# *FEM Multigrid Techniques for Viscoelastic Flow*

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**http://www.mathematik.tu-dortmund.de/lsiii**
**www.featflow.de**

# Multiscale CFD problems

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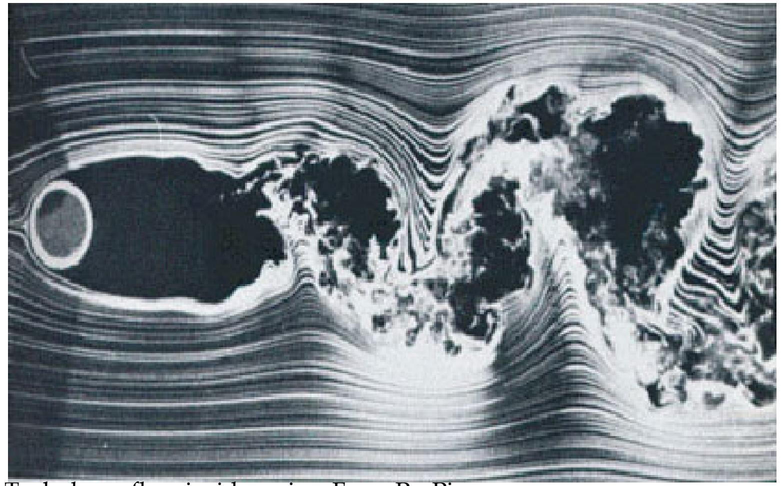
**Inertia turbulence**

☼ Re>>1
☼ Numerical instabilities + problems



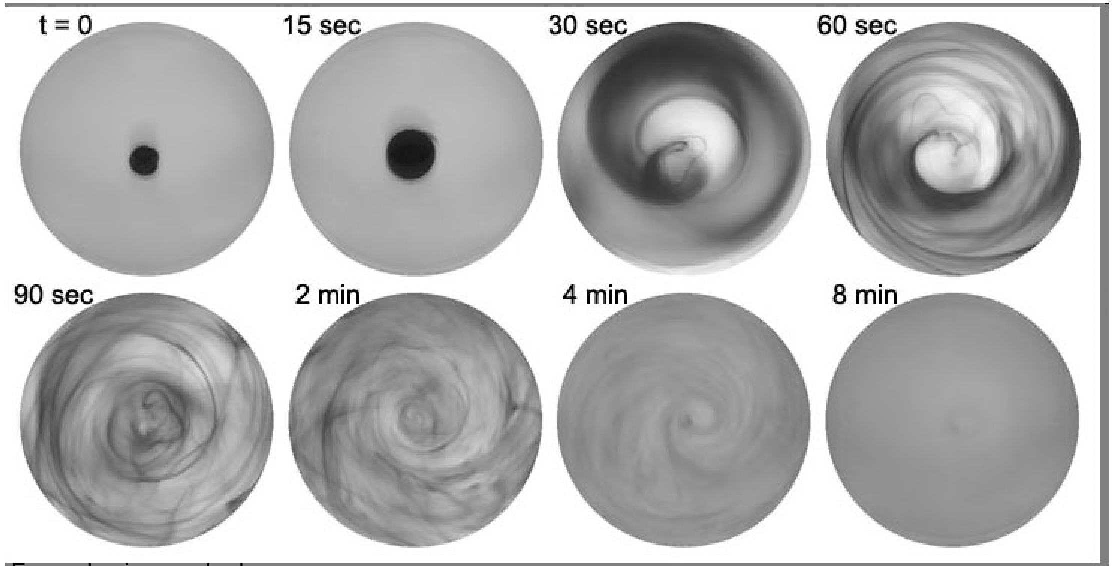
Turbulence flow inside a pipe. From ProPipe

☼ Turbulence Models
☼ Stabilization Techniques

<u>**Characteristics:**</u>
☼ Irregular temporal behaviour and spatially disordered
☼ Broad range of spatial/temporal scales

**Elastic turbulence**

☼ Re<<1, *We*>>1 (less inertia, more elasticity)
☼ Numerical instabilities + problems (HWNP)




From physics.ucsd.edu

☼ Flow models: Oldroyd, Maxwell,…
☼ Stabilization: EEME, EEVS, DEVSS/DG, SD, SUPG,…

# Nonlinear flow models

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$$\begin{cases} \partial_t u + (\nabla u)u = \nabla\cdot\sigma - \nabla p + (1-\gamma\vartheta)j \\ \nabla\cdot u = 0 \\ \partial_t\vartheta + (\nabla u)\vartheta = \nabla\cdot k\nabla\vartheta + \sigma : D \\ \sigma^p + \Lambda\,\delta_t^a\sigma^p = 2\eta_p D, \end{cases}$$

NS

EE

SE

$$\sigma = \sigma^s + \sigma^p, \eta_0 = \eta_s + \eta_p,$$

$$\sigma^s = 2\eta_s(D,\Theta)D,$$

$$D = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$

$$\delta_t^a\sigma^p = \partial_t\sigma + (\nabla u)\sigma + \frac{1-a}{2}\left(\sigma\nabla \boldsymbol{u} + \nabla\boldsymbol{u}^T\sigma\right) - \frac{1+a}{2}\left(\nabla\boldsymbol{u}\,\sigma + \sigma\nabla\boldsymbol{u}^T\right)$$

# Required: I) special models

$$T + \Lambda \delta_t^a T = 2\eta_0 \left( D + \Lambda_r \delta_t^a D \right)$$

**Oldroyd A**

**Oldroyd B**

**Maxwell A**

**Maxwell B**

**Jeffreys**

$$T + \Lambda \delta_t^a T + B(T) = 2\eta D$$

**Phan-Thien Tanner**

**Phan-Thien**

**Giesekus**

# Required: II) special numerics

Special FEM Techniques

Multigrid Solvers

Stabilization for high Re and We numbers

Implicit Approaches

Space-Time Adaptivity

Grid Deformation Methods

Newton Methods

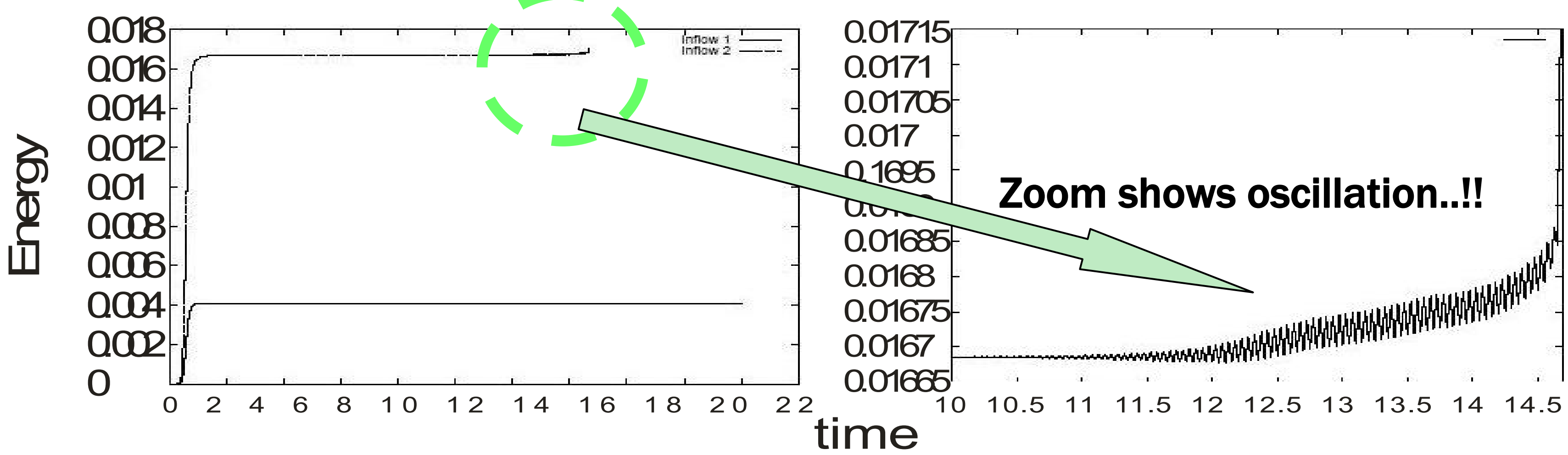
# Problems remain

Different highly developed models

**Oldroyd A/B, Maxwell A/B, Jeffreys, PTT, Giesekus**

… nevertheless, depite „good“ models and „good“ numerics, the HWNP („High Weissenberg Number Problem“) still exists for critical We, resp., De numbers…

**Kinetic Energy for two different velocity inflow**



# Our numerical approach

Fully implicit monolithic FEM
Multigrid solver !

