

FEM Multigrid Techniques for Viscoelastic Flow

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Multiscale CFD Problems

Inertia turbulence



> Numerical instabilities + problems

Turbulence Models Stabilization Techniques



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Turbulence flow inside a pipe. From ProPipe



Characteristics:

- Irregular temporal behaviour and spatially disordered
- Broad range of spatial/temporal scales





Multiscale CFD Problems

...looks like developed turbulence...



Elastic turbulence

- Re<<1, Wi>>1 (less inertia, more elasticity)
- Coil stretching, high stresses
 - > Numerical instabilities + problems

Flow models: Oldroyd, Maxwell,... Stabilization: EEME, EEVS, DEVSS/DG, SD, SUPG,...



Nonlinear Flow Models

Generalized Navier-Stokes equations

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \operatorname{div} \sigma + \nabla p = \rho \mathbf{f}, \text{ div } \mathbf{u} = 0,$$
$$\frac{\partial \Theta}{\partial t} + \mathbf{u} \cdot \nabla \Theta - \operatorname{div} k \nabla \Theta - \mathbf{D} : \sigma = 0,$$
$$\sigma = \sigma^{s} + \sigma^{p}, \ \mathbf{D} = \frac{1}{2} \left(\nabla \mathbf{u} + \nabla \mathbf{u}^{T} \right)$$

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Quasi-Newtonian part $\sigma^{s} = 2\eta_{s} (D, \Theta)D$

Viscoleastic part
$$\sigma^{p} + \Lambda \frac{\delta_{a} \sigma^{p}}{\delta t} = 2\eta_{p} D,$$

$$\frac{\delta_{a} \sigma}{\delta t} = \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \sigma + \frac{1-a}{2} \left(\sigma \nabla \mathbf{u} + \nabla \mathbf{u}^{T} \sigma\right) - \frac{1+a}{2} \left(\nabla \mathbf{u} \sigma + \sigma \nabla \mathbf{u}^{T}\right)$$



Required: I) Special Models

 $T + \Lambda \frac{\delta_a T}{\delta t} = 2\eta_0 \left(D + \Lambda_r \frac{\delta_a D}{\delta t} \right) \left| T + \Lambda \frac{\delta_a T}{\delta t} + B(T) = 2\eta D$

Oldroyd A Oldroyd B Maxwell A **Maxwell B**

Jeffreys

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Phan-Thien Tanner Phan-Thien Giesekus





Required: II) Special Numerics

Special FEM Techniques

Multigrid Solvers

Stabilization for high Re and Wi Numbers

Implicit Approaches

Space-Time Adaptivity

Grid Deformation Methods

Newton Methods





Our Numerical Approach

Fully implicit monolithic multigrid FEM solver



Numerical Techniques

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> The FEM techniques have to handle the following challenging points

- Stable FE spaces for velocity and pressure fields, and velocity and extrastress fields → Q2/P1/? or Q1(nc)/P0/? (new: Q2(nc)/P1/?)
- Special treatment of the convective terms $u \cdot \nabla u$, $u \cdot \nabla \Theta$, $u \cdot \nabla \sigma$ \rightarrow edge-oriented/interior penalty FEM, TVD/FCT
- The presence of the "reactive" terms which are responsible for
 - high Weissenberg number problem (HWNP) \rightarrow LCR
 - blow up phenomena for time dependent solution
- > The (nonlinear) solvers have to deal with different source of nonlinearity
 - nonlinear viscosities \rightarrow Newton method via divided differences
 - the strong coupling of equations \rightarrow monolithic multigrid approach
 - complex geometries and meshes



Newton Solver

Solve for the residual of the nonlinear system algebraic equations

$$R(\mathbf{x}) = 0, \mathbf{x} = (\mathbf{u}, \Theta, \sigma, p)$$

Newton method with damping results in iterations of the form

$$\mathbf{x}^{n+1} = \mathbf{x}^{n} + \boldsymbol{\omega}^{n} \left[\frac{\partial R(\mathbf{x}^{n})}{\partial \mathbf{x}} \right]^{-1} R(\mathbf{x}^{n})$$

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Continuous Newton: on variational level (before discretization)
 The continuous Frechet operator can be calculated

Inexact Newton: on matrix level (after discretization)

