Efficient and robust monolithic multilevel Krylov solver for the solution of stationary incompressible Navier-Stokes equations

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Motivation

To develop an efficient and robust solver for monolithic solution of incompressible Navier-Stokes equations as an alternative to geometric multigrid.

Reason: Geometric multigrid with Vanka or multilevel pressure schur complement (MPSC) approach has problems with anisotropies

Notations

- Linear system:

$$Au = b, \qquad A \in \mathbb{R}^{nxn}, \qquad u, b \in \mathbb{R}^n$$

(A is large, but sparse, nonsymmetric, nonsingular)

- $\sigma(\text{A}) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$: spectrum of A
with eigenvalues in increasing absolute value order.

- $\rho(\text{A}) = max_i\{|\lambda_i|\}$: spectral radius of A

- For hermitian A, $\kappa(\text{A}) := \frac{\lambda_n}{\lambda_1}$ = condition number

- For arbitrary matrices, $\kappa(\text{A}) := \frac{\lambda_n}{\lambda_1}$ = qualtiy of clustering

Convergence of Krylov subspace methods

Assumption (1)

(i) *$A \in \mathbb{R}^{nxn}$ is nonsymmetric and diagonalizable.*

(ii) *$X = [x_1, ...x_n]$ are right eigenvectors of A.*

$$A = X\Lambda X^{-1}$$

$$\Lambda = diag(\lambda_1, ..., \lambda_n)$$

Theorem (Convergence rate)

Let A be defined as in Assumption (1). After j iterations GMRES produces a residual satisfying the following bound:

$$\|r_j\|_2 \leq 2\kappa_2(X)\left(1 - \frac{2}{\sqrt{\kappa(A)} + 1}\right)^j \|r_0\|_2$$

- If A is SPD, $\kappa_2(X) = 1$.

Convergence of Krylov subspace methods

- Convergence depends on $\kappa(A)$
  For A coming from discretization of PDEs, $\kappa(A)$ increases with grid refinements
- Convergence also hampered if some eigenvalues are very close to zero.

Preconditioning

Preconditioned system:

$$\widehat{A}\widehat{u} = \widehat{b}$$

$\widehat{A} := M_1^{-1} A M_2^{-1}, \quad \widehat{u} := M_2 u, \quad \widehat{b} := M_1^{-1} u, \quad M_1, M_2$ *nonsingular*

Left preconditioning: $M_1 = M,\ M_2 = I$

Right preconditioning: $M_1 = I,\ M_2 = M$

Ideal preconditioning $\Rightarrow \kappa(\widehat{A}) \approx 1$

Classical preconditioning

Splitting: $A = D - E - F$

Classical choices for preconditioning:

- $M = D$ **(Jacobi)**
- $M = D - E$ **(Gauß-Seidel)**
- $M = \widetilde{L}\widetilde{U}$, for $A = LU$, $\widetilde{L} \approx L, \widetilde{U} \approx U$ **(ILU)**

$\longrightarrow$ Do not exploit the detailed information about spectrum of A

$\longrightarrow$ $\widehat{A}$ may still have many eigenvalues near to zero hampering the convergence of Krylov methods

Deflation-based preconditioner

Two-level deflation preconditioner:[1]

- Define: $P_D = I - AZE^{-1}Y^T$
  with $E = Y^T AZ$ (Galerkin matrix)
  $Z, Y : \mathbb{R}^r \mapsto \mathbb{R}^n \quad r \ll n$
  Z is called *the deflation subspace*. (Prolongation)
- For nonsingular A, E is invertible if Z is full rank.
- Solve with Krylov method $P_D Au = P_D b$
- We can combine preconditioining and deflation:

$$M^{-1}P_D Au = M^{-1}P_D b$$

[1] J. Frank and C. Vuik. "On the Construction of Deflation-Based Preconditioners". In: *SIAM J. Sci. Comput.* 23.2 (2001), pp. 442–462.

Deflation with eigenvectors

Assumption (2)

(i) $Z = [v_1 ... v_r]$ *and* $Y = [w_1 ... w_r]$ *be the right and left eigenvectors of matrix* A *associated with eigenvalues* $\lambda_{1 \leq i \leq r}$

(ii) $v_i^T w_j = \delta_{ij}$

Theorem (Eigenvector deflation)

Let Z *and* Y *be as defined in Assumption (2). Then*

$$\sigma(P_D A) = \{0, ... 0, \lambda_{r+1}, ..., \lambda_n\}$$

Deflation with eigenvectors

Theorem (Eigenvector deflation)

Let Z and Y be as defined in Assumption (2). Then

$$\sigma(P_D A) = \{0, ...0, \lambda_{r+1}, ..., \lambda_n\}$$

Proof.

Here $E = Y^T AZ = diag\{\lambda_1, ..., \lambda_n\} =: \Lambda_r$.

For $i = 1, ..., r$

$$P_D AZ = (I - AZ\Lambda_r^{-1} Y^T)AZ \quad = AZ - AZ\Lambda_r^{-1} Y^T AZ \quad = AZ - AZ\Lambda_r^{-1}\Lambda_r = 0$$

For $i = r + 1, ..., n$

$$P_D Av_i = (I - AZ\Lambda_r^{-1} Y^T)Av_i \quad = \lambda_i v_i - Z\Lambda_r\Lambda_r^{-1}\lambda_i \underbrace{Y^T v_i}_{=0} = \lambda_i v_i$$

□

Deflation improves convergence

- $\kappa_{eff}(P_D A) = \frac{\lambda_n}{\lambda_{r+1}} \leq \frac{\lambda_n}{\lambda_1} = \kappa(A)$

Theorem (Effective convergence rate)

Let A, Z and Y be as defined in Assumption (1) and (2). Then for any starting vector u_0, GMRES applied to $P_D A u = P_D b$ generates residuals whose 2-norm is bounded by:

$$\|r_{j,D}\|_2 \leq 2\kappa_2(X)\left(1 - \frac{2}{\sqrt{\kappa_{eff}} + 1}\right)^j \|r_{0,D}\|_2$$

Larger deflation subspace $\Rightarrow$ smaller κ_{eff} $\Rightarrow$ E^{-1} expensive

Deflation: Coarse grid issue

- Let E^{-1} be purturbed: $\widetilde{E}^{-1} = diag\left(\frac{1-\epsilon}{\lambda_1}, ..., \frac{1-\epsilon}{\lambda_r}\right)$.
- For $\widetilde{P}_D = I - AZ\widetilde{E}^{-1}Y^T$,

$$\sigma(\widetilde{P}_D A) = \{\epsilon\lambda_1, ..., \epsilon\lambda_r, \lambda_{r+1}, ..., \lambda_n\},$$

and

$$\kappa(\widetilde{P}_D A) = \frac{\lambda_n}{\epsilon\lambda_1}.$$

- If $0 \sim \epsilon < \lambda_r$, (serious convergence problems)
- Exact coarse-grid solve required for fast convergence
- Limit the size of coarse-grid system
- Limit the number of small eigenvalues to be deflated
- Limit its potential

Multilevel Krylov method

- Erlangga and Nabben[2] proposed a *projection-like* method which is **stable** w.r.t. the inaccurate inversion of Galerkin matrix E
- Iterative method based on Krylov subspace can be designed to solve Galerkin system
- Nested iterations $\Longrightarrow$ **Multilevel Krylov method**

[2] Y. Erlangga and R. Nabben. "Multilevel Projection-Based Nested Krylov Iteration for Boundary Value Problems". In: *SIAM J. Sci. Comput.* 30.3 (2008), pp. 1572–1595.