# Numerical techniques



- **The FEM techniques have to handle the following challenging points**
  - ☼ **Stable FE spaces for velocity and pressure fields, and velocity and extra-stress fields → Q2/P1/Q2 or Q1(nc)/P0/Q1(nc) (new: Q2(nc)/P1/Q2(nc))**
  - ☼ **Special treatment of the convective terms** $u \cdot \nabla u$, $u \cdot \nabla \Theta$, $u \cdot \nabla \sigma$
    **→ edge-oriented/interior penalty FEM, TVD/FCT**
  - ☼ high Weissenberg number problem (HWNP) → **LCR**

- **The (nonlinear) solvers have to deal with different source of nonlinearity**
  - ☼ **nonlinear viscosities → Newton method via divided differences**
  - ☼ **the strong coupling of equations → monolithic multigrid approach**
  - ☼ **complex geometries and meshes**

# Newton solver



Solve for the residual of the nonlinear system algebraic equations

$$R(\mathrm{x}) = 0, \quad \mathrm{x} = (\mathrm{u}, \Theta, \sigma, p)$$

Newton method with damping results in iterations of the form

$$\mathrm{x}^{n+1} = \mathrm{x}^{n} + \omega^{n} \left[ \frac{\partial R(\mathrm{x}^{n})}{\partial \mathrm{x}} \right]^{-1} R(\mathrm{x}^{n})$$

☼ Continuous Newton: on variational level (before discretization)

→ The continuous Frechet operator can be calculated

☼ Inexact Newton: on matrix level (after discretization)

→ The Jacobian matrix is **approximated** using finite differences as

$$\left[ \frac{\partial R(\mathrm{x}^{n})}{\partial x} \right]_{ij} \approx \frac{R_i(\mathrm{x}^{n} + \varepsilon \mathrm{e}_j) - R_i(\mathrm{x}^{n} - \varepsilon \mathrm{e}_j)}{2\varepsilon}$$

# Multigrid solver

- Standard geometric multigrid approach
- Full $Q_2$ and $P_1^{disc}$ grid transfer
- Smoother: Local/Global MPSC
  - Local MPSC via Vanka-like smoother

$$\begin{bmatrix} \mathrm{u}^{l+1} \\ \sigma^{l+1} \\ \Theta^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} \mathrm{u}^{l} \\ \sigma^{l} \\ \Theta^{l} \\ p^{l} \end{bmatrix} + \omega^l \sum_{T \in \tau_h} [J]_{|T}^{-1} \begin{bmatrix} \mathrm{Res}_{\mathrm{u}} \\ \mathrm{Res}_{\sigma} \\ \mathrm{Res}_{\Theta} \\ \mathrm{Res}_{p} \end{bmatrix}_{|T}$$

**Monilithic multigrid solver**

  - Global MPSC
    - solve for an intermediate $\tilde{\mathrm{u}}$ (generalized momentum equation)
    - solve for $p$ (pressure Poisson equation)
    - update of $\mathbf{u}$ and $p$
    - solve for $\Theta$ (tracer equation)
    - solve for $\sigma$ (constitutive equation)

**Decoupled multigrid solver**

# Problem reformulation



**Old** → ( u, p, $\boldsymbol{\sigma}^p$ )

$$\left.\begin{aligned} &\rho\left(\frac{\partial \mathrm{u}}{\partial t} + \mathrm{u}\cdot\nabla\mathrm{u}\right) = \nabla p - 2\eta_s \nabla\cdot D - \nabla\cdot\sigma^p, \\ &\nabla\cdot\mathrm{u} = 0, \\ &\Lambda\frac{\delta_a \sigma^p}{\delta t} + \sigma^p - 2\eta_p D = 0, \end{aligned}\right\} \quad \textbf{(1)}$$

**Conformation tensor** → ( u, p, $\boldsymbol{\tau}$ ) This tensor is positive definite by design !!

Replace $\boldsymbol{\sigma}^p$ in (1) with $\sigma^p = \frac{\eta_p}{\Lambda}(\tau - \mathrm{I})$ → special discretization : TVD

$$\left.\begin{aligned} &\rho\left(\frac{\partial \mathrm{u}}{\partial t} + \mathrm{u}\cdot\nabla\mathrm{u}\right) = \nabla p - 2\eta_s \nabla\cdot D - \frac{\eta_p}{\Lambda}\nabla\cdot\tau, \\ &\nabla\cdot\mathrm{u} = 0, \\ &\frac{\delta_a \tau}{\delta t} + \frac{1}{\Lambda}(\tau - \mathrm{I}) = 0, \end{aligned}\right\} \quad \textbf{(2)}$$

# Conformation tensor property



$$\tau(t) = \int_{\infty}^{t} \frac{1}{We} \exp\left(\frac{-(t-s)}{We}\right) F(s,t)\, F(s,t)^{T}\, ds$$

Positive by design,
so we can take its logarithm

**<u>2 observations:</u>**

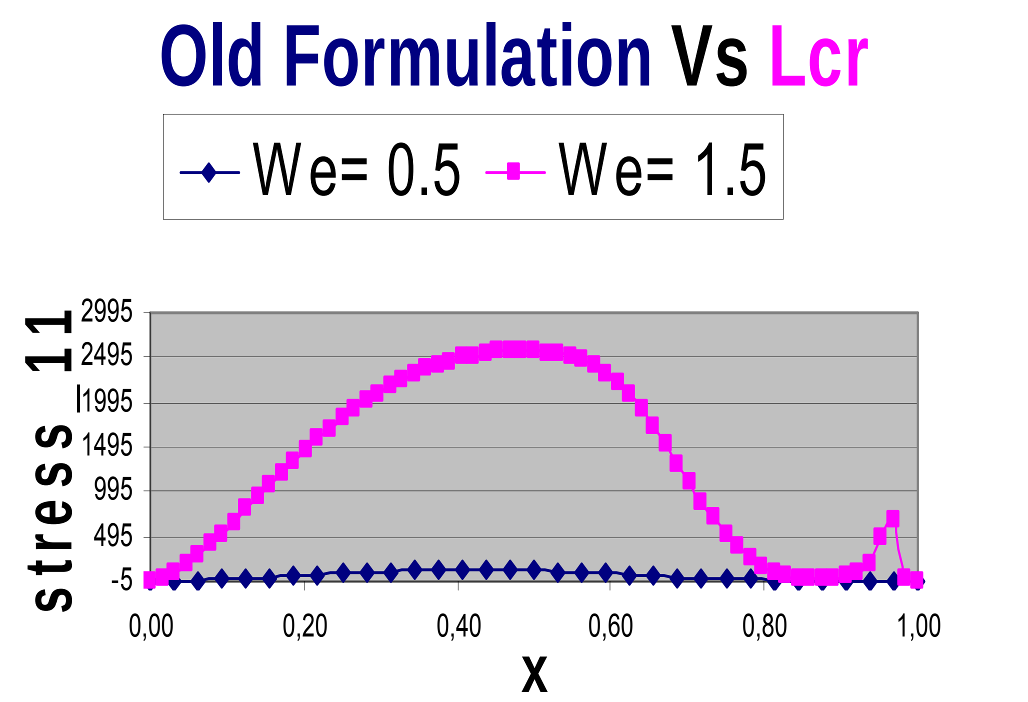
- positive definite → special discretizations like FCT/TVD
- exponential behaviour → approximation by polynomials???

