13.0541961	0.00266864	0.00224555	0.64752019	0.0026781
9	0.01257281	6.57264895	0.00133426	0.00039888	0.23009497	0.0013366
			Re=1d3, c=1d0			
8	0.03685431	9.02403558	0.002668620	0.00171123	0.49209582	0.0026787
9	0.01063014	5.36395474	0.001334261	0.00054886	0.19847427	0.0013367
			Re=1d3, c=1d3			
8	10.6143169	891.529162	0.002668820	0.21236541	61.2296913	0.0026803
9	3.08294602	827.562355	0.001334310	0.06854484	39.5259910	0.0013371

1. Inceasing Reynolds number and the external force diminish the approximate velocity
2. The pressure separation improves significantly the errors for the velocity

Numerical Example: Driven cavity (I)

We compute the kinetic energy, $\frac{1}{2}\int_{\Omega}\|\boldsymbol{u}\|^2\,dx$, for driven cavity problem with and without pressure separation on two different meshes



Two different coarse meshes for driven cavity test

Numerical Example: Driven cavity (II)

structured mesh Level	element	Energy without pressure separation	Energy with pressure separation
		RE=$1\,000$	
4	1024	4.25544672066505E-002	4.80952317761649E-002
5	4096	4.29275374609064E-002	4.58282779716431E-002
6	16384	4.35448330399813E-002	4.48431016405252E-002
7	65536	4.40902268089455E-002	4.45920117331099E-002
8	262144	4.43672829076416E-002	4.45354119388362E-002
9	1048576	4.44721759733802E-002	4.45222739778405E-002
		RE=$5\,000$	
4	1024	4.74420661810366E-002	6.17854505370949E-002
5	4096	4.54274532713127E-002	5.59686370659384E-002
6	16384	4.39420736427947E-002	5.01965232071455E-002
7	65536	4.46217041677444E-002	4.80202879276525E-002
8	262144	4.58956777737660E-002	4.75414996154341E-002
9	1048576	4.67713854860221E-002	4.74551921558477E-002

Clear improvement with pressure separation

Numerical Example: ***Driven cavity (III)***

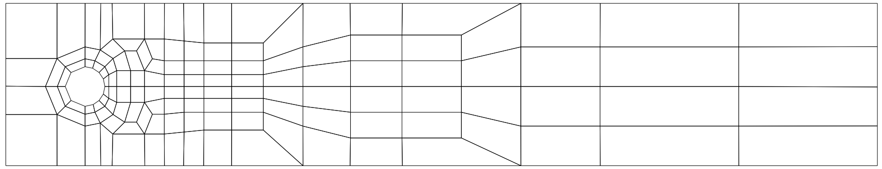
unstructured mesh		Energy	Energy
Level	element	without pressure separation	with pressure separation
		RE=$1\,000$	
4	3392	3.937830914695982E-002	4.258603278981851E-002
5	13568	4.200638598599166E-002	4.399538203747823E-002
6	54272	4.343367659646590E-002	4.442539201306887E-002
7	217088	4.410374815761506E-002	4.451194162495991E-002
8	868352	4.437769488742017E-002	4.452067552798117E-002
9	3473408	4.447552512831641E-002	4.451922842679222E-002
		RE=$5\,000$	
4	3392	3.694454761504827E-002	4.478929109363260E-002
5	13568	3.875516662905162E-002	4.519893868737240E-002
6	54272	4.160819937746986E-002	4.621377280452104E-002
7	217088	4.428310545723057E-002	4.712166739526875E-002
8	868352	4.596571822224903E-002	4.741444989451889E-002
9	3473408	4.683499558639783E-002	4.745377227286869E-002

Clear improvement with pressure separation even for semi-adaptive mesh near the singularity

! A posteriori error estimate and pressure separation, what relation to each other (conflictual, competitive, or mutually completing) ?

