Adaptive higher order temporal discretization scheme with an efficient postprocessing for a time simultaneous implementation

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Introduction I

- about higher order variational time discretizations, namely continuous **Galerkin-Petrov methods** ($cGP(k)$, $k \in \{2, 3\}$)
    - result from the finite element method and arise from a variational approach
    - these schemes give much more accurate results
    - high numerical effort

$\rightarrow$ **adaptive time step control**

- the aim is to calculate a numerical solution with a required accuracy using as few time steps as possible
- use more grid points in the areas where the solution underlies high oscillations
- it is done by sequentially solve one time step after the other
- we need a tool for controlling the length of the time steps
    - $\rightarrow$ error estimators to determine new time step sizes

Introduction II

- **adaptive time step control** based on a **postprocessed solution**
    - a posprocessing procedure for the $cGP(k)$-method, namely cGP-$C^1(k+1)$ method, was introduced in 2011 by Matthies and Schieweck in their work "Higher order variational time discretizations for nonlinear systems of ordinary differential equations"
    - the cGP-$C^1(k+1)$ solution is one order higher than the $cGP(k)$ solution in the L^2-error norm

        $\rightarrow$ we get an error estimator of the $cGP(k)$-method
- to use the full capacity, we prepare the **adaptive** procedure for an in **time simultaneous** usage
    - compute in each adaptive step the solution of the **complete time interval**, determine new step sizes and so rebuild a new time grid

Overview

Nonstationary incompressible Navier-Stokes equation

Find for each time $t \in [0, T]$ a velocity field $\mathbf{u}(t) : \Omega \to \mathbb{R}^d$ and a pressure field $p(t) : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t \mathbf{u} - \epsilon \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = f \quad & \text{in } \Omega \times (0, T], \\ div\ \mathbf{u} = 0 \quad & \text{in } \Omega \times (0, T], \\ \mathbf{u} = g \quad & \text{on } \delta\Omega \times (0, T], \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x) \quad & \text{in } \Omega \text{ for } t = 0, \end{aligned} \tag{1}$$

where ϵ denotes the viscosity, f the body force and $\mathbf{u}_0$ the initial field at time $t = 0$.

- here restricted to the case $d = 2$
- assume homogeneous Dirichlet conditions at the boundary $\delta\Omega$ of a polygonal domain Ω

$cGP(k)$-method for the Navier-Stokes equation I

- for the time discretization, we decompose the time interval $I = [0, T]$ into subintervals $I_n := [t_{n-1}, t_n]$, $n = 1, \ldots, N$ with a time step size $\tau_n := t_n - t_{n-1}$

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$$

- time-continuous ansatz space and time-discontinuous test space

$$\begin{aligned} X_\tau^k &:= \{\mathbf{u} \in C(I, V) : \mathbf{u}|_{I_n} \in \mathbb{P}_k(I_n, V) \qquad \forall\, n = 1, \ldots, N\} \\ Y_\tau^k &:= \{\mathbf{v} \in L^2(I, V) : \mathbf{v}|_{I_n} \in \mathbb{P}_{k-1}(I_n, V) \quad \forall\, n = 1, \ldots, N\} \end{aligned}$$

- time discrete pressure p_τ has an analogous ansatz space $\tilde{X}_\tau^k$ and test space $\tilde{Y}_\tau^k$, where V is replaced by $Q = L_0^2(\Omega)$
- variational formulation in order to get a time discrete problem

$cGP(k)$-method for the Navier-Stokes equation II

time discrete solution $\mathbf{u}_\tau|_{I_n}$ and $p_\tau|_{I_n}$

$$\mathbf{u}_\tau|_{I_n}(t) := \sum_{j=0}^{k} \mathbf{U}_n^j \phi_{n,j}(t), \qquad p_{\tau|I_n}(t) := \sum_{j=0}^{k} P_n^j \phi_{n,j}(t)$$