# Exponential behaviour

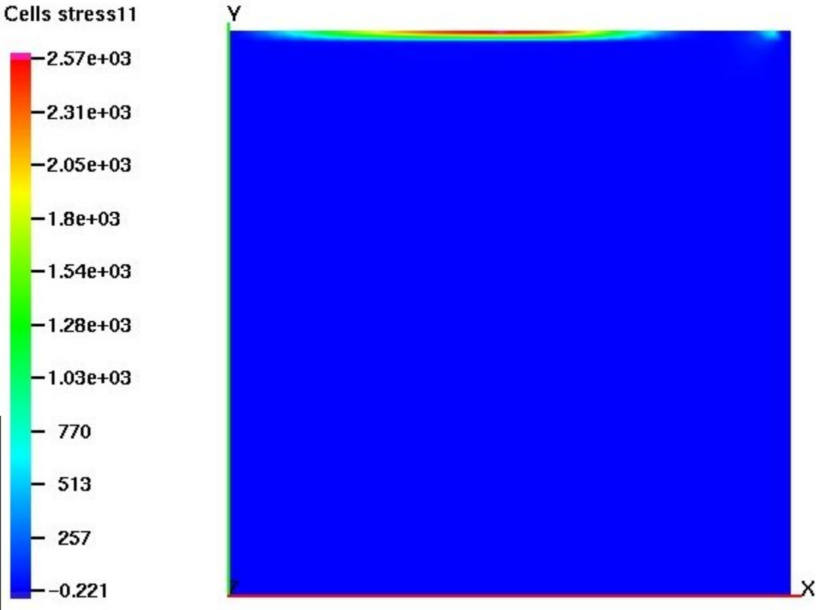


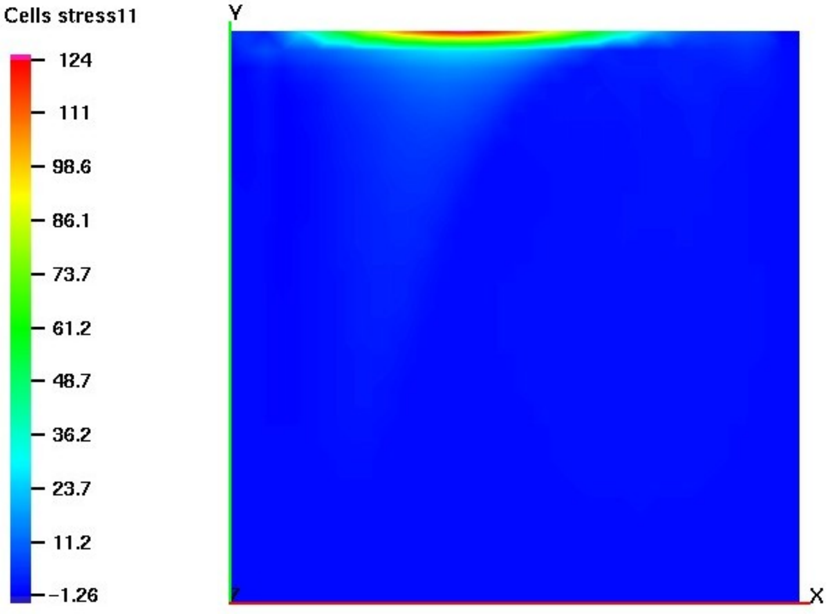
Driven cavity example:

as We number changes from We=0.5 to We=1.5, the stress value jumps very high




Cutline of Stress_11 component at y = 1.0



# Problem reformulation



**M. Behr** → ( u, p, ψ)

Replace $\tau$ in (2) with $\tau = \exp\psi$

$$\left.\begin{aligned}
&\rho\left(\frac{\partial \mathrm{u}}{\partial t} + \mathrm{u}\cdot\nabla \mathrm{u}\right) = \nabla p - 2\eta_s \nabla\cdot D - \frac{\eta_p}{\Lambda}\nabla\cdot(\exp\psi),\\
&\nabla\cdot\mathrm{u} = 0,\\
&\frac{\delta_{\mathrm{a}}(\exp\psi)}{\delta t} + \frac{1}{\Lambda}\left(\exp\psi - \mathrm{I}\right) = 0,
\end{aligned}\right\} \quad (3)$$

Gradient of exponential of ψ → ???

Solvers → ???

# Log. Conf. Reformulation



**Experiences:**

☼ Stresses grow exponentially

☼ Conformation stress is positive by design

**Idea („Kupferman Trick“):**

☼ Decompose the velocity gradient inside the stretching part

$$\nabla u = \Omega + B + N \sigma_c^{-1}$$

☼ Take the logarithm as a new variable ($\psi = \log \sigma$) using eigenvalue decomposition

$$\psi = R \log(\lambda_\tau) \, R^T$$

# LCR for Oldroyd-B model

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$$\tau(t) = \int_{\infty}^{t} \frac{1}{We} \exp\left(\frac{-(t-s)}{We}\right) F(s,t)\, F(s,t)^{T}\, ds$$

Oldroyd-B

$$\Lambda \frac{\delta_a \sigma^p}{\delta t} + \sigma^p - 2\eta_p D = 0,$$

$$\sigma^p = \frac{\eta_p}{We}(\tau - I)$$

$$\frac{\delta_a \tau}{\delta t} + \frac{1}{We}(\tau - I) = 0,$$

$$\nabla u = \Omega + B + N\sigma_c^{-1}$$

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right)\tau - (\Omega\tau - \tau\Omega) + 2B\tau = \frac{1}{\Lambda}(I - \tau)$$

$$\tau = \exp\, \psi \qquad \psi = R \log(\lambda_\tau)\, R^T$$

LCR

$$\partial_t \psi + (\nabla u)\psi - (\Omega.\psi - \psi.\Omega) + 2B = \frac{1}{We}(\exp(-\psi) - I)$$

# LCR for Oldroyd-B model

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$$
\begin{cases}
\partial_t u + (\nabla u)u = \nabla \cdot \sigma - \nabla p + (1-\gamma\vartheta)j \\
\nabla \cdot u = 0 \\
\partial_t \psi + (\nabla u)\psi - (\Omega.\psi - \psi.\Omega) + 2B = \dfrac{1}{We}\left(\boldsymbol{exp(-\psi) - I}\right)
\end{cases}
$$

NS

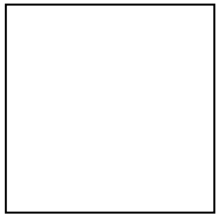
SE

**Standard discretization techniques →EO-FEM, TVD**
**Standard nonlinear (Newton) and linear (MG) solvers**

**→ Increases the critical We number dramatically !!**

Driven cavity NS SE

- Velocity profile at the upper wall: $u_x = 8(1 + \tanh(8(t - 0.5)))x^2(1 - x)^2$
- For $t > 1$, $u_{in} = 16$
- Dirichtlet Bc's everywhere
- Stress field: Natural Bc's

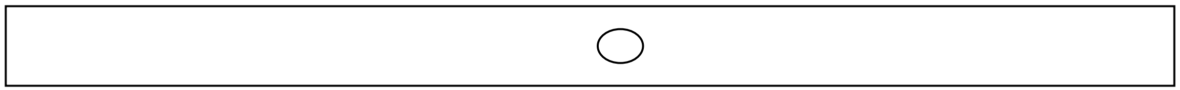


4 to 1 contraction NS SE

- Velocity profile at the inlet: $u_x = \frac{3}{128}\left(16 - y^2\right)$
- Out flow: Natural Bc's
- Stress field: Natural Bc's