Numerical Example: Flow around obstacle

- Flow around cylinder 2D

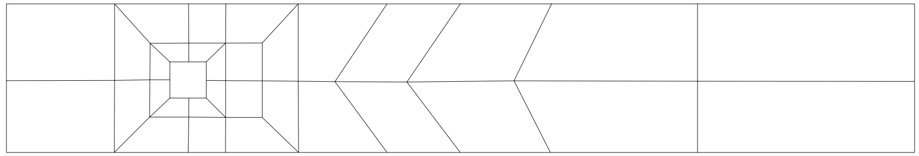


Force Level	C_{drag}	C_{lift}	C_{drag}	C_{lift}
	without pressure separation		with pressure separation	
Re=20 / reference values: $C_D = 5.5795$, $C_L = 0.01061$				
4	0.55803E+01	0.10143E-01	0.55703E+01	0.10350E-01
5	0.55789E+01	0.10435E-01	0.55707E+01	0.10462E-01
6	0.55793E+01	0.10559E-01	0.55747E+01	0.10548E-01
7	0.55795E+01	0.10601E-01	0.55771E+01	0.10588E-01
8	0.55795D+01	0.10614D-01	0.55783D+01	0.10605D-01
Re=50 / reference values: $C_D = 3.6940$, $C_L = -0.010730$				
4	0.37237E+01	-0.10959E-01	0.37041E+01	-0.11045E-01
5	0.37013E+01	-0.10794E-01	0.36897E+01	-0.10860E-01
6	0.36961E+01	-0.10749E-01	0.36908E+01	-0.10786E-01
7	0.36949E+01	-0.10741E-01	0.36925E+01	-0.10758E-01
8	0.36946D+01	-0.10739D-01	0.36935D+01	-0.10747D-01

1. No improvement with pressure separation
2. The "flow" is less dominated by the “ smooth pressure"

Numerical Example: Flow around obstacle

- Flow around square 2D



Force Level	C_{drag} without pressure separation	C_{lift} without pressure separation	C_{drag} with pressure separation	C_{lift} with pressure separation
		RE=20		
5	0.63864E+01	0.69694E-01	0.64587E+01	0.71458E-01
6	0.64239E+01	0.70204E-01	0.64640E+01	0.71122E-01
7	0.64463E+01	0.70646E-01	0.64705E+01	0.71153E-01
8	0.64572E+01	0.70908E-01	0.64728E+01	0.71201E-01
9	0.64624D+01	0.71039D-01	0.64726D+01	0.71217D-01
		RE=50		
5	0.41397E+01	0.23774E-01	0.41619E+01	0.24433E-01
6	0.41223E+01	0.23730E-01	0.41385E+01	0.24202E-01
7	0.41248E+01	0.23796E-01	0.41381E+01	0.24102E-01
8	0.41296E+01	0.23830E-01	0.41396E+01	0.24012E-01
9	0.41328D+01	0.23846D-01	0.41399D+01	0.23951D-01

1. Significant improvement with the pressure separation
2. The "flow" is more dominated by the pressure due to the corner singularity

Numerical Example: Flow around obstacle 3D

Reynolds number 20		without pressure separation		with pressure separation	
Level	cells	C_{drag}	$C_{\text{l}ift}$	C_{drag}	$C_{\text{l}ift}$
Flow around cylinder		reference values: $C_{\text{d}rag} = 7.75895$, $C_{\text{l}ift} = 6.87927E-2$			
3	8192	0.76277E+01	0.38110E-01	0.77676E+01	0.52552E-01
4	65536	0.77550E+01	0.54334E-01	0.77247E+01	0.63255E-01
5	524288	0.77438E+01	0.63013E-01	0.77342E+01	0.67294E-01
6	4194304	0.77447E+01	0.67372E-01	0.77556E+01	0.68589E-01
Flow around square		reference values: $C_{\text{d}rag} = 6.185331$, $C_{\text{l}ift} = 9.40136e-3$			
3	6144	0.59160E+01	-0.12441E-02	0.61155E+01	0.32743E-02
4	49152	0.61549E+01	0.47570E-02	0.61447E+01	0.80229E-02
5	393216	0.61829E+01	0.77422E-02	0.61602E+01	0.91252E-02
6	3145728	0.61861E+01	0.87470E-02	0.61721E+01	0.93316E-02

- Significant improvement with pressure separation for flow around obstacle with circular cross-section in contrast to 2D case

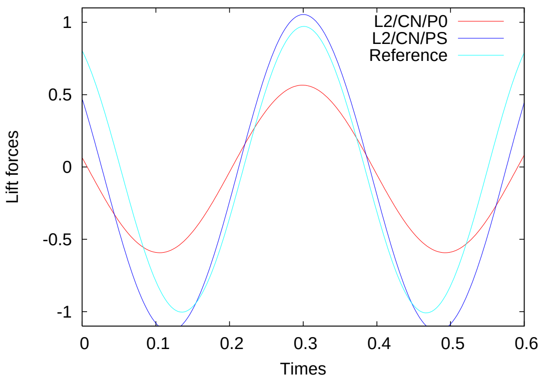
! The improvement in the lift and drag is only due to the improvement in the velocity or also to the improvement in the pressure ?