where the coefficients $(\mathbf{U}_n^j, P_n^j)$ are elements of the Hilbert space $V \times Q$ and the ansatz functions $\phi_{n,j} \in \mathbb{P}_k(I_n, V)$ are the Lagrange basis functions with respect to $k+1$ suitable nodal points $t_{n,j} \in I_n$ satisfying

$$\phi_{n,j}(t_{n,i}) = \delta_{i,j}, \quad i, j = 0, \ldots, k.$$

- $t_{n,j}$ are the quadrature points of the $(k+1)$-point Gauß-Lobatto formula where $t_{n,0} = t_{n-1}$
- the initial condition is equivalent to

$$\mathbf{U}_n^0 = \mathbf{u}_\tau|_{I_{n-1}}(t_{n-1}) \;\; if \;\; n \geq 2 \qquad or \qquad \mathbf{U}_n^0 = \mathbf{u}_0 \;\; if \;\; n = 1$$

Discretization in space and time

- discretize each of the I_n-problems in space with finite elements
- M denotes the mass matrix, L the discrete Laplacian matrix and B_u, B_v the gradient matrices
- the convection matrix N with a discrete velocity field $\mathbf{w}_h(\underline{\mathbf{w}})$
- replace the coefficients $\mathbf{U}_n^j \in \mathbf{V}$ and $P_n^j \in Q_h$ by the space discrete coefficients $\underline{\mathbf{U}}_{n,h}^j = (\underline{U}_{n,h}^j, \underline{V}_{n,h}^j)$ and $\underline{P}_{n,h}^j$ with

$$\mathbf{u}_{n,h}(t_{n,j}) = \underline{\mathbf{U}}_{n,h}^j, \qquad p_{n,h}(t_{n,j}) = \underline{P}_{n,h}^j, \qquad j = 0, \dots, k$$

- where $t_{n,j} = T_n(\hat{t}_j), j = 0, \dots, k$ with $t_{n,0} = t_{n-1}$ and $t_{n,k} = t_n$

$cGP(2)$-method for the Navier-Stokes equation I

$cGP(2)$-method

For a given initial value $\underline{\mathbf{U}}_n^0 = (\underline{U}_n^0, \underline{V}_n^0)$ and $\underline{P}_n^0$ solve the following system to find $\underline{U}_n^1, \underline{U}_n^2, \underline{V}_n^1, \underline{V}_n^2$ and $\underline{P}_n^1, \underline{P}_n^2$ such that

$$\begin{bmatrix} A(u,v) & 0 & B_u \\ 0 & A(u,v) & B_v \\ B_u^T & B_v^T & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ p \end{bmatrix} = \begin{bmatrix} R_u \\ R_v \\ 0 \end{bmatrix},$$

where

$$A(u,v) = \begin{bmatrix} M + \frac{\tau_n}{2} L + \frac{\tau_n}{2} N(u^1, v^1) & \frac{1}{4} M \\ -4M & 2M + \frac{\tau_n}{2} L + \frac{\tau_n}{2} N(u^2, v^2) \end{bmatrix},$$

$$B_u = \begin{bmatrix} B_1 & 0 \\ 0 & B_1 \end{bmatrix}, \quad B_v = \begin{bmatrix} B_2 & 0 \\ 0 & B_2 \end{bmatrix},$$

$cGP(2)$-method for the Navier-Stokes equation II

$cGP(2)$-method

$$R_u = \begin{bmatrix} \frac{\tau_n}{2}(F_n^1 + \frac{1}{2}F_n^0) + \frac{5}{4}M\underline{U}_n^0 - \frac{\tau_n}{4}(L + N(\underline{\mathbf{U}}_n^0)\underline{U}_n^0 - \frac{\tau_n}{4}B_1\underline{P}_n^0 \\ \frac{\tau_n}{2}(F_n^2 - F_n^0) - 2M\underline{U}_n^0 + \frac{\tau_n}{2}(L + N(\underline{\mathbf{U}}_n^0)\underline{U}_n^0 + \frac{\tau_n}{2}B_1\underline{P}_n^0 \end{bmatrix},$$