Planar flow around cylinder NS SE

- Velocity profile at the inlet: $u_x = 1.5\left(1 - y^2 / 4\right)$
- Out flow: Natural Bc's
- Stress field: Natural Bc's



# Driven cavity

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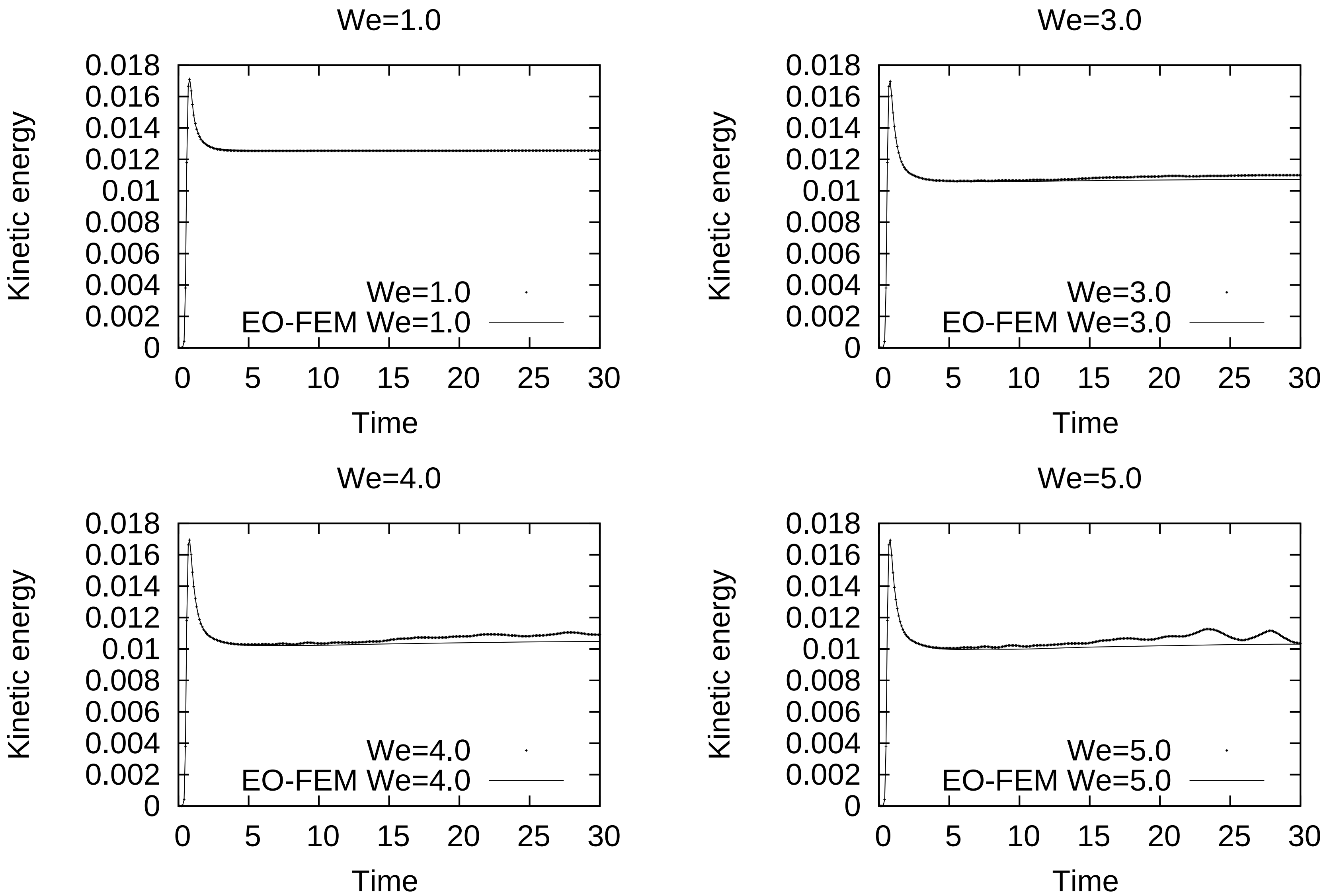


As We increases, numerical instabilities become visible.
EOFEM is able to throw this

# Driven cavity

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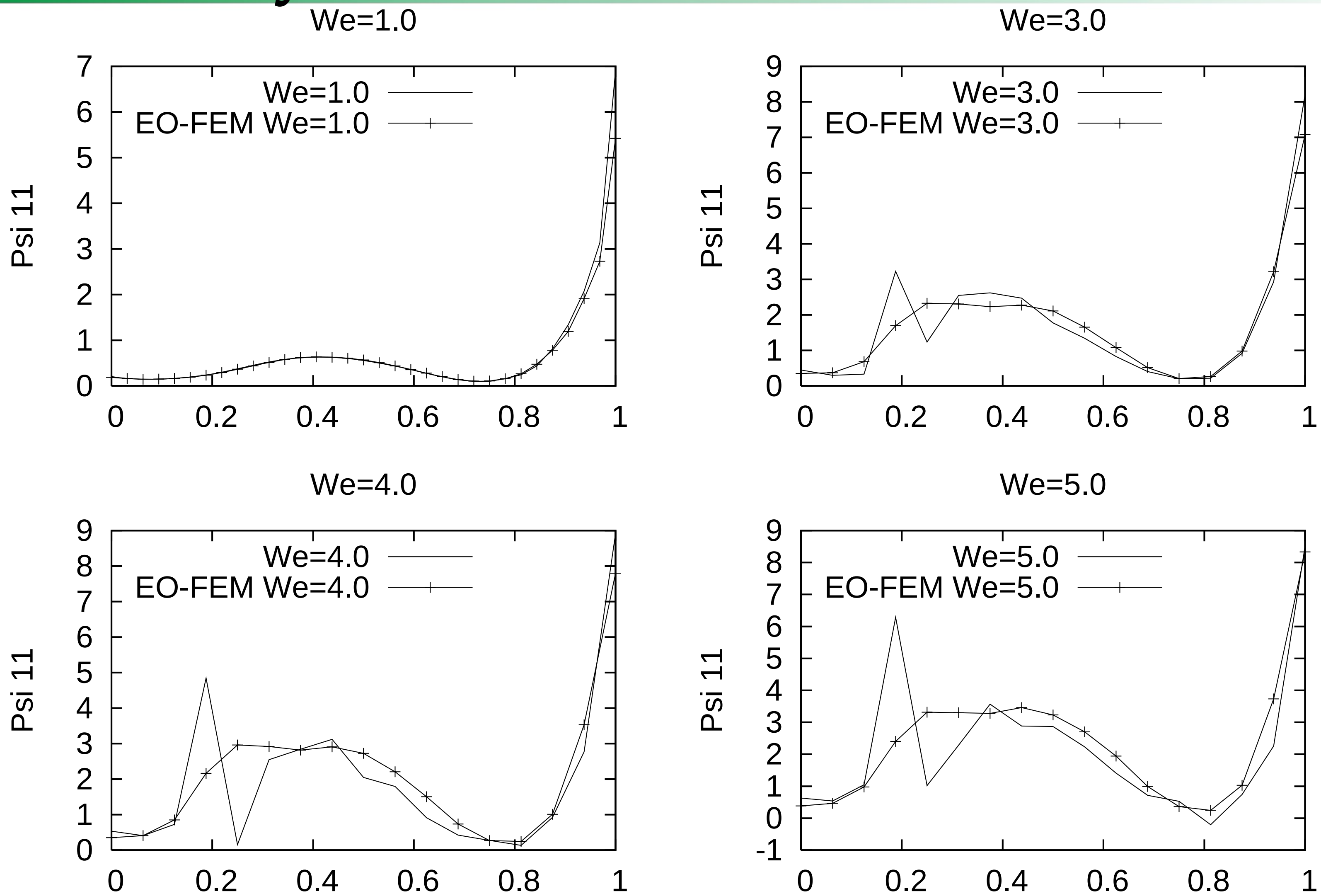
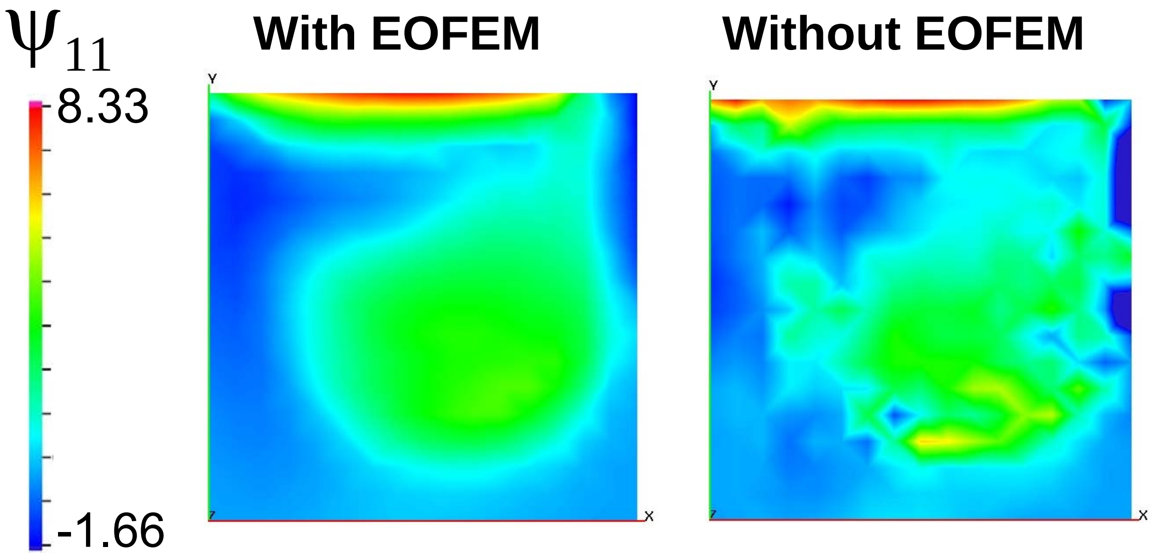