Power method and Wielandt's deflation

Consider preconditioned system: $\widehat{A}\widehat{u} = \widehat{b}$

- Applying Power method to $\widehat{A}^{-1}$ gives (λ_1, z_1) eigenpair of $\widehat{A}$
- *Wielandt* deflation: deflate λ_1 to zero.

$$\widehat{A}_1 = \widehat{A} - \lambda_1 z_1 y^T, \quad y^T z_1 = 1, \quad y : \textit{arbitrary}.$$

$$\sigma(\widehat{A}_{1,\lambda_1}) = \{0, \lambda_2, ..., \lambda_n\}$$

- Applying Power method to $\widehat{A}_1^{-1}$ gives (λ_2, z_2) eigenpair of $\widehat{A}$
- Deflating λ's to zero not the only choice

$$\widehat{A}_{1,\gamma_1} = \widehat{A} - \gamma_1 z_1 y^T, \quad y^T z = 1, \quad \gamma_1 \in \mathbb{R}$$

$$\sigma(\widehat{A}_{1,\gamma_1}) = \{\lambda_1 - \gamma_1, \lambda_2, ..., \lambda_n\}$$

Towards stable deflation process

Let $(\lambda_i, z_i)_{1 \leq i \leq r}$ be r already calculated eigenpairs of $\widehat{A}$. Let

$$\widehat{A}_r = \widehat{A} - Z\Gamma_r Y^T, \quad \Gamma_r = diag(\gamma_1, ..., \gamma_r), \quad Y^T Z = I_r.$$ Then,

$$\sigma(\widehat{A}_r) = \{\lambda_1 - \gamma_1, ..., \lambda_r - \gamma_r, ..., \lambda_n\}$$

For $1 \leq i \leq r$,

- if $\gamma_i = \lambda_i$, then $\sigma(\widehat{A}_r) = \{0, ..., 0, \lambda_{r+1} ..., \lambda_n\}$
- if $\gamma_i = \lambda_i - \lambda_n$, then $\sigma(\widehat{A}_r) = \{\lambda_n, ..., \lambda_n, \lambda_{r+1} ..., \lambda_n\}$

Both choices result in similar spectra.

Towards stable deflation process

Let $Y = [y_1 ... y_r]$ be left eigenvectors of A.

For $\gamma_i = \lambda_i$: $\Gamma_r = \widehat{E} = diag(\lambda_1, ..., \lambda_r)$

$$
\begin{aligned}
\widehat{A}_r &= \widehat{A} - Z\widehat{E}Y^T \\
&= \widehat{A} - Z\widehat{E}\widehat{E}^{-1}\widehat{E}Y^T \\
&= \widehat{A} - \widehat{A}Z\widehat{E}^{-1}Y^T\widehat{A} \qquad \because Z\widehat{E} = \widehat{A}Z \text{ and } \widehat{E}Y^T = Y^T\widehat{A} \\
&= (I_n - \widehat{A}Z\widehat{E}^{-1}Y^T)\widehat{A} \qquad (:= P_D\widehat{A}) \\
&= \widehat{A}(I_n - Z\widehat{E}^{-1}Y^T\widehat{A}) \qquad (:= \widehat{A}Q_D)
\end{aligned}
$$

$P_D^2 = P_D$ Left *projection* preconditioner

$Q_D^2 = Q_D$ Right *projection* preconditioner

Towards stable deflation process

For $\gamma_i = \lambda_i - \lambda_n$: $\Gamma_r = \widehat{E} - \lambda_n I_r$

$$\begin{aligned}
\widehat{A}_r &= \widehat{A} - Z(\widehat{E} - \lambda_n I_r)Y^T \\
&= \widehat{A} - Z\widehat{E}Y^T + \lambda_n Z Y^T \\
&= \widehat{A} - Z\widehat{E}\widehat{E}^{-1}\widehat{E}Y^T + \lambda_n Z\widehat{E}^{-1}\widehat{E}Y^T \\
&= \widehat{A} - \widehat{A}Z\widehat{E}^{-1}Y^T\widehat{A} + \lambda_n Z\widehat{E}^{-1}Y^T\widehat{A} \qquad \because Z\widehat{E} = \widehat{A}Z \text{ and } \widehat{E}Y^T = Y^T\widehat{A} \\
&= (I_n - \widehat{A}Z\widehat{E}^{-1}Y^T + \lambda_n Z\widehat{E}^{-1}Y^T)\widehat{A} \qquad (:= P_N\widehat{A}) \\
&= \widehat{A}(I_n - Z\widehat{E}^{-1}Y^T\widehat{A} + \lambda_n Z\widehat{E}^{-1}Y^T) \qquad (:= \widehat{A}Q_N)
\end{aligned}$$

Thus: $P_N = P_D + \lambda_n Z\widehat{E}^{-1}Y^T$

Stable deflation

- Let $\widetilde{E}^{-1} = diag\left(\frac{1-\epsilon}{\lambda_1}, ..., \frac{1-\epsilon}{\lambda_r}\right)$.

$$\sigma(\widetilde{P}_D\widehat{A}) = \{\epsilon\lambda_1, ..., \epsilon\lambda_r, \lambda_{r+1}, ..., \lambda_n\}$$

$$\Updownarrow$$

$$\sigma(\widetilde{P}_N\widehat{A}) = \{(1-\epsilon)\lambda_n + \epsilon\lambda_1, ..., (1-\epsilon)\lambda_n + \epsilon\lambda_r, \lambda_{r+1}, ..., \lambda_n\}$$

- For $0 < \epsilon < \lambda_1 < \lambda_{r+1}$:

$$\kappa(\widetilde{P}_D\widehat{A}) = \frac{\lambda_n}{\epsilon\lambda_1} > \frac{(1-\epsilon)\lambda_n + \epsilon\lambda_r}{\lambda_{r+1}} = \kappa(\widetilde{P}_N\widehat{A})$$

- In $\widetilde{P}_N\widehat{A}$, λ_{r+1} remains the smallest eigenvalue: insensitive to inexact inversion of $\hat{E}$ (inversion by iterative method)

Stable deflation with general vectors

Theorem

Let Z,Y be "arbitrary" rectangular matrices of rank r.

