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The „Tensor Diffusion“-approach for simulating viscoelastic fluids with special emphasis on the „no solvent“-case

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Introduction

Simulation of viscoelastic fluids without solvent

Application in mind

- processing of (pure) rubber melts („Kautschuk“), industrial partner: ARLANXEO

- simulating viscoelastic fluids …
  - … consisting of wide relaxation time spectrum (over several decades)
    - → „multi mode“-approach for adequate modelling
    - → High Weissenberg Number Problem (HWNP)
  - … **without „solvent contribution“**
- **plus:** solver often intended for **direct steady-state** solutions (relevant for applications)

- governing equations

$$-2\,\eta_s \nabla \cdot \boldsymbol{D}(\boldsymbol{u}) - \nabla \cdot \boldsymbol{\sigma} + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

plus constitutive equation for stress tensor $\boldsymbol{\sigma}$

Modelling approaches

1. **differential** material model (for multiple „modes“ $\Lambda_k, k = 1, \dots, K$)

$$\boldsymbol{\sigma} = \sum_{k=1}^{K} \boldsymbol{\sigma}_k \,, \qquad (\boldsymbol{u} \cdot \nabla)\boldsymbol{\sigma}_k - \nabla \boldsymbol{u}^T \cdot \boldsymbol{\sigma}_k - \boldsymbol{\sigma}_k \cdot \nabla \boldsymbol{u} + \boldsymbol{f}\left(\Lambda_k, \eta_{p,k}, \boldsymbol{\sigma}_k\right) = 2\frac{\eta_{p,k}}{\Lambda_k}\boldsymbol{D}(\boldsymbol{u})$$

- model function: $\boldsymbol{f}\left(\Lambda_k, \eta_{p,k}, \boldsymbol{\sigma}_k\right) = \frac{1}{\Lambda_k}\left(\boldsymbol{\sigma}_k + \alpha_k \frac{\Lambda_k}{\eta_{p,k}} \boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma}_k\right)$ for $\alpha_k \in [0,1]$

2. **integral** material model („Deformation Fields Method“, c.f. Hulsen et al.)

$$\boldsymbol{\sigma} = \int_0^\infty M(s)\, \boldsymbol{g}\left(\boldsymbol{B}(s)\right) ds$$

for $s \in [0, \infty[, \boldsymbol{B}(0) = \boldsymbol{I}$: $\qquad \frac{\partial}{\partial s}\boldsymbol{B}(s) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B}(s) - \nabla \boldsymbol{u}^T \cdot \boldsymbol{B}(s) - \boldsymbol{B}(s) \cdot \nabla \boldsymbol{u} = 0$

- (multi-mode) memory function: $M(s) = \sum_{k=1}^K M_k(s) = \sum_{k=1}^K \frac{\eta_{p,k}}{\Lambda_k^2} \exp\left(-\frac{s}{\Lambda_k}\right)$

- in both cases for $\eta_s = 0$ („no solvent“):

Operator-Splitting not applicable vs. monolithic approach difficult

Monolithic approach

- „no solvent" in Stokes equations (plus constitutive)

$$\cancel{-\,2\,\eta_s \nabla\cdot \boldsymbol{D}(\boldsymbol{u})} - \nabla\cdot\boldsymbol{\sigma} + \nabla p = 0, \qquad \nabla\cdot\boldsymbol{u} = 0$$

- **differential** models for **single-mode**

$$\begin{pmatrix} 0 & B & -C \\ B^T & 0 & 0 \\ D & 0 & K(\boldsymbol{u}) \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \\ \boldsymbol{\sigma} \end{pmatrix} = \begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_p \\ \boldsymbol{r}_\sigma \end{pmatrix}$$

  - standard Krylov-space methods: no diagonal preconditioning
  - multigrid
    - no diagonal smoothers applicable
    - Vanka-like smoothers behave unstable
  - stability problems w.r.t. additional LBB for $\boldsymbol{u}, \boldsymbol{\sigma}$