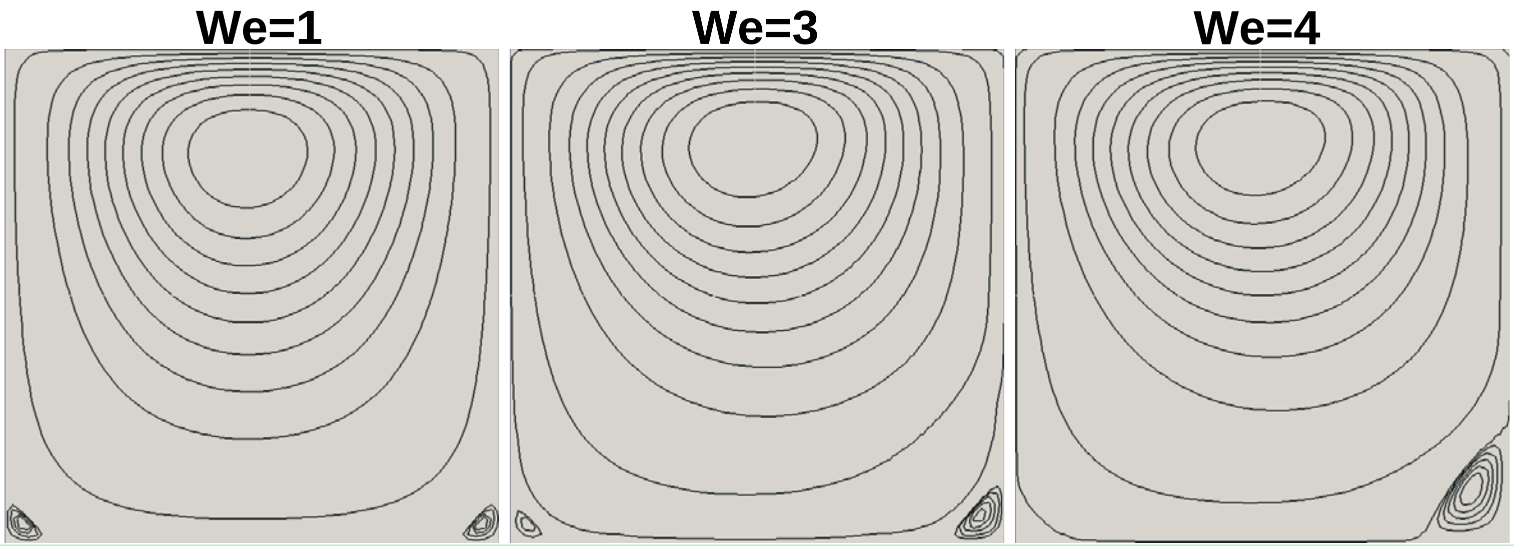

As We increases, numerical instabilities become visible.
EOFEM is able to throw this

# Driven cavity



We=5




As We increases, numerical instabilities become visible.
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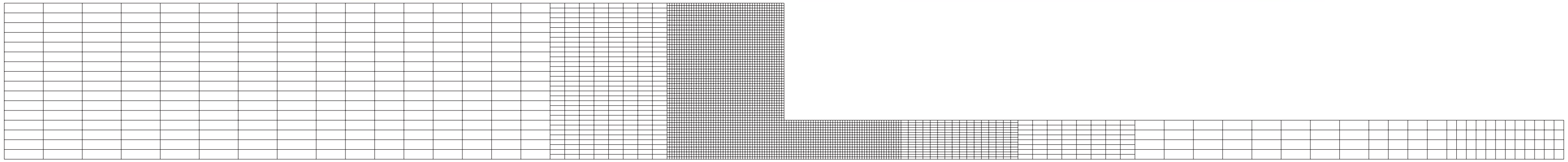
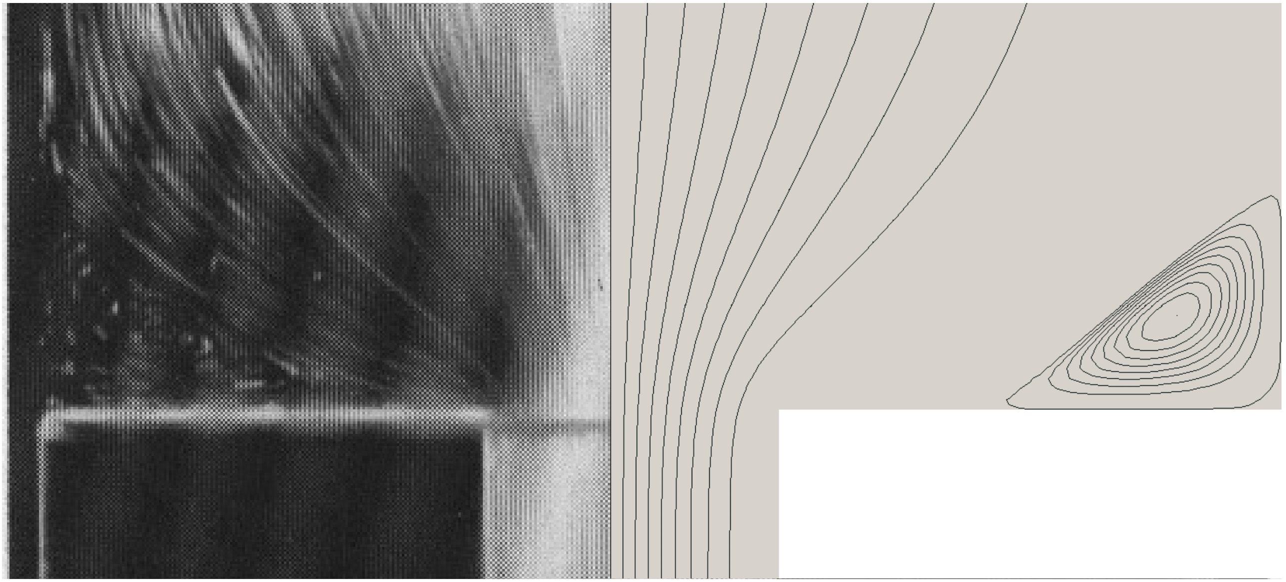
# 4:1 Contraction