$$R_v = \begin{bmatrix} \frac{\tau_n}{2}(G_n^1 + \frac{1}{2}G_n^0) + \frac{5}{4}M\underline{V}_n^0 - \frac{\tau_n}{4}(L + N(\underline{\mathbf{U}}_n^0)\underline{V}_n^0 - \frac{\tau_n}{4}B_2\underline{P}_n^0 \\ \frac{\tau_n}{2}(G_n^2 - G_n^0) - 2M\underline{V}_n^0 + \frac{\tau_n}{2}(L + N(\underline{\mathbf{U}}_n^0))\underline{V}_n^0 + \frac{\tau_n}{2}B_2\underline{P}_n^0 \end{bmatrix},$$

with

$$u = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} \underline{U}_n^1 \\ \underline{U}_n^2 \end{bmatrix}, \;\; v = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} \underline{V}_n^1 \\ \underline{V}_n^2 \end{bmatrix}, \;\; p = \begin{bmatrix} p^1 \\ p^2 \end{bmatrix} = \begin{bmatrix} \tau\underline{P}_n^1 \\ \tau\underline{P}_n^2 \end{bmatrix},$$

and $\underline{\mathbf{U}}_{n+1}^0 := \underline{\mathbf{U}}_n^2$.

C^1-continuous Galerkin-Petrov, cGP-$C^1(k+1)$-method

- named after its discrete solution which is a C^1-function in time
- the polynomial order increases to $k+1$ without increasing the total number of unknowns
- if we use in the $cGP(k)$-method a reduced numerical time integration as the k-point Gauß-Lobatto formula we achieve the cGP-$C^1(k)$-method

the cGP-$C^1(k+1)$-method can be computed from the original $cGP(k)$-method with a simple postprocessing step with low computational cost

$cGP - C^1(3)$-method for the Navier-Stokes equation

Postprocessing

The solution of the cGP-$C^1(3)$-method at some time $t \in I_n$ is given by

$$\mathbf{u}_{h,\tau}^{cGP\text{-}C^1}(t) = \mathbf{u}_{h,\tau}^{cGP}(t) + a_n \zeta_n(t), \qquad p_{h,\tau}^{cGP\text{-}C^1}(t) = p_{h,\tau}^{cGP}(t) + b_n \zeta_n'(t)$$

with the polynomial $\zeta_n(t) := \frac{\tau_n}{2}\hat{\zeta}_(T_n^{-1}(t))$ and $\hat{\zeta} \in \mathbb{P}_{k+1}$ which uses the 3 Gauss-Lobatto points $\hat{\zeta}(\hat{t}_j) = 0, j = 0, \ldots, 2$ and $\hat{\zeta}'(\hat{t}_2) = 1$. The coefficient $\mathbf{a}_n \in \mathbf{V}$ and $b_n \in Q_h$ are the solutions of

$$\begin{bmatrix} M & 0 & B_1 \\ 0 & M & B_2 \\ B_1^T & B_2^T & 0 \end{bmatrix} \begin{bmatrix} a_{n,1} \\ a_{n,2} \\ b_n \end{bmatrix} = \begin{bmatrix} F_n^2 \\ G_n^2 \\ 0 \end{bmatrix}$$

$$- \begin{bmatrix} A + N(\mathbf{U}_n^2) & 0 & B_1 \\ 0 & A + N(\mathbf{U}_n^2) & B_2 \\ B_1^T & B_2^T & 0 \end{bmatrix} \begin{bmatrix} \underline{U}_n^2 \\ \underline{V}_n^2 \\ \underline{P}_n^2 \end{bmatrix} - \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{X}_{n,1} \\ \mathcal{X}_{n,2} \\ 0 \end{bmatrix},$$

where $\mathcal{X}_{n,1}, \mathcal{X}_{n,2} \in \mathbb{R}^{m_h}$ denote the nodal components of $\mathbf{u}_{h,t}'(t_n)$

Remarks on the postprocessing and on a time-simultaneous implementation

- solve the nonlinear $cGP(2)$ system time-simultaneously by performing an outer nonlinear iteration
  - use a global-in-time Newton-method