- not applicable for **integral** models

Operator-Splitting

1. for given $\boldsymbol{\sigma}^n$, solve **Stokes-problem** for $\boldsymbol{u}^{n+1}, p^{n+1}$ with non-zero RHS

$$\cancel{-2\,\eta_s \nabla \cdot \boldsymbol{D}\,(\boldsymbol{u}^{n+1})} + \nabla p^{n+1} = \nabla \cdot \boldsymbol{\sigma}^n, \qquad \nabla \cdot \boldsymbol{u}^{n+1} = 0$$

2. calculate $\boldsymbol{\sigma}^{n+1}$ from **differential** ($\boldsymbol{\sigma}_k^{n+1}$ for „multi-mode“ $\boldsymbol{\sigma} = \sum_{k=1}^{K} \nabla \cdot \boldsymbol{\sigma}_k$)

$$(\boldsymbol{u}^{n+1} \cdot \nabla)\boldsymbol{\sigma}^{n+1} - \nabla \boldsymbol{u}^{n+1^T} \cdot \boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^{n+1} \cdot \nabla \boldsymbol{u}^{n+1} + \boldsymbol{f}\left(\Lambda, \eta_p, \boldsymbol{\sigma}^{n+1}\right) = 2\frac{\eta_p}{\Lambda}\boldsymbol{D}(\boldsymbol{u}^{n+1})$$

or **integral** constitutive equation / DFM

$$\boldsymbol{\sigma}^{n+1} = \int_0^\infty M(s)\, \boldsymbol{g}\big(\boldsymbol{B}(s)\big)\, ds$$

$$\frac{\partial}{\partial s}\boldsymbol{B}(s) + (\boldsymbol{u}^{n+1} \cdot \nabla)\boldsymbol{B}(s) - \nabla \boldsymbol{u}^{n+1^T} \cdot \boldsymbol{B}(s) - \boldsymbol{B}(s) \cdot \nabla \boldsymbol{u}^{n+1} = 0$$

- Stokes problem without diffusive part

→ not applicable for „no solvent“ - independently of actual model!

The „Tensor Diffusion“-approach

Assumption: „Tensor Diffusion“ $\boldsymbol{\mu}$ via $\boldsymbol{\sigma} = \boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u})$

- direct computation of $(\boldsymbol{u}, p)$-solution via „Tensor Stokes“-problem

$$-\nabla \cdot \big(\boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u})\big) + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

  - $\boldsymbol{\sigma}$ computed in postprocessing
  - → solve **linear „Stokes“-problem** instead of highly nonlinear system

- **monolithic** solution approach for „no-solvent“ (differential models only, single-mode)

$$\begin{pmatrix} -T(\boldsymbol{\mu}) & B & 0 \\ B^T & 0 & 0 \\ D & 0 & K(\boldsymbol{u}) \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \\ \boldsymbol{\sigma} \end{pmatrix} = \begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_p \\ \boldsymbol{r}_\sigma \end{pmatrix}$$

  - no „zero-diagonal“ in Stokes-part
  - → diagonal preconditioners / smoothers applicable
  - → improved Vanka-behaviour
  - → stabilizing character of „Tensor Diffusion“ w.r.t. LBB for $\boldsymbol{u}, \boldsymbol{\sigma}$

Assumption: „Tensor Diffusion“ $\boldsymbol{\mu}$ via $\boldsymbol{\sigma} = \boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u})$

→ prototypical **Operator-Splitting** in iterative methods

1. for given $\boldsymbol{\mu}^n$ (e.g. $\boldsymbol{\mu} = \boldsymbol{I}$ for $n = 0$) determine $(\boldsymbol{u}^{n+1}, p^{n+1})$ from „Tensor Stokes“-problem

$$-\nabla \cdot \left(\boldsymbol{\mu}^n \cdot \boldsymbol{D}(\boldsymbol{u}^{n+1})\right) + \nabla p^{n+1} = 0, \qquad \nabla \cdot \boldsymbol{u}^{n+1} = 0$$