If

$$\sigma(P_D\widehat{A}) = \{0, ...0, \mu_{r+1}, ..., \mu_n\},$$

then

$$\sigma(P_N\widehat{A}) = \{\lambda_n, ...\lambda_n, \mu_{r+1}, ..., \mu_n\}$$

Remark

$$\underline{\mu_n \leq \lambda_n} \Rightarrow \quad \kappa(P_N\widehat{A}) := \frac{\lambda_n}{\mu_{r+1}} \geq \frac{\mu_n}{\mu_{r+1}} =: \kappa(P_D\widehat{A}).$$

But there exists ω *s.t.* $\mu_n = \lambda_n$

$$\sigma(P_N\widehat{A}) = \{\mu_n, ...\mu_n, \mu_{r+1}, ..., \mu_n\}$$

with $P_N = P_D + \omega\lambda_n Z\widehat{E}^{-1}Y^T$

Stable deflation: Additional remarks

- $P_N^2 \neq P_N$
- P_N is nonsingular
- $P_N\widehat{A} \neq \widehat{A}Q_N$, but $\sigma(P_N\widehat{A}) = \sigma(\widehat{A}Q_N)$
- $P_N\widehat{A}$ is **nonsymmetric** even if $\widehat{A}$ is symmetric. A Krylov method for nonsymmetric matrices has to be used.

Deflated Preconditioned System

Right preconditioned system	Left preconditioned system
$AM^{-1}Q_N Q_N^{-1}Mu = b$ $\widehat{A}Q_N\widehat{u} = \widehat{b}$	$P_N M^{-1}Au = P_N M^{-1}b$ $P_N\widehat{A}\widehat{u} = \widehat{b}$

Implementation: Two level

Solve $\widehat{A} Q_N \widehat{u} = \widehat{b}$ using GMRES.

Algorithm Two level deflated right preconditioned GMRES($\widehat{A}, \widehat{b}, Q_N$)

Choose $u_{0,h} (= 0), \omega, \lambda_n$
Compute $r_{0,h} = \widehat{b}_{0,h} - \widehat{A}_h \widehat{u}_{0,h}$, and $v_1 = r_{0,h} / \|r_{0,h}\|_2$
for $i = 1, ..., k$ **do**
    $x_i = Q_N v_i$
    $w = \widehat{A}_h x_i$
    Orthogonalization
end for
Set $X_k := [x_1, ..., x_k]$ and $\widehat{H}_k = \{h_{i,j}\}_{1 \leq i \leq j+1; 1 \leq j \leq k}$
Compute $y_k = argmin_y \left\| \beta e_1 - \widehat{H}_k y \right\|_2$ and $u_k = u_0 + M^{-1} X_k y_k$

Implementation: Two level

$$x_i = Q_N v_i = (I - Z\widehat{E}_H^{-1} Y^T \widehat{A}_h + \omega\lambda_n Z\widehat{E}^{-1} Y^T) v_i$$
$$= v_i - Z\widehat{E}_H^{-1} Y^T (A_h M_h^{-1} v_i - \omega\lambda_n v_i)$$

$$x_i = Q_N v_i = \begin{cases} s & = A_h M_h^{-1} v_i \\ v_H & = Y^T (s - \omega\lambda_n v_i) & (Restriction) \\ \widetilde{v_H} & = \widehat{E}_H^{-1} v_H & (Coarse - grid\ solve) \\ t & = Z\widetilde{v}_H & (Prolongation) \\ x_i & = v_i - t & (Correction) \end{cases}$$

Similar processes to multigrid but different interpretation.

- Modifies the Arnoldi vector , and NOT the error vector.
- M is not a smoother ($M = I$ works if matrix is well-conditioned)

Implementation: Multilevel Krylov Method

Algorithm Multilevel Krylov subspace method (MLKM)

Given $A^{(l)}, Z^{(l,l+1)}, Y^{(l,l+1)}$, choose $\omega^{(l)}, \lambda_n^{(l)}$ for $l = m, \ldots, 1$

With an initial guess $u_0^{(m)} = 0$, compute $r_{0,h} = \widehat{b}_{0,h} - \widehat{A}_h \widehat{u}_{0,h}$

while $i \leq$ maxitr **do**

  for i = 1,...,k **do**

    $v_1 = r_{0,h} / \left\| r_{0,h} \right\|_2$

    if $(l) > 1$ **then**

      $s = A_h M_h^{-1} v_i$

      $v_H = Y^T (s - \omega \lambda_n v_i)$

      $\widetilde{v}_H = GMRES(\widehat{A}_H, v_H, Q_N)$

      $t = Z \widetilde{v}_H$

      $x_i = v_i - t$

    end if

    $w = A_h M_h^{-1} x_i$

    Orthogonalization

  end for

  Set solution subspace X and extract the solution from it.

end while

Multilevel components

- Z and Y

  $\longrightarrow$ Full rank

  $\longrightarrow$ Piecewise constant interpolation or standard multigrid grid transfer operators

- λ_n

  Expensive to compute, but an approximate is sufficient:

  $\longrightarrow$ by Gerschgorin's theorem

  $\longrightarrow$ $\lambda_n = 1$ mostly worked with Z and Y being multigrid inter-grid transfer operators

2D Convection-diffusion equation

2D convection-diffusion equation with vertical winds:

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial y} - \frac{1}{Pe}\nabla \, . \, \nabla \boldsymbol{u} &= \mathbf{f}, \quad \boldsymbol{\Omega} \; = \; (\mathbf{0}, \mathbf{1})^{\mathbf{2}} \\
\mathbf{f} &= -\,3xy(-x^2y + 2x^2 + 2y^2) \\
\boldsymbol{u}(0, y) &= 0, \quad \boldsymbol{u}(1, y) = y^3 \\
\boldsymbol{u}(x, 0) &= 0, \quad \boldsymbol{u}(x, 1) = x^3 \\
\boldsymbol{u}_{exact} &= x^3y^3
\end{aligned}$$