- do the linear postprocessing time parallel
  - problems are independent of each other
  - we just need the time discrete solutions on the discrete time points of the $cGP(2)$-method

Adaptive time step control: velocity

Postprocessing:

- both methods coincide at the endpoints of the time intervals t_n

$\Rightarrow$ cGP-(2) and cGP-$C^1(3)$-method are superconvergent of fourth order at the discrete time points t_n

- the discretization error in the L^2-norm of the cGP-$C^1(k+1)$-method is one order higher in the whole time interval than of the $cGP(k)$-method

$\Rightarrow$ cGP-(2)-method is convergent of third order in the L^2-norm and cGP-$C^1(3)$-method is superconvergent of fourth order in the L^2-norm

- because of the $cGP(k)$-method and the cGP-$C^1(k+1)$-method we achieve two solutions with a different order in the L^2-norm

$\rightarrow$ **error estimator** of the analytical velocity $cGP(k)$-error

Post-processing for high order pressure

- post-processing leads to superconvergence of order 2k for pressure

$\Rightarrow$ take the $cGP - C^1(3)$ pressure solution $p_{h,\tau}^{cGP-C^1}(t_n)$ as $\underline{P}_{n+1}^0$

- we gain a superconvergence of order 2k in the whole time interval for the velocity, but not for the pressure

$\Rightarrow$ construct the cubic Lagrangian polynomial passing through four Gaussian Lobatto points to achieve a superconvergence of order 2k

  - like in "A note on accurate and efficient higher order Galerkin time stepping schemes for the nonstationary Stokes equations" by Hussain, Schieweck and Turek [4]
  - for the $cGP(2)-$method we have 3 points in each subinterval I_n
  - take one additionally from a neighbouring subinterval

$\rightarrow$ **error estimator** of the analytical pressure $cGP(k)$-error

error estimators for the $cGP(k)$-method

velocity error estimator η^u

The adaptive time stepping is based on the $L^2(I, V_h)$-error norm of the numerical solution $\mathbf{u}_{h,\tau}^{cGP}(t)$ and its post-processed solution $\mathbf{u}_{h,\tau}^{cGP\text{-}C^1}(t)$ as an per unit step **scaled estimator** of the analytical error of the $cGP(k)$-method

$$\eta_n^u = ||\mathbf{u}_{h,\tau}^{cGP}(t) - \mathbf{u}_{h,\tau}^{cGP\text{-}C^1}(t)||_{L^2(I_n,V_h)} \frac{1}{\sqrt{\tau_n}}, \quad t \in I_n,$$

for the time interval I_n with $n = 1, 2, \ldots, N$.

pressure error estimator η^p

The adaptive time stepping is based on the absolute pointwise error of the numerical solution $p_{h,\tau}^{cGP}(t)$ and the cubic Lagrange interpolated solution $p_{h,\tau}^{cGP\text{-}cLI}(t)$ as a **pointwise estimator** of the analytical error of the $cGP(k)$- method

$$\eta_n^p = |p_{h,\tau}^{cGP}(t) - p_{h,\tau}^{cGP-cLI}(t)|, \quad t \in I_n \backslash \{t_{n-1}, \frac{t_{n-1}+t_n}{2}, t_n\}$$

error estimators for the lift coefficients

lift error estimator η^{lift}

We can also estimate a pointwise error of the lift values c_{lift}^{cGP} by using the lift coefficients $c_{lift}^{cGP\text{-}C^1/cLI}$

$$\eta_n^{lift} = |c_{lift}^{cGP}(t) - c_{lift}^{cGP\text{-}C^1/cLI}(t)|, \quad t \in I_n \backslash \{t_{n-1}, \frac{t_{n-1}+t_n}{2}, t_n\},$$

where the lift coefficient with the mean velocity, the caracteristic length of the flow and the lift forces are defined by

$$c_{lift} = \frac{2}{U_{mean}^2 L} F_L, \qquad F_L = -\int_S (p\epsilon \frac{\delta u_t}{\delta n} n_x + p n_y) dS.$$

The lift coefficient $c_{lift}^{cGP\text{-}C^1/cLI}$ uses the $cGP - C^1$ velocity solution and the $cGP - cLI$ pressure solution.