2. for $\boldsymbol{u}^{n+1}$, determine stress tensor $\boldsymbol{\sigma}^{n+1}$ from **integral or differential** model
3. determine (tensor) viscosity $\boldsymbol{\mu}^{n+1}$ via $\boldsymbol{\sigma}^{n+1} = \boldsymbol{\mu}^{n+1} \cdot \boldsymbol{D}(\boldsymbol{u}^{n+1})$

- now, **non-vanishing velocity coupling** in momentum equation

→ Operator-Splitting applicable

Proof of concept

- **main idea:** postulate stress-decomposition $\boldsymbol{\sigma} = \boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u})$ exists

→ incorporate into viscoelastic simulations (besides Stokes / constitutive equations)

- show feasibility of presented „Tensor Diffusion“-approach by
  1. solve full **differential model** in $(\boldsymbol{u}, \boldsymbol{\sigma}, p)$ (already **realized inside FEATFLOW**)
  2. calculating „Tensor Diffusion“ $\boldsymbol{\mu}$
     a) analytically
     b) based on differential $(\boldsymbol{u}, \boldsymbol{\sigma}, p)$-solution
  3. solve pure **„Tensor Stokes“-problem** for $(\boldsymbol{u}, p)$ with above $\boldsymbol{\mu}$

→ „Tensor Stokes“-solution should give same as full 3-field $(\boldsymbol{u}, \boldsymbol{\sigma}, p)$-formulation!

1D-flow configurations

$$\boldsymbol{u} = \binom{u}{v} = \binom{u(y)}{0}, \qquad \boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{pmatrix}, \qquad \frac{\partial \boldsymbol{\sigma}}{\partial x} = \frac{\partial \boldsymbol{B}}{\partial x} = 0$$

1. analytical solutions for $\boldsymbol{B}$ inserted into **stress integral** (single-mode UCM)

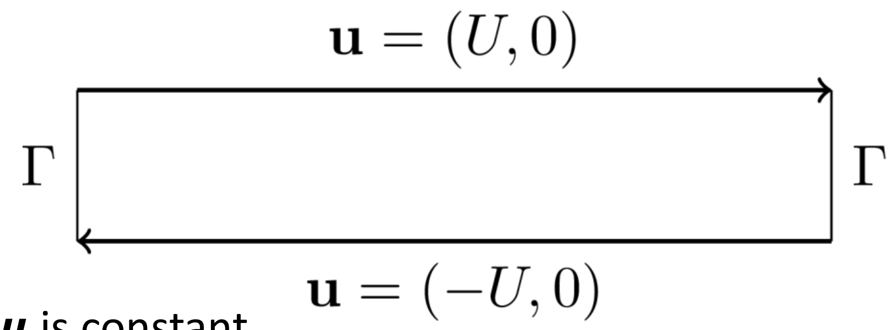
$$\boldsymbol{\sigma} = \int_0^\infty \frac{\eta_p}{\Lambda^2} \exp\left(-\frac{s}{\Lambda}\right)(\boldsymbol{B}(s) - \boldsymbol{I})ds = \left\{2\int_0^\infty \frac{\eta_p}{\Lambda^2} \exp\left(-\frac{s}{\Lambda}\right)\begin{pmatrix} s & s^2\frac{\partial u}{\partial y} \\ 0 & s \end{pmatrix} ds\right\} \boldsymbol{D}(\boldsymbol{u})$$

→ (unsymmetric) tensor–valued quantity relating $\boldsymbol{\sigma}$ to $\boldsymbol{D}(\boldsymbol{u})$: $\boldsymbol{\mu}(\boldsymbol{u}) = 2\eta_p \begin{pmatrix} 1 & 2\Lambda\frac{\partial u}{\partial y} \\ 0 & 1 \end{pmatrix}$