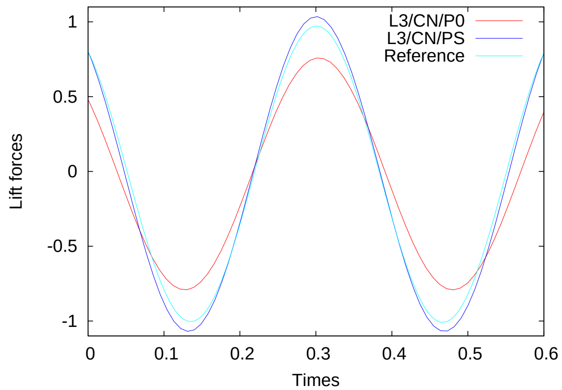
Numerical Example: Nonstationary flow around cylinder 2D

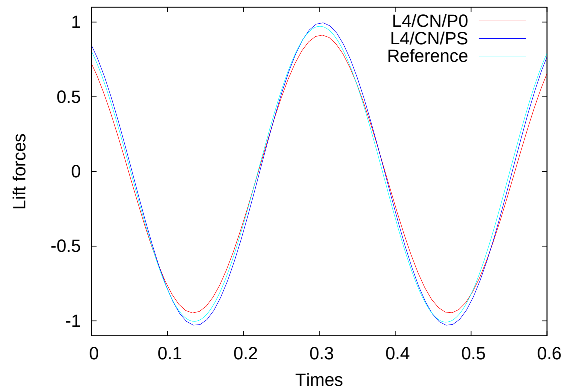
- Increase the Reynolds number, $RE = 100$, to analyze
  1. The dependency of the solution stability on RE
  2. The dependency of the a priori error contant on RE

Level	Δt	Max. amp.	Min. amp.	Strouhal	Max. amp.	Min. amp.	Strouhal
		without pressure separation			with pressure separation		
2	1d-3	0.56570	-0.59274	0.25839	1.06364	-1.123600	0.27548
	3d-3	0.56576	-0.59288	0.25839	1.22990	-1.126479	0.27777
	1d-2	0.56621	-0.58289	0.25641	1.30503	-1.179772	0.27778
3	1d-3	0.75874	-0.79120	0.29762	0.93465	-0.985156	0.29762
	3d-3	0.75823	-0.79084	0.28490	1.03266	-1.069062	0.29239
	1d-2	0.75787	-0.78942	0.30303	1.06471	-1.097485	0.30303
4	1d-3	0.91257	-0.94652	0.29761	0.99825	-1.038190	0.29498
	3d-3	0.91115	-0.94657	0.29239	0.99559	-1.028638	0.30030
	1d-2	0.91054	-0.94145	0.30303	1.00137	-1.009727	0.30303
5	1d-3	0.96380	-0.99826	0.30030	0.98527	-1.020072	0.30030
	3d-3	0.96278	-0.99883	0.30030	0.98717	-1.020784	0.30030
	1d-2	0.95174	-0.98505	0.30303	0.98817	-1.003275	0.30303

- Lift coefficient for level 2, 3, and 4 using Crack-Nicolson "CN" with pressure separation "PS" and without pressure separation "P0" for $RE = 100$







Pressure Separation brings significant improvement in the amplitude and the frequency

Standing vortex $Re = \infty$

- Consider the incompressible Navier-Stokes equations for inviscid flow ($Re = \infty$) in a unit square