2D Convection-diffusion equation

Convergence criterion: Relative error $\leq 10^{-6}$

	Q_1		Q_2	
Level	L2-Error	H1-Error	L2-Error	H1-Error
4	3.89E-03	6.72E-02	3.61E-05	1.87E-03
5	9.76E-04	3.34E-02	4.51E-06	4.67E-04
6	2.44E-04	1.67E-02	5.64E-07	1.167E-04
7	6.10E-05	8.35E-03	8.55E-08	2.92E-05
8	1.52E-05	4.18E-03	3.74E-08	7.78E-06

2D Convection-diffusion equation

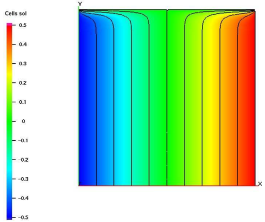
2D convection-diffusion equation with vertical winds:

$$\frac{\partial \boldsymbol{u}}{\partial y} - \frac{1}{Pe}\nabla \,.\, \nabla \boldsymbol{u} = \mathbf{f}, \quad \boldsymbol{\Omega} = (\mathbf{0},\mathbf{1})^2$$

$$\mathbf{f} = 0$$

$$\boldsymbol{u}(0,y) = -1/2, \quad \boldsymbol{u}(1,y) = 1/2$$

$$\boldsymbol{u}(x,0) = x - 1/2, \quad \boldsymbol{u}(x,1) = 0$$



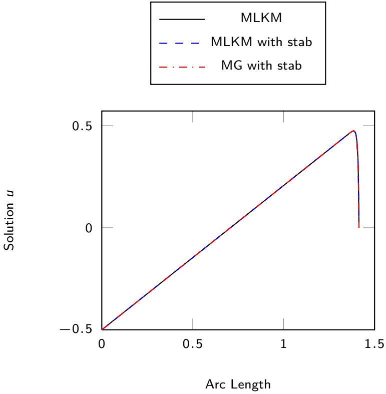
- MLKM (4,2,2) $\quad \lambda_n = 1.0$
- MG (4 smoothing steps), Exact coarse-grid solve
- Convergence criterion: relative error $\leq 10^{-6}$

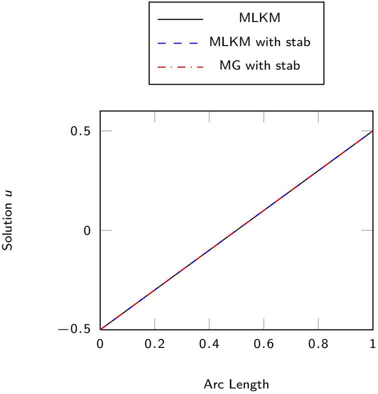
2D Convection-diffusion equation

Preconditioner/ smoother: **Jacobi**, FEM discr.: Q_1

Time taken by solvers at level 9 also shown

Level/ Pe	20	50	100	200
6 MK	9	12	21	47
MG	4	div	div	div
7 MK	9	9	12	25
MG	4	div	div	div
8 MK	9	9	9	13
MG	4	div	div	div
9 MK	9 (0.16)	9 (0.16)	9 (0.16)	9 (0.16)
MG	4 (0.11)	div	div	div
9 MK (stab)	9 (0.24)	9 (0.24)	9 (0.24)	9 (0.24)
MG (stab)	5 (0.21)	5 (0.21)	6 (0.26)	7 (0.32)





Solution for $Pe = 200$ along cut-lines [(0,0)-(1,1)] and [(0,0.5)-(1,0.5)]

2D Convection-diffusion equation

Preconditioner/ smoother: **Gauß-Seidel**, FEM discr.: Q_1

Time taken by solvers at level 9 also shown

Level/ Pe	**20**	**50**	**100**	**200**
6 MK	9	11	21	42
MG	4	div	div	div
7 MK	8	9	12	20
MG	3	div	div	div
8 MK	8	8	9	12
MG	4	div	div	div
9 MK	8 (0.16)	8 (0.16)	8 (0.16)	9 (0.18)
MG	3 (0.10)	div	div	div
9 MK (stab)	8 (0.17)	8 (0.17)	9 (0.19)	11 (0.23)
MG (stab)	4 (0.22)	4 (0.23)	5 (0.28)	5 (0.28)

2D Convection-diffusion equation

Preconditioner/ smoother: **Jacobi**,   FEM discr.: Q_2

Time taken by solvers at level 9 also shown

Level/ Pe	**20**	**50**	**100**	**200**
6 MK	11	4	16	34
MG	4	div	div	div
7 MK	11	11	12	20
MG	5	5	div	div
8 MK	11	11	11	14
MG	5	5	div	div
9 MK	11 (1.0)	11 (1.0)	11 (1.0)	11 (1.0)
MG	5 (0.87)	5 (0.87)	div	div
MG (stab)	5(2.20)	6(2.60)	8(3.42)	10(4.28)

2D Convection-diffusion equation

Preconditioner/ smoother: **Gauß-Seidel**, FEM discr.: Q_2

Time taken by solvers at level 9 also shown

Level/ Pe	20	50	100	200
6 MK	11	12	15	26
MG	3	4	div	div
7 MK	11	11	12	17
MG	3	3	div	div
8 MK	10	11	11	13
MG	3	3	div	div
9 MK	10 (1.10)	10 (1.10)	11 (1.22)	11 (1.20)
MG	3 (0.65)	3 (0.65)	div	div
MG (stab)	5(3.20)	7(4.45)	9(5.79)	44(27.74)

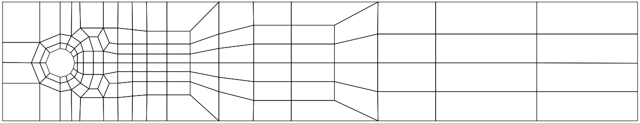
2D Convection-diffusion equation

Convection-diffusion equation on **unstructured grid**

$$\frac{\partial \boldsymbol{u}}{\partial x} - \frac{1}{Pe}\nabla \,.\, \nabla \boldsymbol{u} \;=\; 0, \quad \Omega \;=\; [0,1]^2$$

Boundary conditions:

$$\boldsymbol{u}(0,y) \;=\; 0, \quad \boldsymbol{u}(x,0) \;=\; 0 \quad \boldsymbol{u}(x,0.41) \;=\; 0$$
$$\boldsymbol{u}(x,y) = 1 \text{ for (x,y) on circle}, \quad \partial_n \boldsymbol{u}(2.2,y) \;=\; 0$$



Level	DOF (Q_1)
6	133952
7	534144
8	2133248
9	8526336

2D Convection-diffusion equation

Preconditioner/ smoother: **Jacobi**, FEM discr.: Q_1

Time taken by solvers at level 9 also shown

Level/ Pe	**100**	**200**	**500**	**1000**
6 MK	18	19	28	56
MG	15	div	div	div
7 MK	17	17	19	32
MG	15	div	div	div
8 MK	16	16	16	21
MG	17	div	div	div
9 MK	15 (39)	15 (39)	15 (39)	16 (42)
MG	18 (86)	div	div	div
MG (stab)	18 (182)	div	div	div

2D Convection-diffusion equation

Preconditioner/ smoother: **Gauß-Seidel**, FEM discr.: Q_1

Time taken by solvers at level 9 also shown

Level/ Pe	**100**	**200**	**500**	**1000**	**1000** (stab)
6 MK	14	14	44	286	77
MG	5	div	div	div	52
7 MK	14	14	21	138	45
MG	5	div	div	div	48
8 MK	15	15	15	71	28
MG	5	div	div	div	55
9 MK	15 (51)	15 (52)	15 (54)	28 (107)	18(103)
MG	5 (34)	div	div	div	not conv.