2. for **differential** (single-mode) UCM: $\boldsymbol{\sigma} = \begin{pmatrix} 2\Lambda\eta_p \left(\frac{\partial u}{\partial y}\right)^2 & \eta_p \frac{\partial u}{\partial y} \\ \eta_p \frac{\partial u}{\partial y} & 0 \end{pmatrix} = \boldsymbol{\mu}(\boldsymbol{u}) \cdot \boldsymbol{D}(\boldsymbol{u})$

- methodology can be analogously applied to **Giesekus / PSM →** $\boldsymbol{\mu_{21} \neq 0}$

Shear flow



- **velocity field** $\boldsymbol{u} = \begin{pmatrix} u(y) \\ v \end{pmatrix} = \begin{pmatrix} Uy \\ 0 \end{pmatrix}$

→ $\boldsymbol{D}(\boldsymbol{u})$ and $\boldsymbol{\sigma}$ are constant → viscosity tensor $\boldsymbol{\mu}$ is constant

- solve „Tensor Stokes“-problem with $\boldsymbol{\mu}$ prescribed globally

$$-\nabla \cdot \left(\boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u})\right) + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

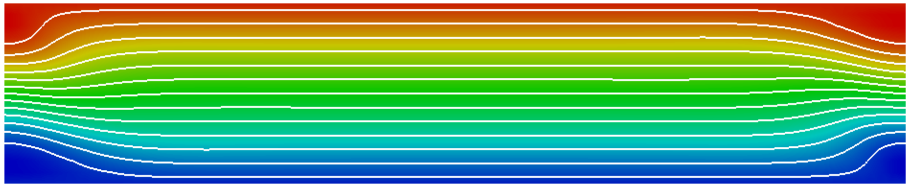
- test configuration: set perturbed velocity profile on Γ

$$\tilde{\boldsymbol{u}} = \begin{pmatrix} w(y) \\ 0 \end{pmatrix}, \qquad w(y) = Uy + \kappa y(1 - y^2)$$

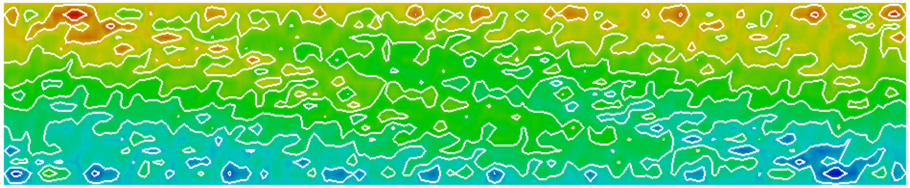
→ analytical flow profiles should be recovered inside of the channel

Shear flow

- for „low“ relaxation times Λ, analytical x-velocity is recovered
- for higher Λ, flow shows defects (due to unsymmetric $\boldsymbol{\mu}$!)