- similarly for the drag coefficients

Adaptive strategies: iteration process

Iteration process

1. for all time steps t_n, $n = 1, 2, \ldots, N$:
   1.1 solve the cGP block system
   1.2 do the post-processing step to get the solution from the cGP-C^1-method
   1.3 calculate the error estimators η_n
2. check whether the error estimators are under a given tolerance and if so break
3. determine the new time step sizes and generate the new time grid

Adaptive strategy

Strategy

The optimal time step size is determined by a controller like

$$\tau_n^{i+1} = \tilde{\tau}_n \theta \left(\frac{TOL}{\tilde{\eta}_n^u}\right)^{\frac{1}{p}} \qquad \text{for } \tilde{\eta}_n^u > 0.75 \cdot TOL \ \text{ or } \ \tilde{\eta}_n^u, \tilde{\eta}_n^p \leq 0.01 \cdot TOL,$$

$$\tau_n^{i+1} = \tilde{\tau}_n \theta \left(\frac{TOL}{\tilde{\eta}_n^p}\right)^{\frac{1}{p}} \qquad \text{for } \tilde{\eta}_n^p > 0.75 \cdot TOL \ \text{ and } \ \tilde{\eta}_n^u < 0.75 \cdot TOL,$$

$$\tau_n^{i+1} = \tilde{\tau}_n \qquad \text{else},$$

with θ is a security parameter and $p = k$ which depends on the order of the method $O(\tau^k)$. The interpolated values for the step sizes and error estimators are given with $\tilde{\tau}_n$ and $\tilde{\eta}_n$ in the $i+1$ iteration for the time point t_n^{i+1}. To prevent the step sizes from shrinking/growing too much, the already computed time step size is matched with

$$\tau_n^{i+1} = \min\{2\tilde{\tau}_n, \max(0.05\tilde{\tau}_n, \tau_n^{i+1})\}.$$

Numerical tests

Test problem for the heat equation

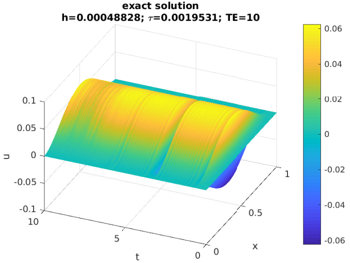
We consider the heat equation for the space domain $\Omega = [0,1]$ and the time interval $I = [0,10]$ with the prescribed exact solution

$$u(x,t) = (x(1-x))^2 sin(\pi N(t)t),$$

$$\textbf{with} \;\; N(t) = (9-(t-3)^2)sin(\frac{\pi t}{6})$$

and the associated data f and $u_0(x) = u(x,0)$.

- finite differences as space discretization
- $cGP(2)/cGP\text{-}C^1(3)$ method as time discretization