Corner vortex, no lip

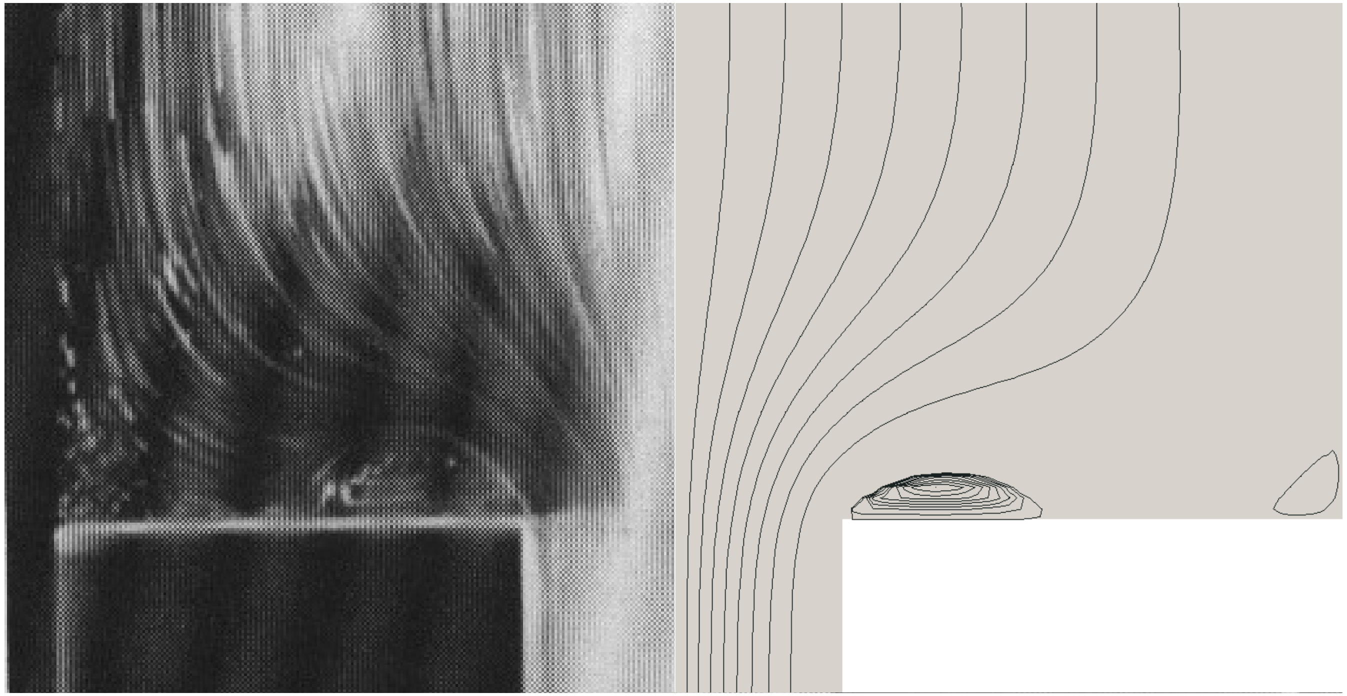
Experiment vs Numeric



# 4:1 Contraction

Lip vortex appears

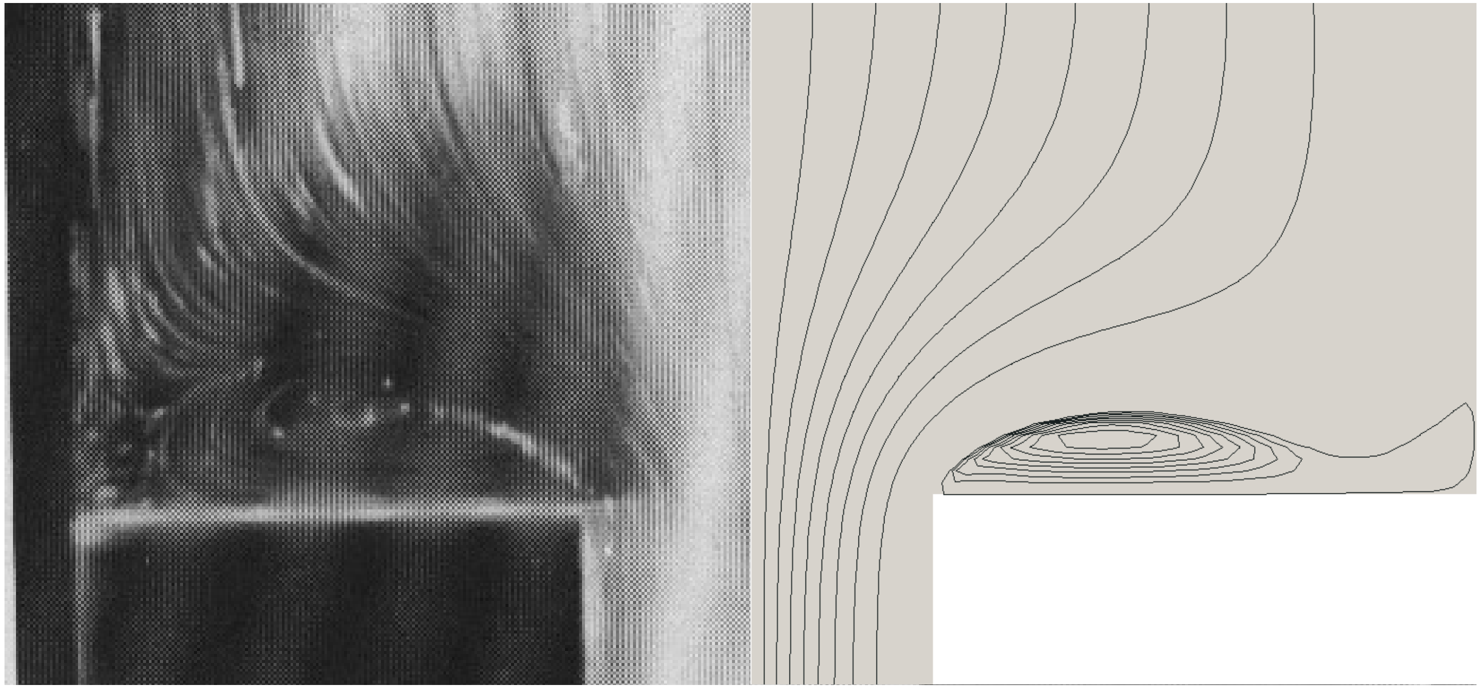
Experiment vs Numeric



# 4:1 Contraction

Lip vortex grows

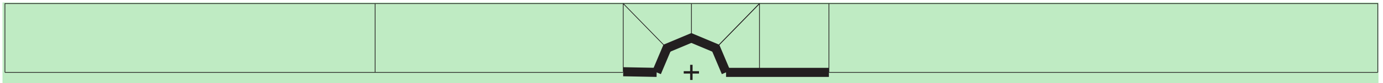
Experiment vs Numeric



*Qualitatively close*

# Planar flow around cylinder



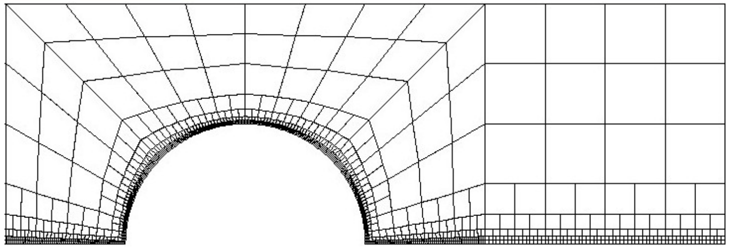


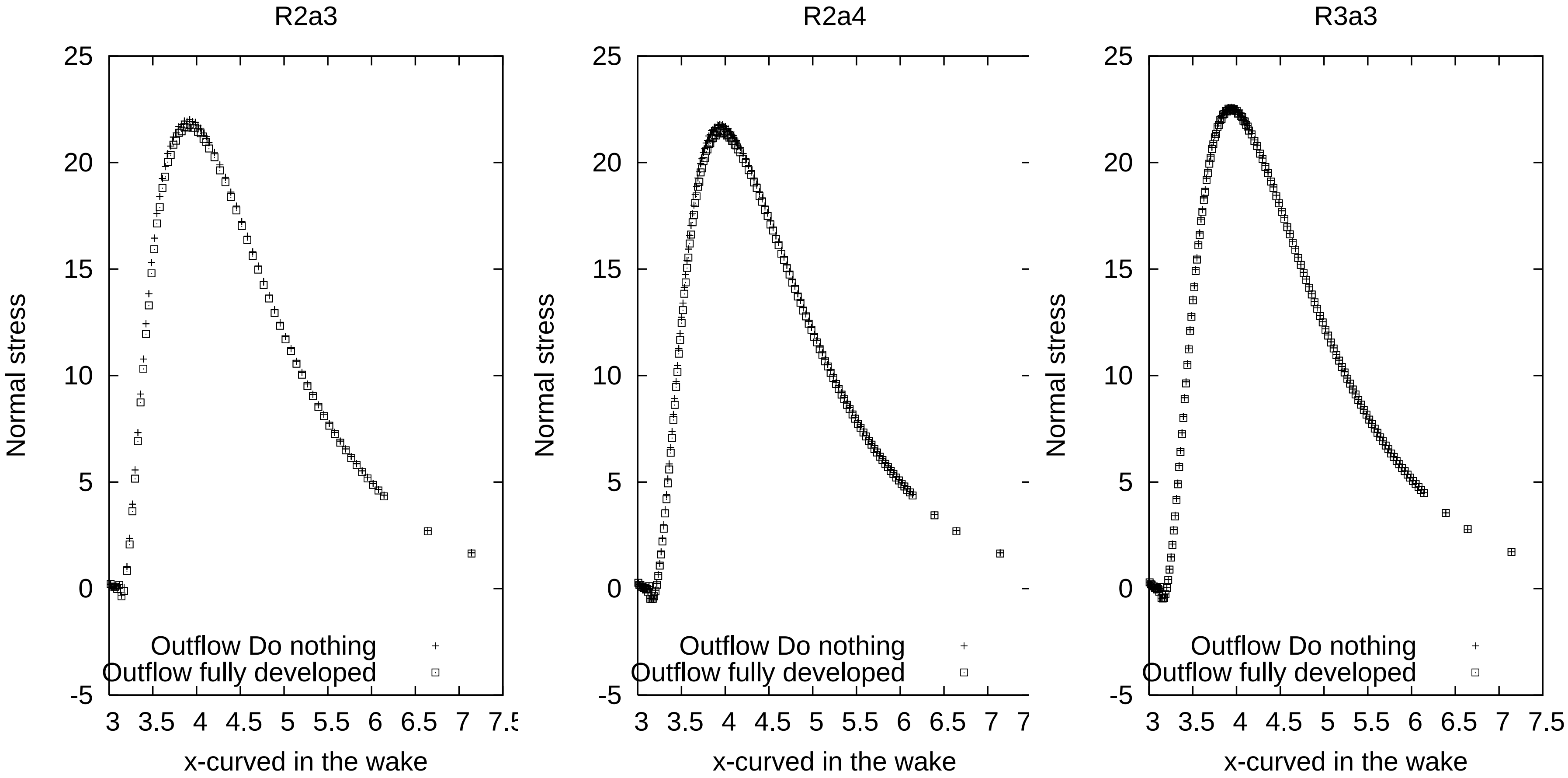
| We | 0.01 | | 0.1 | | 1.0 | |
|---|---|---|---|---|---|---|