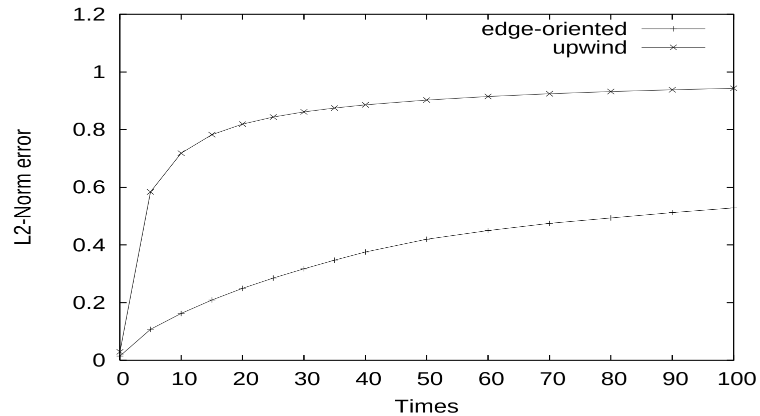
$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = 0, \quad \nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega = (0,1) \times (0,1). \tag{1}$$

$$r = \sqrt{(x-0.5)^2 + (y-0.5)^2}, \quad \boldsymbol{u}_r = 0, \quad \boldsymbol{u}_\theta = \begin{cases} 5r, & r < 0.2, \\ 2 - 5r, & 0.2 \le r \le 0.4, \\ 0, & r > 0.4, \end{cases} \tag{2}$$

! Which discretization schemes preserve the original vortex ?

- The L_2-Norm for the error w.r.t. the mesh refinement and time beside a long time simulation on level 3.

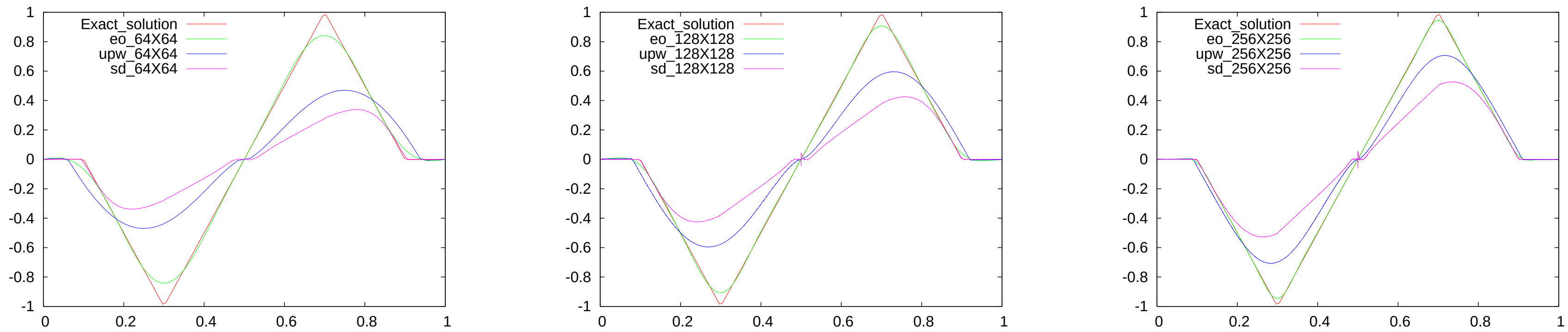
Stab.		EO			UPW	
Level	T=1	T=2	T=3	T=1	T=2	T=3
6	0.0551	0.0695	0.0823	0.250	0.383	0.471
7	0.0252	0.0337	0.0401	0.154	0.249	0.324
8	0.0115	0.0153	0.0184	0.087	0.151	0.204
9	0.0052	0.0070	0.0084	0.049	0.087	0.120
10	0.0024	0.0032	0.0039	0.027	0.049	0.068