$u(y), \quad \Lambda = 10.0$



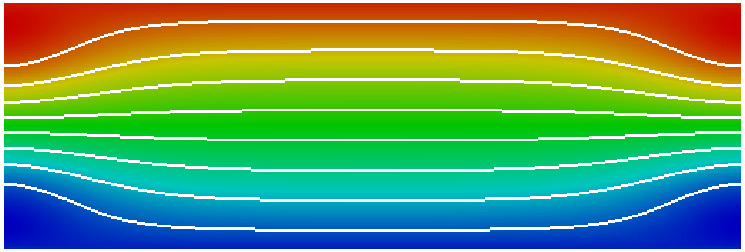
$u(y), \quad \Lambda = 50.0$

- consider **symmetrized** **„Tensor Stokes“**-problem

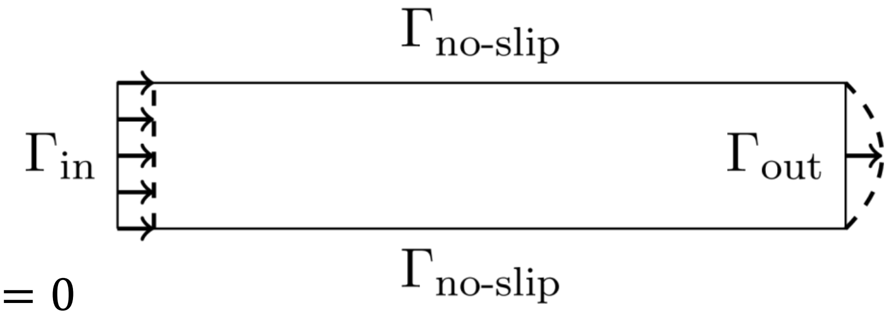
$$-\frac{1}{2}\nabla\cdot(\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})+\boldsymbol{D}(\boldsymbol{u})\cdot\boldsymbol{\mu}^T)+\nabla p=0, \qquad \nabla\cdot\boldsymbol{u}=0$$

→ x-velocity looks reasonable even for very large Λ!

$u(y), \Lambda = 1000.0$



Poiseuille flow

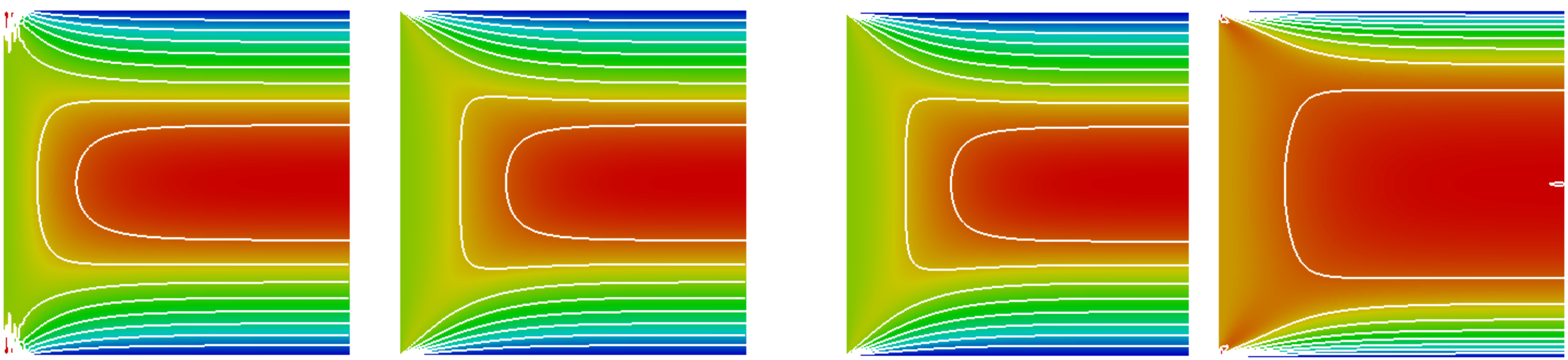


- solve symmetrized „Tensor Stokes“-problem

$$-\frac{1}{2}\nabla \cdot (\boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{D}(\boldsymbol{u}) \cdot \boldsymbol{\mu}^T) + \nabla p = 0, \nabla \cdot \boldsymbol{u} = 0$$

- „fully developed“ Tensor Diffusion $\boldsymbol{\mu}$**,** „perturbed“ velocity profile on inflow edge

$\Lambda = 1.0$ $\Lambda = 50.0$ $\Lambda = 1.0, U_{\max} = 0.099$ $\Lambda = 50.0, U_{\max} = 0.083$