2D Anisotropic diffusion

Operator based anisotropy:

$$-\nabla \cdot (\mathbf{G}\ \nabla \boldsymbol{u}) = \mathbf{f}$$

where $\mathbf{G}$ introduces anisotropic diffusion along some vector field $\boldsymbol{v} = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$

$$\mathbf{G} = \begin{pmatrix} (\alpha - \beta)\ v_1 v_1 + \beta & (\alpha - \beta)\ v_1 v_2 \\ (\alpha - \beta)\ v_1 v_2 & (\alpha - \beta)\ v_2 v_2 + \beta \end{pmatrix}$$

- Unit square domain with regular quad mesh
- Q_1 FEM discretization
- MLKM (4,2,2) $\quad \lambda_n = 1.0$
- MG (4 smoothing steps), Exact coarse-grid solve
- Convergence criterion: relative error $\leq 10^{-6}$

Level	DOF
6	4225
7	16641
8	66049
9	263169

Numerical results: Axis parallel anisotropy

$$\mathbf{v} = (1 \quad 0)^T \quad \Rightarrow \qquad \mathbf{G} = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$$

$\alpha = 1, \ \beta > 1$

Convergence rates: **Jacobi**

Lev/ β	**50**	**100**	**500**
6 MK	0.73	0.80	0.86
MG	div	div	div
7 MK	0.74	0.79	0.88
MG	div	div	div
8 MK	0.73	0.79	0.88
MG	div	div	div
9 MK	0.73	0.78	0.88
MG	div	div	div

Convergence rates: **Gauß-Seidel**

Lev/ β	**50**	**100**	**500**
6 MK	0.60	0.67	0.79
MG	0.46	0.64	0.87
7 MK	0.59	0.67	0.81
MG	0.46	0.65	0.89
8 MK	0.59	0.67	0.82
MG	0.46	0.65	0.90
9 MK	0.59	0.67	0.82
MG	0.47	0.66	0.90

Convergence rates: **ILU**

Lev/ β	**50**	**100**	**500**
6 MK	0.14	0.13	0.06
MG	3E-5	3E-5	1E-12
7 MK	0.14	0.14	0.10
MG	8E-5	3E-5	1E-07
8 MK	0.14	0.13	0.12
MG	12E-5	7E-5	7E-6
9 MK	0.13	0.13	0.12
MG	13E-5	8E-5	2E-5

Numerical results: Rotated anisotropy

- Mixed 2nd order derivatives in action
- No renumbering strategy produce block diagonal structure
- No more a piece of cake for ILU
- $\boldsymbol{v} = (1 \quad 1)^T, \ \alpha = 1, \ \beta > 1$

Convergence rates: **Jacobi**

Lev/ β	100	500	1000
6 MK	0.39	0.42	0.43
MG	0.60	0.72	0.73
7 MK	0.39	0.43	0.43
MG	0.62	0.78	0.81
8 MK	0.39	0.43	0.44
MG	0.61	0.82	0.86
9 MK	0.39	0.43	0.45
MG	0.61	0.84	0.88

Convergence rates: **Gauß-Seidel**

Lev/ β	100	500	1000
6 MK	0.41	0.44	0.46
MG	0.45	0.59	0.62
7 MK	0.41	0.46	0.47
MG	0.47	0.68	0.71
8 MK	0.41	0.46	0.48
MG	0.46	0.73	0.78
9 MK	0.41	0.46	0.48
MG	0.46	0.75	0.82

Convergence rates: **ILU**

Lev/ β	100	500	1000
6 MK	0.22	0.27	0.28
MG	0.21	0.35	0.37
7 MK	0.22	0.29	0.31
MG	0.24	0.46	0.50
8 MK	0.21	0.30	0.32
MG	0.24	0.53	0.60
9 MK	0.21	0.30	0.32
MG	0.23	0.56	0.65

Numerical results: Rotated anisotropy

- $\mathbf{v} = (1 \quad 0.3)^T$, $\alpha = 1$, $\beta > 1$

Convergence rates: **Jacobi**

Lev/ β	100	500	1000
6 MK	0.69	0.73	0.74
MG	0.64	0.77	0.79
7 MK	0.69	0.74	0.75
MG	0.65	0.82	0.84
8 MK	0.69	0.74	0.76
MG	0.65	0.85	0.88
9 MK	0.69	0.74	0.76
MG	0.65	0.86	0.90

Convergence rates: **Gauß-Seidel**

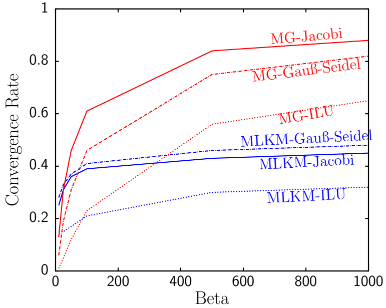
Lev/ β	100	500	1000
6 MK	0.63	0.68	0.69
MG	0.42	0.59	0.62
7 MK	0.62	0.68	0.69
MG	0.43	0.67	0.71
8 MK	0.62	0.68	0.69
MG	0.43	0.71	0.77
9 MK	0.62	0.68	0.70
MG	0.43	0.73	0.80

Convergence rates: **ILU**

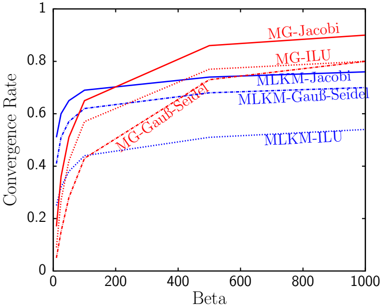
Lev/ β	100	500	1000
6 MK	0.44	0.50	0.51
MG	0.49	0.74	0.77
7 MK	0.44	0.51	0.52
MG	0.55	0.74	0.77
8 MK	0.44	0.51	0.53
MG	0.56	0.76	0.79
9 MK	0.44	0.51	0.54
MG	0.57	0.77	0.80

Numerical results: Rotated anisotropy

$\boldsymbol{v} = (1 \;\; 1)^T$



$\boldsymbol{v} = (1 \;\; 0.3)^T$



2D Anisotropic diffusion

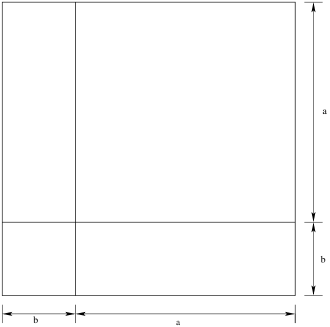
Grid based anisotropy:

$$-\Delta \boldsymbol{u} = \mathbf{f} \qquad \Omega = [0,1]^2$$

with homogeneous Dirichlet boundary conditions.