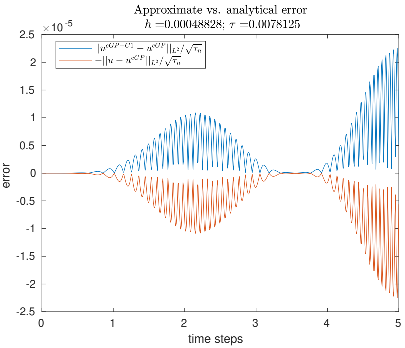


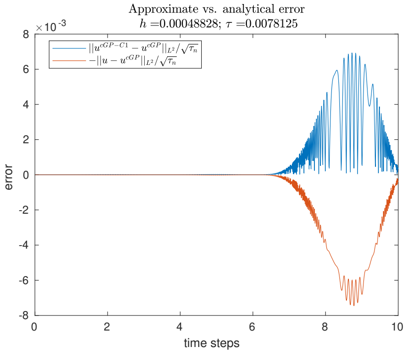
Heat equation results: EOC I

	$cGP(2)$				$cGP\text{-}C^1(3)$			
$\frac{1}{\tau}$	$\lvert e_u \rvert_{L^\infty}$	EOC	$\lvert\lvert e_u \rvert\rvert_{L^2}$	EOC	$\lvert \tilde{e}_u \rvert_{L^\infty}$	EOC	$\lvert\lvert \tilde{e}_u \rvert\rvert_{L^2}$	EOC
64	3.94e-01	1.41	2.00e-01	1.60	3.94e-01	1.41	2.00e-01	1.61
128	2.71e-03	7.18	7.28e-03	4.78	2.71e-03	7.18	5.19e-03	5.27
256	1.26e-04	4.43	8.66e-04	3.07	1.26e-04	4.43	3.34e-04	3.95
512	7.36e-06	4.09	1.08e-04	3	7.36e-06	4.09	2.09e-05	4
1024	4.54e-07	4.02	1.36e-05	3	4.54e-07	4.02	1.31e-06	4
		4		3		4		4

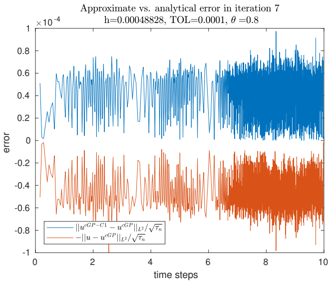
Table: L^2-error and L^∞-error with a spatial step size $h = 4.882812e - 04$

Error estimator and analytical error with $T = \{5, 10\}$

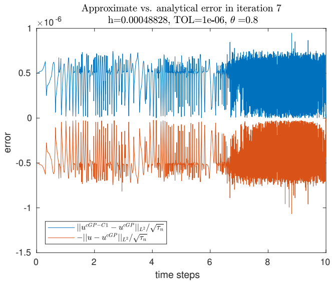




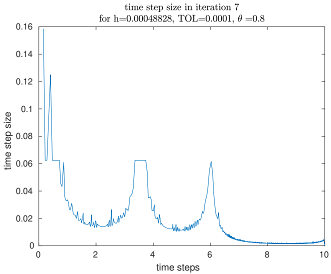
adaptive Strategy with different tolerances

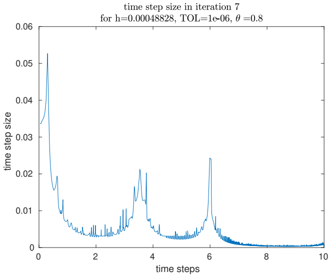


(a) $TOL = 10^{-4}$, $steps = 1852$



(b) $TOL = 10^{-6}$, $steps = 8560$





Nonstationary incompressible Navier-Stokes equation

Test problem

On $\Omega = (0,1)^2$ and with $\epsilon = 1$, the prescribed velocity field $\mathbf{u} = (u, v)$ is

$$u(x,y,t) := x^2(1-x)^2[2y(1-y)^2 - 2y^2(1-y)]sin(10\pi t),$$
$$v(x,y,t) := -[2x(1-x)^2 - 2x^2(1-x)]y^2(1-y)^2 sin(10\pi t),$$

and the pressure distribution is

$$p(x,y,t) := -(x^3 + y^3 - 0.5)(1.5 + 0.5 sin(10\pi t)).$$

The initial data is $\mathbf{u}_0(x,y) = \mathbf{u}(x,y,0)$.

- $cGP(2)/cGP\text{-}C^1(3)$ method as time discretization
- Q_2/P_1^{disc} as space discretization
- time interval $I = [0,1]$
- to solve the nonlinear problem we apply the fixed-point iteration

Navier-Stokes equation results: EOC II

	$cGP(2)$				$cGP\text{-}C^1(3)$			
$\frac{1}{\tau}$	$\|e_u\|_{L^\infty}$	EOC	$\|\|e_u\|\|_{L^2}$	EOC	$\|\tilde{e}_u\|_{L^\infty}$	EOC	$\|\|\tilde{e}_u\|\|_{L^2}$	EOC
10	6.95E-04		4.21E-04		6.95E-04		4.89E-04	
20	4.01E-05	4.12	7.81E-05	2.43	4.01E-05	4.12	2.73E-05	4.16
40	3.09E-06	3.70	1.05E-05	2.89	3.09E-06	3.70	1.68E-06	4.02
80	2.03E-07	3.93	1.34E-06	2.97	2.03E-07	3.93	1.08E-07	3.96
160	4.28E-08	2.25	1.71E-07	2.97	4.28E-08	2.25	3.02E-08	1.84
		4		3		4		4