| Linear Tol | 0.1 | 0.01 | 0.1 | 0.01 | 0.1 | 0.01 |
| R1 | 9/2 | 5/3 | 10/1 | 7/3 | 14/1 | 10/3 |
| R2 | 9/3 | 5/5 | 10/2 | 7/4 | 16/2 | 10/5 |
| R3 | 9/3 | 5/6 | 10/3 | 7/5 | 16/2 | 11/5 |
| R4 | 9/3 | 5/6 | 10/3 | 9/5 | 13/3 | 11/5 |

*stable Newton and multigrid !*

# Planar flow around cylinder








*Natural outflow condition is sufficient !*

# Planar flow around cylinder

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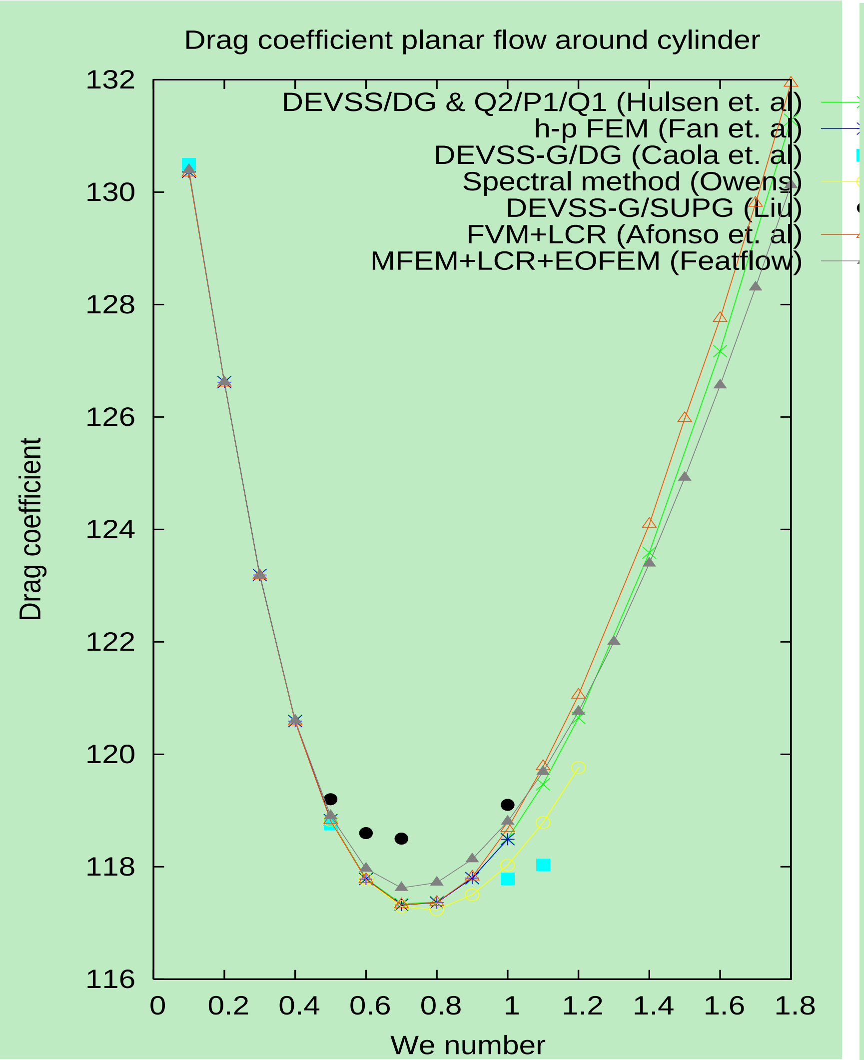