- Q_1 FEM discretization
- MLKM (4,2,2) $\quad \lambda_n = 1.0$
- MG (4 smoothing steps), Exact coarse-grid solve
- Convergence criterion: relative error $\leq 10^{-6}$

Aspect ratio = a / b.



Numerical results: Grid anisotropy

Convergence rates: **Jacobi**

Lev/ AR	**7**	**15**	**31**
6 MK	0.74	0.82	0.84
MG	div	div	div
7 MK	0.74	0.83	0.87
MG	div	div	div
8 MK	0.74	0.84	0.88
MG	div	div	div
9 MK (Time)	0.84 (3.50)	0.82 (6.61)	0.88 (9.51)
MG (Time)	div	div	div

Convergence rates: **Gauß-Seidel**

Lev/ AR	**7**	**15**	**31**
6 MK	0.56	0.69	0.76
MG	0.41	0.75	0.89
7 MK	0.57	0.72	0.79
MG	0.43	0.78	0.92
8 MK	0.57	0.72	0.80
MG	0.42	0.80	0.93
9 MK (Time)	0.57 (2.16)	0.72 (4.15)	0.80 (5.72)
MG (Time)	0.42 (2.33)	0.80 (9.29)	0.94 (29.7)

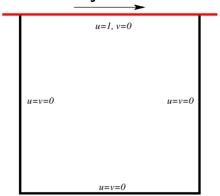
2D stationary incompressible NS Equation

$$\left.\begin{aligned} -\ \frac{1}{Re}\Delta\ \boldsymbol{u} + (\boldsymbol{u}.grad)\boldsymbol{u} +\ grad\ p = 0 \\ \operatorname{div}\boldsymbol{u} = 0 \end{aligned}\right\}$$

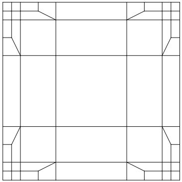
- Benchmark comparisons for code validation
- Solver performance comparison on anisotropic meshes
- Full Vanka as preconditioner/ smoother

Benchmark: Lid driven cavity Flow

Boundary conditions



Coarse mesh



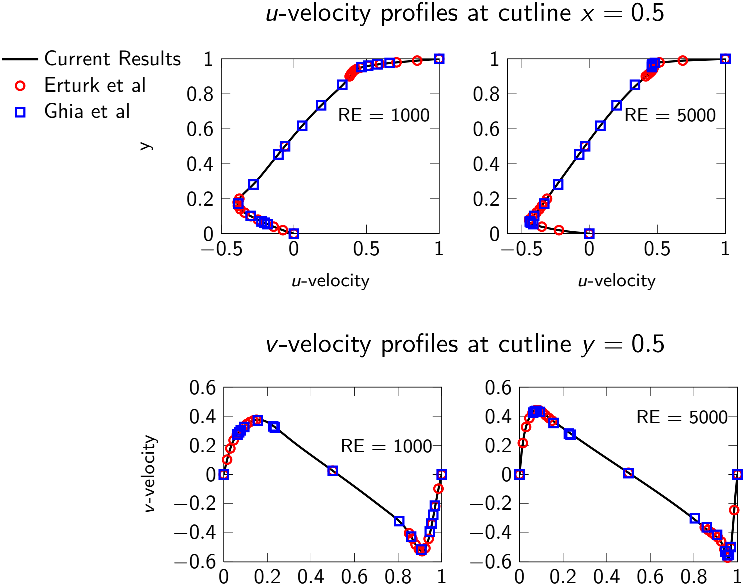
	Re=1000			Re=5000		
cells	Present	Ref.[3]	Ref.[4]	Present	Ref.[3]	Ref.[4]
4096	4.4605E-02	4.4590E-02	4.5828E-02	4.9229E-02	4.9571E-02	5.5969E-02
16384	4.4525E-02	4.4524E-02	4.4843E-02	4.7684E-02	4.7691E-02	5.0196E-02
65536	4.4519E-02	4.4519E-02	4.4592E-02	4.7446E-02	4.7465E-02	4.8020E-02
262144	4.4518E-02	4.4518E-02	4.4535E-02	4.7429E-02	4.7430E-02	4.7541E-02

Table: Kinetic energies

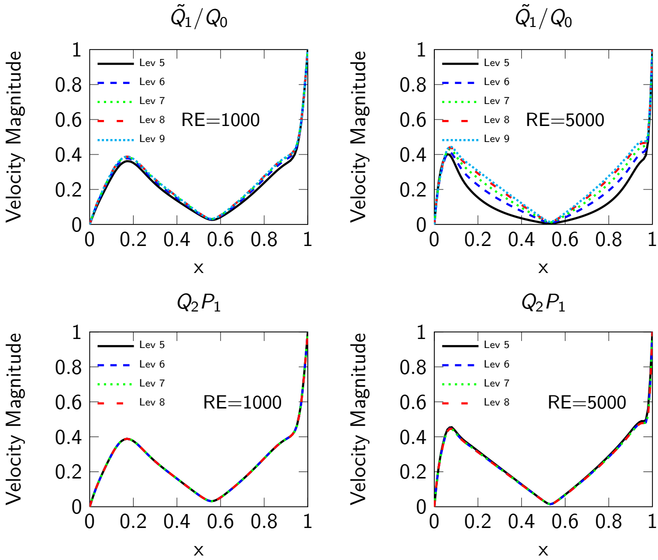
[3] H. Damanik. *FEM Simulation of Non–isothermal Viscoelastic Fluids, PhD thesis*. May 2011.

[4] S. Turek, A. Ouazzi, and J. Hron. "On pressure separation algorithms (PSEPA) for improving the accuracy of incompressible flow simulations". In: *International Journal for Numerical Methods in Fluids* 59.4 (2008), pp. 387–403.