Table: velocity L^2-error and L^∞-error for space mesh level 7

Navier-Stokes equation results: EOC I

	$cGP(2)$				$cGP\text{-}C^1(3)$			
$\frac{1}{\tau}$	$\lvert e_p \rvert_{L^\infty}$	EOC	$\lVert e_p \rVert_{L^2}$	EOC	$\lvert \tilde{e}_p \rvert_{L^\infty}$	EOC	$\lVert \tilde{e}_p \rVert_{L^2}$	EOC
10	1.85E-03		7.82E-03		1.85E-03		9.82E-03	
20	1.15E-04	4.01	2.97E-03	1.40	1.15E-04	4.01	7.75E-04	3.66
40	3.84E-05	1.58	3.83E-04	2.96	3.84E-05	1.58	5.46E-05	3.83
80	3.74E-05	0.04	5.14E-05	2.90	3.74E-05	0.04	1.93E-05	1.50
160	3.74E-05	0.00	1.78E-05	1.53	3.74E-05	0.00	1.91E-05	0.02

Table: pressure L^2-error and L^∞-error for space mesh level 7 with $\underline{P}_{n+1}^0 = \underline{P}_n^2$

	$cGP(2)$				$cGP\text{-}C^1(3)$			
$\frac{1}{\tau}$	$\lvert e_p \rvert_{L^\infty}$	EOC	$\lVert e_p \rVert_{L^2}$	EOC	$\lvert \tilde{e}_p \rvert_{L^\infty}$	EOC	$\lVert \tilde{e}_p \rVert_{L^2}$	EOC
10	1.85E-03		7.82E-03		1.85E-03		9.83E-03	
20	1.09E-04	4.09	2.97E-03	1.40	1.09E-04	4.09	7.76E-04	3.66
40	8.76E-06	3.64	3.83E-04	2.96	8.76E-06	3.64	5.15E-05	3.91
80	8.42E-07	3.38	4.87E-05	2.98	8.42E-07	3.38	3.29E-06	3.97
160	5.32E-07	0.66	6.14E-06	2.99	5.32E-07	0.66	4.84E-07	2.77
		4		3		4		4

Table: pressure L^2-error and L^∞-error for space mesh level 7 with $\underline{P}_{n+1}^0 = p_{h,\tau}^{cGP-C^1}(t_n)$

DFG flow around cylinder benchmark 2D-3

- flow configuration can be found at http://www.mathematik.tu-dortmund.de/ featflow/en/benchmarks/cfdbenchmarking.html
- Navier-Stokes equation (5) with source term f=0, viscosity $\epsilon = 10^{-3}$, and the final time $T = 8$
- Finite element space discretization with Q_2/P_1^{disc}