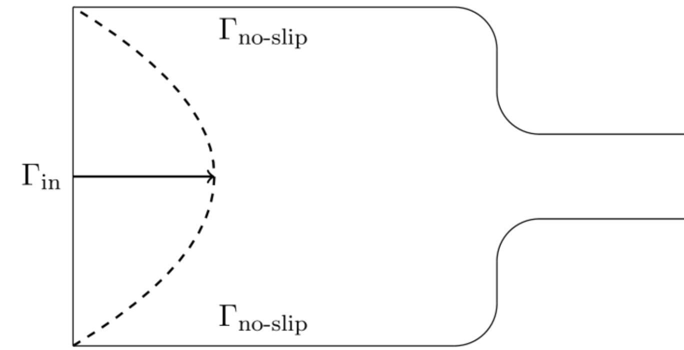


UCM Giesekus, $\alpha = 0.1$

→ via solving purely **linear (Tensor-)Stokes** problem:

fully developed **viscoelastic(!) flow** recovered away from (perturbed) inflow!

Four-field formulation



- „Tensor Diffusion“ not known (semi-)analytically
- how to calculate $\boldsymbol{\mu}$ from differential $(\boldsymbol{u}, \boldsymbol{\sigma}, p)$-solution?
- extend single-mode differential models for „no solvent“

$$-\nabla \cdot \boldsymbol{\sigma} + \nabla p = 0 \quad \textit{or} \quad -\frac{1}{2}\nabla \cdot \left(\boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{D}(\boldsymbol{u}) \cdot \boldsymbol{\mu}^{T}\right) + \nabla p = 0,$$

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{\sigma} - \nabla \boldsymbol{u}^{T} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \nabla \boldsymbol{u} + \boldsymbol{f}\left(\Lambda, \eta_p, \boldsymbol{\sigma}\right) = 2\frac{\eta_p}{\Lambda}\boldsymbol{D}(\boldsymbol{u}),$$

$$\boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{\sigma} = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

→ **1st alternative:** $(\boldsymbol{u}, \boldsymbol{\sigma}, p)$-solution does not depend on $\boldsymbol{\mu}$ („postprocessing fashion“)

→ **2nd alternative:** four-field formulation of „Tensor Stokes“-problem,

$(\boldsymbol{u}, \boldsymbol{\sigma}, p)$ coupled with $\boldsymbol{\mu}$

Four-field formulation

- discrete operators and nonlinear systems via **FEM** with $Q_2/P_1^{\text{disc}}/\ Q_2/\ Q_0$
- plus „**Edge-Oriented FEM**“ stabilization (EOFEM) for $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\mu}\ )$

→ solution via monolithic Newton-**multigrid**-scheme (even for higher Λ)

- for $\omega_n \in]0{,}1], \boldsymbol{x} = (\boldsymbol{u}, p, \boldsymbol{\sigma}, \boldsymbol{\mu})^T$:

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \omega_n \boldsymbol{A}(\boldsymbol{u}^n, \boldsymbol{\sigma}^n, \boldsymbol{\mu}^n)^{-1} \boldsymbol{r}(\boldsymbol{u}^n, p^n, \boldsymbol{\sigma}^n, \boldsymbol{\mu}^n\ )$$

- residual $\boldsymbol{r}$ and Jacobian matrix $\boldsymbol{A}$ problem-dependent

→ original approach

$$\boldsymbol{A}_C = \begin{pmatrix} J_u & B & -C & 0 \\ B^T & 0 & 0 & 0 \\ D + K_u(\boldsymbol{\tau}) & 0 & K_\sigma(\boldsymbol{w}) + J_\sigma & 0 \\ 0 & 0 & M & D(\boldsymbol{w}) + J_\mu \end{pmatrix}, \qquad \boldsymbol{r}_C = \begin{pmatrix} Bq - C\boldsymbol{\tau} \\ B^T\boldsymbol{w} \\ D\boldsymbol{w} + K(\boldsymbol{w})\boldsymbol{\tau} \\ M\boldsymbol{\tau} + D(\boldsymbol{w})\boldsymbol{\nu} \end{pmatrix}$$