Benchmark: Lid driven cavity Flow

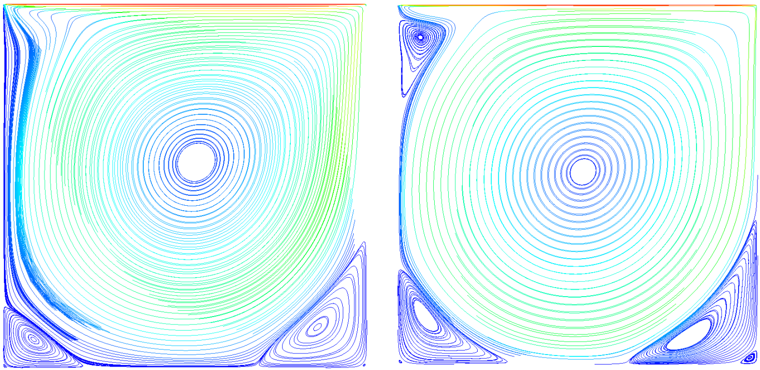


Benchmark: Lid driven cavity Flow



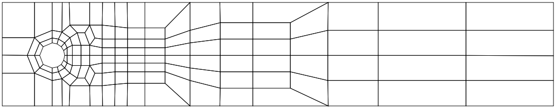
Velocity magnitude profiles at horizontal line passing through geometric center.

Benchmark: Lid driven cavity Flow



Stream lines contours for $RE = 1000$ (left) and $RE = 5000$ (right).

Benchmark: Flow around cylinder



Lev.	NEL
1	520
2	2080
3	8320
4	33280
5	133120

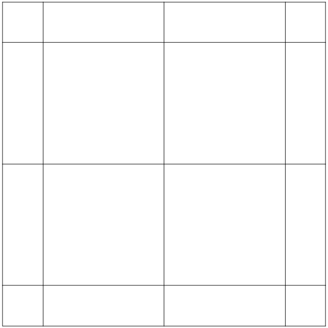
Q_2P_1			$\tilde{Q}_1/Q_0$		
Lev.	Drag/Lift	Ref.[5]	Lev.	Drag/Lift	Ref.[6]
1	5.5456/ 0.00904	5.5424/ 0.00945	2	5.6102/ 0.008590	5.6186/ 0.008874
2	5.5671/ 0.01043	5.5672/ 0.01047	3	5.5935/ 0.009619	5.5901/ 0.010022
3	5.5761/ 0.01057	5.5761/ 0.01057	4	5.5841/ 0.010258	5.5823/ 0.010436
4	5.5786/ 0.01060	5.5786/ 0.01060	5	5.5808/ 0.010508	5.5803/ 0.010566

Table: Results for **RE = 20**. For Q_2P_1 no stabilization was used, for $\tilde{Q}_1/Q_0$ edge oriented jump stablization ($\gamma = 0.01$) was used.

[5] H. Damanik. *FEM Simulation of Non–isothermal Viscoelastic Fluids, PhD thesis*. May 2011.

[6] A. Ouazzi. *Finite element simulation of nonlinear fluids with application to granular material and powder, PhD thesis*. Dec. 2005.

LDC flow with anisotropic grid



Lev.	NEL
1	16
5	4096
6	16384
7	65536
8	262144

LDC flow with anisotropic grid

	Multigrid		MLKM	
Lev	Fixed Pt	Newton	Fixed Pt.	Newton
		Re = 1		
5	6/1	3/2	8/4	4/7
6	5/1	3/2	8/4	4/6
7	5/1	3/2	7/4	4/6
8	5/1	3/2	7/4	4/7
		Re=1000		
5	18/2	5/5	18/4	4/9
6	15/2	5/4	15/4	4/8
7	12/1	4/3	13/4	4/9
8	10/1	5/3	12/4	4/8
		Re=2000		
5	33/2	5/6	28/5	5/12
6	22/2	4/7	22/4	5/10
7	17/2	5/7	18/4	5/10
8	14/1	5/6	15/4	4/9

Table: Results for $\tilde{Q}_1/Q_0$ pair on grid with AR= **1**

LDC flow with anisotropic grid

	Multigrid		MLKM	
Lev	Fixed Pt	Newton	Fixed Pt.	Newton
		Re = 1		
5	div.	div.	10/6	5/12
6	div.	div.	9/7	5/13
7	div.	div.	9/8	5/14
8	div.	div.	10/8	5/17
		Re=1000		
5	div.	div.	28/8	6/24
6	div.	div.	19/8	5/24
7	div.	div.	16/7	5/19
8	div.	div.	16/7	5/17
		Re=2000		
5	div.	div.	34/13	6/36
6	div.	div.	23/11	5/36
7	div.	div.	21/8	5/26
8	div.	div.	20/7	5/20

Table: Results for $\tilde{Q}_1/Q_0$ pair on grid with AR= **12**

LDC flow with anisotropic grid

	Multigrid		MLKM	
Lev	Fixed Pt	Newton	Fixed Pt.	Newton
		Re $= 1$		
5	div.	div.	10/11	5/21
6	div.	div.	11/13	5/26
7	div.	div.	11/14	5/29
8	div.	div.	10/16	5/36
		Re=1000		
5	div.	div.	33/16	6/57
6	div.	div.	22/16	5/57
7	div.	div.	17/12	5/38
8	div.	div.	19/10	5/37
		Re=2000		
5	div.	div.	not conv.	12/119
6	div.	div.	31/43	6/79
7	div.	div.	26/32	6/70
8	div.	div.	25/20	6/42

Table: Results for $\tilde{Q}_1/Q_0$ pair on grid with AR= **24**

LDC flow with anisotropic grid

	Multigrid		MLKM	
Lev	Fixed Pt	Newton	Fixed Pt.	Newton
		Re = 1		
5	5/1	3/2	8/4	4/8
6	5/1	3/2	8/4	4/8
7	5/1	3/2	8/4	4/8
		Re=1000		
5	12/2	4/5	14/6	4/16
6	10/1	4/3	13/5	4/12
7	7/1	4/2	12/5	4/9
		Re=2000		
5	19/3	div.	21/8	5/30
6	15/2	div.	17/5	4/17
7	11/2	div.	16/5	4/13

Table: Results for Q_2/P_1 pair on grid with AR= $\mathbf{1}$

LDC flow with anisotropic grid

	Multigrid		MLKM	
Lev	Fixed Pt	Newton	Fixed Pt.	Newton
		Re = 1		
5	div.	div.	9/19	5/35
6	div.	div	10/22	5/45
7	div.	div	10/26	5/49
		Re=1000		
5	div.	div.	29/8	5/41
6	div.	div.	14/9	5/47
7	div.	div.	14/11	4/49
		Re=2000		
5	div.	div.	not conv.	5/88
6	div.	div.	43/10	5/59
7	div.	div.	19/8	5/42