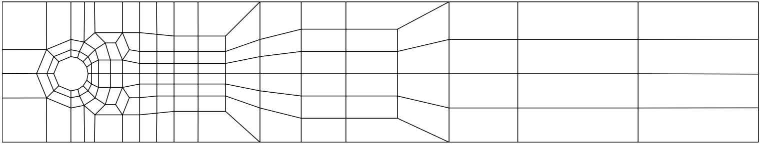


Figure: initial grid on level 1

Velocity error estimator and analytical error

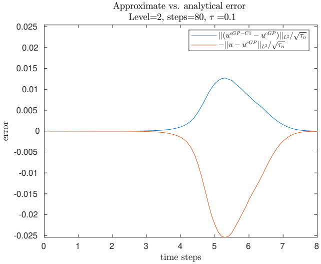


Figure: Iteration 1

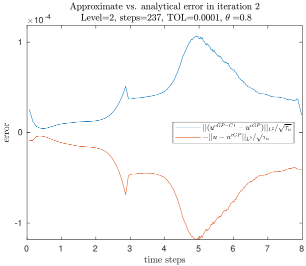


Figure: Iteration 2

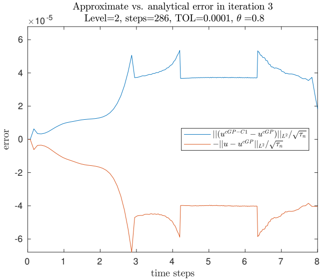


Figure: Iteration 3

Pressure error estimator and analytical error

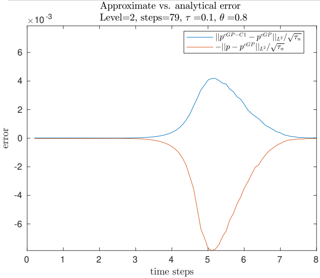


Figure: Iteration 1

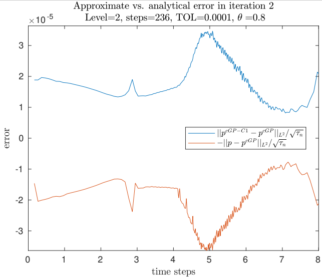


Figure: Iteration 2

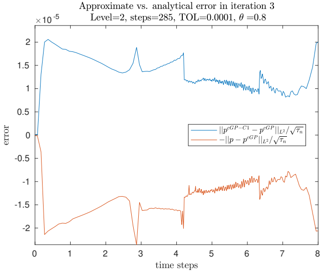
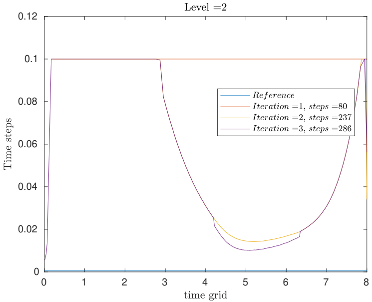
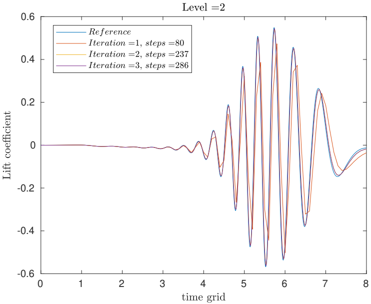


Figure: Iteration 3

Time grids and lift values in the adaptive iterations

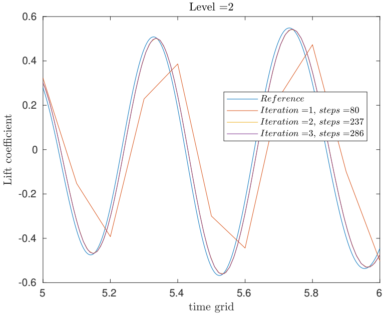


(a) time steps sizes

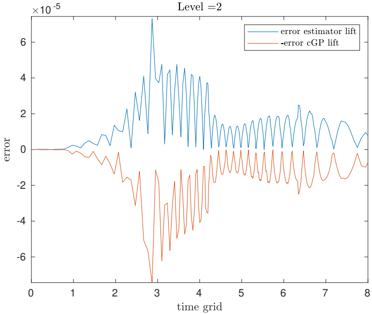


(b) Lift coefficients

Lift values and error estimator for the adaptive iterations



(c) Lift coefficients



(d) Lift error estimator in iteration 3

Conclusion, remarks and outlook

Conclusion:

- higher order variational time discretization for velocity and pressure and cheap error estimators for velocity and pressure
- the time step control works fine and the length of the time step represent the dynamics of the solution
- we built a new grid in each iteration
- we can use some time parallel methods or time simultaneous methods because we solve adaptive over the complete time interval

Remarks and outlook:

- modify the adaptive strategy for relative and absolute errors
- use a combination of velocity, pressure and lift error estimators to determine the required time step size

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