 \rightarrow The Jacobian matrix is **approximated** using finite differences as

$$\left[\frac{\partial R(\mathbf{x}^{n})}{\partial x}\right]_{ij} \approx \frac{R_{i}(\mathbf{x}^{n} + \varepsilon \mathbf{e}_{j}) - R_{i}(\mathbf{x}^{n} - \varepsilon \mathbf{e}_{j})}{2\varepsilon}$$



Multigrid Solver

- Standard geometric multigrid approach
- > Full Q_2 , $\tilde{Q_1}$, P_1^{disc} and P_0 grid transfer
- Smoother: Local/Global MPSC
 - Local MPSC via Vanka-like smoother

Coupled multigrid solver

$$\begin{bmatrix} \mathbf{u}^{l+1} \\ \sigma^{l+1} \\ \Theta^{l+1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^{l} \\ \sigma^{l} \\ \Theta^{l} \\ p^{l} \end{bmatrix} + \omega^{l} \Sigma_{T \in \tau_{h}} \begin{bmatrix} K + J \end{bmatrix}_{T}^{-1} \begin{bmatrix} \operatorname{Res}_{u} \\ \operatorname{Res}_{\sigma} \\ \operatorname{Res}_{\Theta} \\ \operatorname{Res}_{p} \end{bmatrix}_{T}$$

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- Global MPSC
 - solve for an intermediate \$\tilde u (generalized momentum equation)
 - solve for p (pressure Poisson equation)
 - update of **u** and *p*
 - solve for Θ (tracer equation)
 - solve for σ (constitutive equation)

Decoupled multigrid solver





Viscoelastic Models

Different highly developed models

Oldroyd A/B, Maxwell A/B, Jeffreys, PTT, Giesekus

...nevertheless, despite "good" models and "good" Numerics, the HWNP ("High Weissenberg Number problem") stills exists for critical Wi, resp., De numbers...





Problem Reformulation

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Old
$$\rightarrow$$
 (u, p, σ^{p})

$$\rho(\frac{\partial u}{\partial t} + u \cdot \nabla u) = \nabla p - 2\eta_{s} \nabla \cdot D - \nabla \cdot \sigma^{p},$$

$$\nabla \cdot u = 0,$$

$$\Lambda \frac{\delta_{a} \sigma^{p}}{\delta t} + \sigma^{p} - 2\eta_{p} D = 0,$$
(1)

Conformation tensor \rightarrow (u, p, τ) This tensor is positive definite by design !! Replace σ^{p} in (1) with $\sigma^{p} = \frac{\eta_{p}}{\Lambda}(\tau - I) \rightarrow$ special discretization : TVD

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = \nabla p - 2\eta_s \nabla \cdot D - \frac{\eta_p}{\Lambda} \nabla \cdot \tau,$$

$$\nabla \cdot \mathbf{u} = 0,$$

$$\frac{\delta_a \tau}{\delta t} + \frac{1}{\Lambda} (\tau - \mathbf{I}) = 0,$$
(2)





Properties of Conformation Tensor

$$\tau(X,t) = \int_{\infty}^{t} \frac{\eta_{p}}{\Lambda} \exp\left(\frac{-(t-s)}{\sqrt{\Lambda}}\right) F(s,t) F(s,t)^{T} ds$$
Positive by design,
so we can take its logarithm

2 observations:

- positive definite \rightarrow special discretizations like FCT/TVD
- exponential behaviour \rightarrow approximation by polynomials???





Driven Cavity

Cutline of Stress_11 component at y = 1.0



→ We= 0.5 → We= 1.5









Problem Reformulation

M. Behr \rightarrow (u, p, ψ)

Replace τ in (2) with $\tau = \exp \psi$

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = \nabla p - 2\eta_s \nabla \cdot D - \frac{\eta_p}{\Lambda} \nabla \cdot (\exp \psi),$$

$$\nabla \cdot \mathbf{u} = 0,$$

$$\frac{\delta_a(\exp \psi)}{\delta t} + \frac{1}{\Lambda} (\exp \psi - \mathbf{I}) = 0,$$
(3)

Gradient of exponential of $\psi \rightarrow ???$

Solvers \rightarrow ???