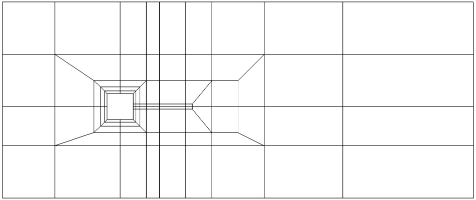
Table: Results for Q_2/P_1 pair on grid with AR= **12**

LDC flow with anisotropic grid

	Multigrid		MLKM	
Lev	Fixed Pt	Newton	Fixed Pt.	Newton
		Re = 1		
5	div.	div.	10/28	5/54
6	div.	div.	10/42	5/80
7	div.	div.	11/51	5/100
		Re=1000		
5	div.	div.	15/12	5/52
6	div.	div.	14/19	5/90
7	div.	div.	13/25	5/107
		Re=2000		
5	div.	div.	not conv.	8/141
6	div.	div.	50/11	6/92
7	div.	div.	20/19	6/109

Table: Results for Q_2/P_1 pair on grid with AR= **24**

Flow around square with anisotropic grid



Lev.	DOF
1	466
3	7,024
4	27,808
5	110,565
6	441,472

Flow around square with anisotropic grid

Lev	MLKM (Drag/Lift)	Ref.[7] (Drag/Lift)
	Re = 5	
3	5.0462+1 / 0.1297+1	4.8239+1 / 0.1241+1
4	5.1685+1 / 0.1329+1	5.0098+1 / 0.1292+1
5	5.2100+1 / 0.1341+1	5.1009+1 / 0.1316+1
6	5.2250+1 / 0.1345+1	5.1507+1 / 0.1330+1
	Re = 50	
3	6.0601+0 / 0.2029-0	6.0175+0 / 0.1904-0
4	6.1880+0 / 0.2070-0	6.0820+0 / 0.2000-0
5	6.2306+0 / 0.2091-0	6.1274+0 / 0.2047-0
6	6.2461+0 / 0.2100-0	6.1651+0 / 0.2072-0
	Re = 500	
3	1.7955+0 / -0.1235-1	1.7795+0 / -0.7125-2
4	1.7279+0 / -0.6910-2	1.7195+0 / -0.5231-2
5	1.6974+0 / -0.4840-2	1.6843+0 / -0.4375-2
6	1.6886+0 / -0.4090-2	1.6733+0 / -0.4459-2

Table: Drag and lift values on mesh with $AR = 10$

[7] S. Turek. *Efficient Solvers for Incompressible Flow Problems: An Algorithmic and Computational Approach*. Springer-Verlag, 1999.

Flow around square with anisotropic grid

	Multigrid		MLKM	
Lev	Iterations	Time	Iterations	Time
		Re = 5		
3	5/2	0.31	7/5	0.30
4	5/2	0.74	7/5	0.99
5	6/2	3.15	6/7	3.74
6	5/2	12.38	6/7	16.24
		Re= 50		
3	7/1	0.33	8/6	0.35
4	6/2	0.83	7/6	1.08
5	5/2	2.81	6/6	3.78
6	4/2	10.78	6/7	16.60
		Re= 500		
3	15/2	1.50	15/7	1.31
4	12/2	4.72	13/6	4.43
5	12/2	19.89	11/6	14.98
6	11/2	77.50	10/7	62.58

Table: Results for $\tilde{Q}_1/Q_0$ pair on grid with AR= **10**

Flow around square with anisotropic grid

	Multigrid (16)*		MLKM	
Lev	Iterations	Time	Iterations	Time
		Re = 5		
3	5/1	0.34	7/6	0.32
4	5/1	1.07	7/8	1.19
5	5/2	5.12	8/10	5.75
6	5/2	28.51	7/12	25.77
		Re= 50		
3	7/1	0.43	7/6	0.32
4	6/1	1.09	7/8	1.27
5	5/2	4.65	7/10	5.28
6	5/2	28.48	6/11	22.52
		Re= 500		
3	15/1	1.58	15/8	1.43
4	13/2	6.80	15/9	5.96
5	11/2	23.87	13/12	24.69
6	10/2	103.69	11/19	121.07

Table: Results for $\tilde{Q}_1/Q_0$ pair on grid with AR = **20**

Flow around square with anisotropic grid

	Multigrid (64)*		MLKM	
Lev	Iterations	Time	Iterations	Time
		Re = 5		
3	4/1	0.62	8/6	0.36
4	4/1	2.04	8/12	1.71
5	5/2	15.38	8/17	8.72
6	5/3	157.54	8/31	65.80
		Re= 50		
3	6/1	0.89	8/6	0.36
4	5/1	3.02	7/12	1.54
5	5/1	14.19	7/17	7.64
6	5/3	136.54	7/29	53.84
		Re= 500		
3	15/1	3.03	17/10	1.95
4	13/5	52.40	15/20	11.40
5	11/2	86.80	14/33	67.21
6	div.	-	13/67	442.29

Table: Results for $\tilde{Q}_1/Q_0$ pair on grid with AR= **40**

Summary and outlook

- A multilevel Krylov subspace method is presented as an alternative to multigrid
- MLKM, like multigrid, has asymptotically mesh-independent and (almost) Reynolds number independent convergence behavior.
- For scalar problems, MLKM/(Jacobi/ Gauß-Seidel) worked better than multigrid/(Jacobi/ Gauß-Seidel).
- MLKM is more robust than MG towards anisotropies.
- For convection dominated flows, MLKM performs better than MG, in a sense, that it can work without stabilization for comparatively high Re/Pe flows.

Summary and outlook

- Application to shear dependent flows.

  Highly viscous flows with low Re numbers $\Longrightarrow$ Monolithic solver

  Locally changing diffusions $\Longrightarrow$ anisotropic diffusion

- Good candidate for parallel implementation

  Matrix-vector multiplication

  Jacobi preconditioner $\Longrightarrow$ Vanka as block Jacobi

Thanks for the patience

Questions

Let $\widehat{A}$ is SPD and $Y = Z$.

$P_D\widehat{A}$	$=$	$\widehat{A}$	$-$	$\widehat{A}Z\widehat{E}^{-1}Z^T\widehat{A}$
(symmetric)		(SPD)		(positive semi definite)
μ_i		λ_i		

$$\Rightarrow \quad \boxed{\mu_i \le \lambda_i}$$