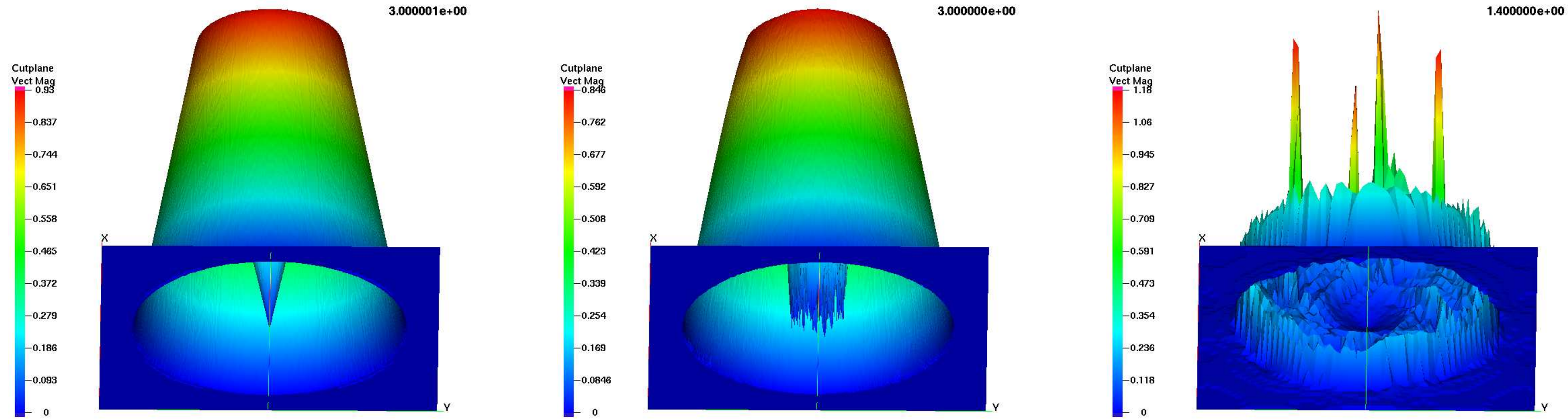
Edge-oriented stabilization preserves ”perfectly” the solution with high accuracy

Standing vortex $Re = \infty$

- Cutline of the x-velocity component for the 'Standing Vortex' problem



- 3D presentation of the norm of the velocity (VECT MAG) for the 'Standing Vortex' problem: edge-oriented (left) vs. streamline-diffusion ($\delta^* = 0.5$) (middle) and central (right)

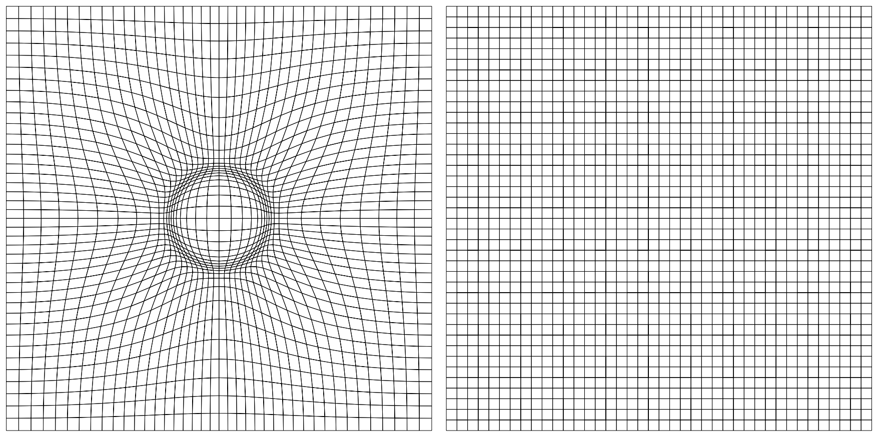


- **EO and the Entropy condition !?**
- **Can EO be seen as a scheme which just filter the non-desirable spurious mode**

Flow with interfaces

- **Common observation**
  1. Spurious velocities concentrated in the vicinity of the interface
  2. Reduction of the convergence order
- **Remedy**
  1. Grid deformation ?

     Adaptive mesh versus the simple one with the same number of elements

     ◇ The mesh resolves the interface

     ◇ The mesh preserves the same connectivities for efficiency
  2. Pressure Separation ?
  3. Edge-oriented FEM ?