LCR Formulation (I)

Experiences:

- Stresses grow exponentially
- Conformation stress is positive by design
- Stretching part creates numerical problem

$$\left(\nabla \mathbf{u} \, \boldsymbol{\sigma} + \boldsymbol{\sigma} \, \nabla \mathbf{u}^T \right)$$

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Idea ("Kupferman Trick"):

• Decompose the velocity gradient inside the stretching part

$$\nabla u = \Omega + B + N\sigma_c^{-1}$$

• Take the logarithm as a new variable (ψ =log σ) using eigenvalue problem

$$\psi = R \log(\lambda_{\tau}) R^{T}$$

LCR Formulation (II) $\tau = \exp \psi$

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Inside the constitutive equation (2), decompose $\nabla u = \Omega + B + N \sigma_c^{-1}$

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right) \tau - \left(\Omega \tau - \tau \Omega\right) + 2B\tau = \frac{1}{\Lambda} \left(\mathbf{I} - \tau\right)$$
(4)

Matrix B is purely extension: Responsible for the stretching

$$\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \quad \text{... it is commutable !!!} \qquad B \ \tau + \tau \ B = 2B \ \tau$$
$$\boxed{\begin{array}{c} \partial \tau \\ \partial t \end{array}} = 2B \ \tau \qquad \Longrightarrow \qquad \frac{\partial \psi}{\partial t} = 2B \end{array}$$

Matrix Ω is purely rotation: Responsible for the rotating

$$\begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} \quad \text{mit is symmetric !!!} \quad (\Omega \ \tau - \tau \ \Omega)_{ij} = (\Omega \ \tau - \tau \ \Omega)_{ji}$$
$$\frac{\partial \psi}{\partial t} = (\Omega \ \psi - \psi \ \Omega)$$



LCR Formulation (III)

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 $\frac{\partial(\exp\psi)}{\partial t} = \frac{\partial\psi}{\partial t} \exp\psi$

As in M. Behr, replace in (4) $\tau = \exp \psi$ decouples $2B\tau$

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = \nabla p - 2\eta_s \nabla \cdot D - \frac{\eta_p}{\Lambda} \nabla \cdot \exp \psi,$$

$$\nabla \cdot \mathbf{u} = 0,$$

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right) \psi - \left(\Omega \psi - \psi \Omega\right) + 2B = \frac{1}{\Lambda} \left(e^{-\psi} - \mathbf{I}\right),$$
(5)

Note: Divergence of exponential of ψ is calculated explicitly using eigenvalue problem !!

Standard discretization techniques \rightarrow EO, TVD Standard nonlinear (Newton)and linear (MG) solvers

 \rightarrow Increases the critical Wi number dramatically !!





Numerical Results (steady problem tests)

1. Driven cavity

Velocity profile at the upper wall: $v_{in} x^2 (1-x)^2$ Dirichtlet Bc's everywhere Stress field: Neuman Bc's $v_{in} = 16$



Velocity profile at the inlet:

Out flow: Neuman Bc's

Stress field: Neuman Bc's

$$\frac{3}{128}v_{in}\left(16-y^2\right)$$

$$v_{in} = 1.0$$









Numerical Results (unsteady problem test)

Driven cavity

Velocity profile at the upper wall:

$$v_{in} x^{2} (1-x)^{2}$$

$$v_{in} = 8(1 + \tanh(8(t-0.5)))$$
For t > 1, v_{in} = 16



Dirichtlet Bc's everywhere Stress field: Neuman Bc's



Stream function

t = 8

We = 1



Increasing Wi number shifts the stream to the left



We = 3

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 Ψ_{11}



Increasing Wi number increases psi by a factor of 1





 σ_{11}

t = 8





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Velocities



Increasing Wi number does not give much impact to the velocity field



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Summary

With LCR, we are now able to simulate much higher Wi numbers

- \rightarrow Wi ~ 1.0 for 4 to 1 configuration
- \rightarrow Wi ~ 0.5 for square

NEW:

- → Wi >> 4.5 for 4 to 1 contraction (steady state)
- → Wi >> 1.5 for square (steady state)

Additional stabilization will help for high Re + Wi numbers

→ LCR + Edge Oriented/TVD stabillization

Application to other viscoelastic flow models