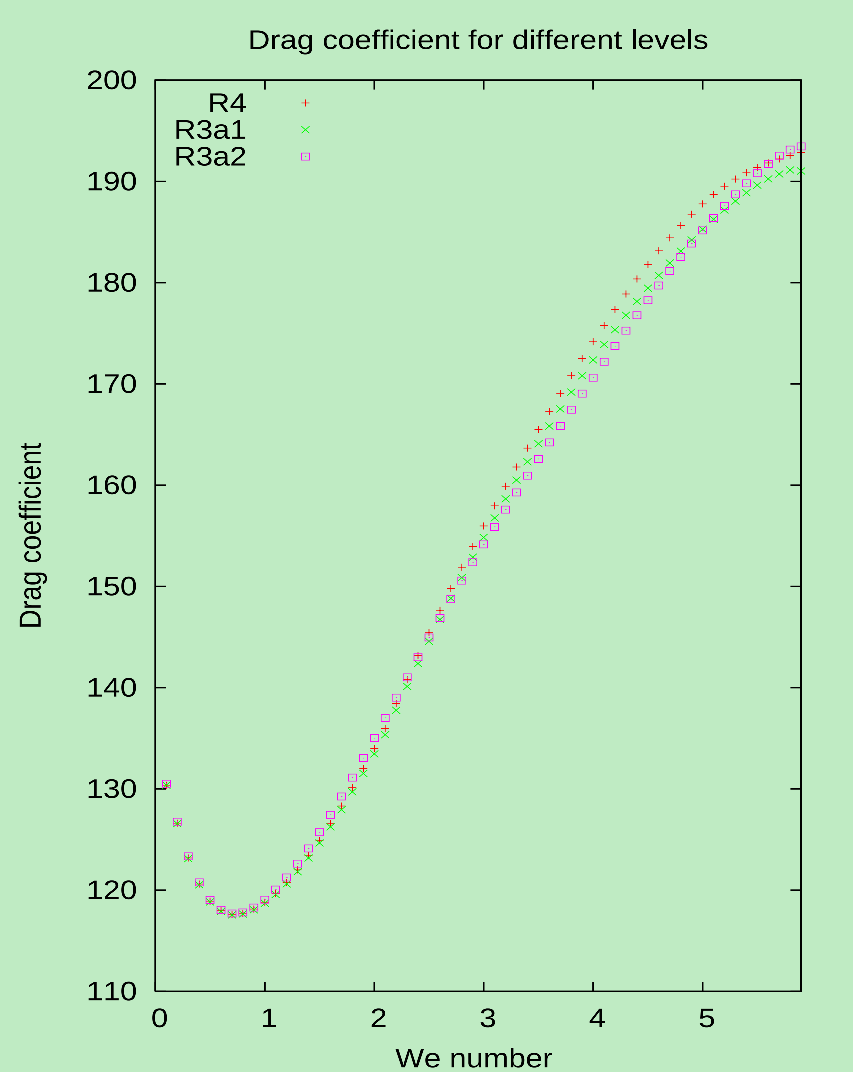


*We can go further than ever before*

# Complex model

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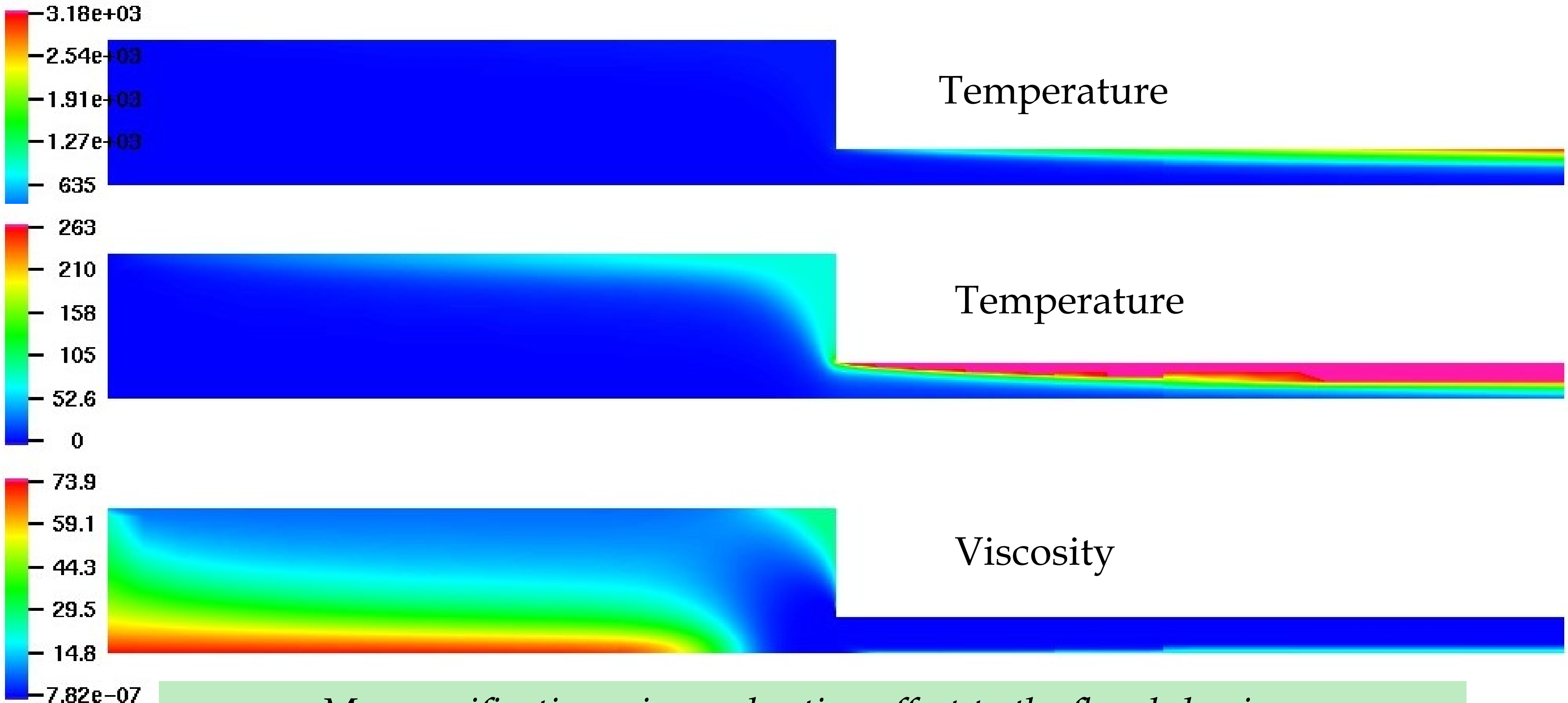
NS EE SE

$$\eta = \eta_0 \exp\left(a_1 + \frac{a_2}{a_3 + \theta}\right)\left(b_1 + b_2\|D\|\right)^{-b^3}$$




*More verification: viscous heating effect to the flow behaviour*

We present a tool to simulate complex fluid flow with:

- Monolithic FEM
- Geometric Multigrid
- Edge oriented stabilization
- LCR for Oldroyd-B model
- Local adaptivity
- 2nd order accurate time integrator

Future work:

- Further investigation of Newton – multigrid solver
- Validation of NS EE SE
- Further complex viscoelastic fluid model
- Further coupling with structure part (FSI)