Flow with interfaces: Static bubble

- This test case models a perfectly stationary circular bubble at equilibrium
- Ideally, we should have
  1. A zero velocity field
  2. According to the Laplace-Young law

$$p_{in} = p_{out} + \sigma / r$$

p_{in}	The pressure inside bubble
p_{out}	The pressure outside the bubble
σ	The surface tension coefficient
r	The radius of the bubble

Static bubble: Errors on a simple mesh

Level	$\lvert p_{in} - p_{out}\rvert/(\frac{\sigma}{r})$	$\lVert \boldsymbol{u} - \boldsymbol{u}_h \rVert_0$	$\lvert \boldsymbol{u} - \boldsymbol{u}_h \rvert_{1,h}$	N/MG	$\lvert p_{in} - p_{out}\rvert/(\frac{\sigma}{r})$	$\lVert \boldsymbol{u} - \boldsymbol{u}_h \rVert_0$	$\lvert \boldsymbol{u} - \boldsymbol{u}_h \rvert_{1,h}$	N/MG
	without pressure separation				with pressure separation			
	without edge-oriented FEM							
4	0.954349	0.00260818914	0.207652409	5/1	0.9923660	0.00155247296	0.118638289	5/1
5	0.979682	0.00097177495	0.153784641	5/1	0.9975795	0.00060710946	0.0917261177	5/1
6	0.992961	0.00036200902	0.112884238	4/1	1.0012544	0.00023717308	0.0694932176	4/1
7	0.997166	0.00013827272	0.082118238	4/1	1.0010944	9.7099099E-05	0.0514852452	4/1
	with edge-oriented FEM with global constant penalty parameter $\gamma = 1d1$							
4	0.951601	3.340218E-05	0.0025474881	6/1	0.9520738	3.330621E-05	0.00256013778	6/1
5	0.979383	1.086225E-05	0.0016437028	5/1	0.9792187	1.214066E-05	0.00181935191	5/1
6	0.992989	4.221967E-06	0.0012491933	5/1	0.9926422	4.712407E-06	0.00140673313	5/1
7	0.997110	1.624452E-06	0.0009130933	4/1	0.9966825	1.725864E-06	0.00103094094	4/1
	with edge-oriented FEM with global constant penalty parameter $\gamma = 1d3$							
4	0.951998	3.385404E-07	2.600357E-05	6/1	0.988809	2.201445E-07	1.644319E-05	6/1
5	0.979198	1.233316E-07	1.846353E-05	5/1	0.997101	8.065997E-08	1.169144E-05	5/1
6	0.992635	4.789259E-08	1.428115E-05	5/1	1.001279	3.218998E-08	9.075670E-06	5/1
7	0.996678	1.753280E-08	1.046342E-05	4/1	1.000998	1.258395E-08	6.466150E-06	4/1
	with edge-oriented FEM with local penalty parameter γ as a function of the monitor/distance function							
4	0.949683	3.617674E-07	2.686804E-05	6/1	0.986366	2.402242E-07	1.756815E-05	6/1
5	0.978834	1.191665E-07	1.730090E-05	5/1	0.996440	9.042803E-08	1.250519E-05	5/1
6	0.992673	4.752538E-08	1.313859E-05	5/1	1.000876	3.748013E-08	9.682061E-06	5/1
7	0.996931	2.010113E-08	9.567174E-06	4/1	1.000757	1.671886E-08	6.858230E-06	4/1

- **Pressure Separation:** Good results for the pressure
- **Edge-oriented FEM:** Excellent results for the velocity with any desired error & no degradation in the performance of the iterative solver

Static bubble: Errors on an adapted mesh

Level	$\lvert p_{in} - p_{out}\rvert/(\frac{\sigma}{r})$	$\lVert \boldsymbol{u} - \boldsymbol{u}_h \rVert_0$	$\lvert \boldsymbol{u} - \boldsymbol{u}_h \rvert_{1,h}$	N/MG	$\lvert p_{in} - p_{out}\rvert/(\frac{\sigma}{r})$	$\lVert \boldsymbol{u} - \boldsymbol{u}_h \rVert_0$	$\lvert \boldsymbol{u} - \boldsymbol{u}_h \rvert_{1,h}$	N/MG
	without pressure separation				with pressure separation			
	without edge-oriented FEM							
4	1.000669	0.0001899205	0.09765440	6/1	1.0019009	0.0001749634	0.04170730	6/1
5	1.000135	3.503739E-05	0.05796067	5/1	1.0009837	5.679786E-05	0.03268579	5/1
6	1.000032	6.628077E-06	0.03782558	4/1	1.0003227	1.897943E-05	0.02315339	3/1
7	1.000000	2.257852E-06	0.02894883	4/1	1.0001409	6.480485E-06	0.01641194	4/1
	with edge-oriented FEM with global constant penalty parameter $\gamma = 1d1$							
4	1.000719	1.872302E-05	0.004474451	5/1	1.0008292	1.559681E-05	0.00334168	5/1
5	1.000336	4.214648E-06	0.002285405	4/2	1.0005137	5.341385E-06	0.00252941	4/2
6	1.000109	1.665666E-06	0.001819288	4/2	1.0001367	2.051831E-06	0.00201336	4/2
7	1.000040	5.368627E-07	0.001158316	4/2	1.0000440	6.515407E-07	0.00128245	4/2
	with edge-oriented FEM with global constant penalty parameter $\gamma = 1d3$							
4	1.000712	2.186715E-07	5.113188E-05	5/1	1.0006502	1.810321E-07	3.810090E-05	5/1
5	1.000347	5.257776E-08	2.710133E-05	4/2	1.0004652	6.212258E-08	2.860246E-05	4/2
6	1.000113	2.139767E-08	2.195185E-05	4/2	1.0001190	2.437015E-08	2.309864E-05	4/2
7	1.000043	6.806535E-09	1.378594E-05	4/2	1.0000350	7.652504E-09	1.453748E-05	4/2
	with edge-oriented FEM with local penalty parameter γ as a function of the monitor/distance function							
4	1.000599	5.221021E-07	0.0001083159	6/1	1.0008286	4.957030E-07	8.887125E-05	5/1
5	1.000277	1.994104E-07	8.527144E-05	4/2	1.0000613	1.726385E-07	6.962604E-05	4/2
6	0.999927	7.079676E-08	6.249968E-05	5/2	0.9997869	7.318976E-08	5.877441E-05	5/2
7	1.000017	2.608346E-08	4.290585E-05	4/2	0.9999271	2.460148E-08	3.811101E-05	4/2

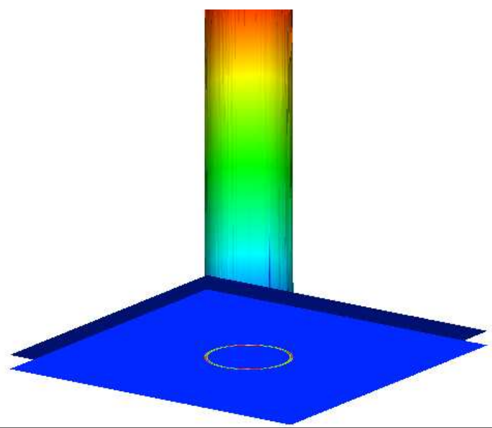
- **Grid deformation:** Good results for the pressure & amelioration in the velocity
- **Edge-oriented FEM:** Excellent results for the velocity with any desired error & no degradation in the performance of the iterative solver

Flow with interface

- **Pressure Separation:** Good results for the pressure mainly seen on non adapted mesh
- **Grid deformation:** Good results for the pressure as well as significant amelioration in the velocity
- **Edge-oriented FEM:**
  1. Excellent results for the velocity with any desired error without any degradation in the performance of the iterative solver for both adapted and non adapted mesh
  2. The penalty mesh-dependent parameter can be applied as global constant as well as a function of the interface or location of the spurious velocity; $\sum_{\text{edge E}} \max[\gamma \nu h_E, \gamma^* h_E^2, \gamma_{\text{dist}} f(\textit{dist}(\Gamma); h_E)) h_E] \int_E [\nabla \boldsymbol{u}][\nabla \boldsymbol{v}] d\sigma$ with ,

     $\gamma_{\text{dist}} \gg 0$ (big enough), $\textit{dist}(\Gamma)$ a distance function to the interface, and f any variant of dirac function

Example of local mesh-dependent penalty parameter for edge-oriented FEM (high values, only, in the vicinity of the interface)



Summary and outlook

A numerical investigation highlighting the effectiveness of pressure separation and edge-oriented FEM to deal with spurious velocities in incompressible flow problems due to

- dominant pressure
- small viscosity
- boundary layer
- interfaces

Both schemes conjoint successfully to solve the issue of spurious velocities for a large class of incompressible flow problems, nevertheless more should be done concerning

- pressure separation and higher order FEM
- pressure separation and time dependent problems
- grid deformation and non analytical monitor function