Four-field formulation

- discrete operators and nonlinear systems via **FEM** with $Q_2/P_1^{\text{disc}}/\ Q_2/\ Q_0$
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- for $\omega_n\ \in]0,1],\boldsymbol{x}=(\boldsymbol{u},p,\boldsymbol{\sigma},\boldsymbol{\mu})^T$:

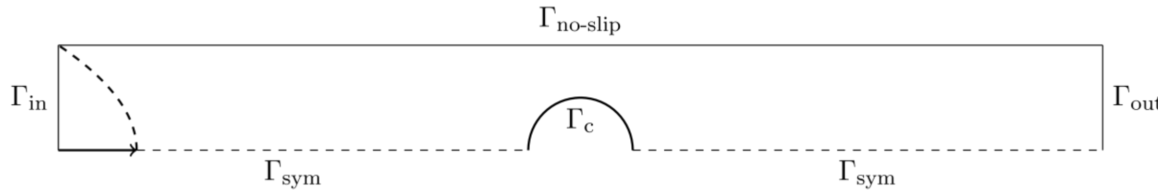
$$\boldsymbol{x}^{n+1}=\boldsymbol{x}^n-\omega_n\boldsymbol{A}(\boldsymbol{u}^n,\boldsymbol{\sigma}^n,\boldsymbol{\mu}^n)^{-1}\boldsymbol{r}(\boldsymbol{u}^n,p^n,\boldsymbol{\sigma}^n,\boldsymbol{\mu}^n\ )$$

- residual $\boldsymbol{r}$ and Jacobian matrix $\boldsymbol{A}$ problem-dependent

→ original approach, „Tensor Stokes“-problem

$$\boldsymbol{A}_T=\begin{pmatrix}-T_u(\boldsymbol{v})+J_u & B & 0 & -T_\mu(\boldsymbol{w})\\ B^T & 0 & 0 & 0\\ D+K_u(\boldsymbol{\tau}) & 0 & K_\sigma(\boldsymbol{w})+J_\sigma & 0\\ 0 & 0 & M & D(\boldsymbol{w})+J_\mu\end{pmatrix},\qquad \boldsymbol{r}_T=\begin{pmatrix}Bq-T(\boldsymbol{w})\boldsymbol{v}\\ B^T\boldsymbol{w}\\ D\boldsymbol{w}+K(\boldsymbol{w})\boldsymbol{\tau}\\ M\boldsymbol{\tau}+D(\boldsymbol{w})\boldsymbol{v}\end{pmatrix}$$

Flow around cylinder



- drag coefficient calculated via $C_D(\boldsymbol{T}) = \frac{2}{U_{\text{mean}}^2 R} \int_{E_c} \left(T_{xx} n_1 + T_{xy} n_2\right) \frac{\partial \varphi}{\partial x} \, dx$

→ problem-dependent total stress tensor:

$$\boldsymbol{T}_C = -p\boldsymbol{I} + 2\eta_S \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{\sigma}, \qquad \boldsymbol{T}_T = -p\boldsymbol{I} + 2\eta_S \boldsymbol{D}(\boldsymbol{u}) + \frac{1}{2}\left(\boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{D}(\boldsymbol{u}) \cdot \boldsymbol{\mu}^T\right)$$

Oldroyd-B ($\eta_s = 0.59$)

Λ	$C_D(\boldsymbol{T}_C)$	N_C	$C_D(\boldsymbol{T}_T)$	N_T	Ref.
0.1	130.342	2	130.348	3	130.36
0.6	117.695	3	117.970	3	117.78

Giesekus ($\alpha = 0.1, \eta_s = 0.59$)

Λ	$C_D(\boldsymbol{T}_C)$	N_C	$C_D(\boldsymbol{T}_T)$	N_T	Ref.
5.0	85.210	3	85.243	6	85.22
10.0	83.047	4	83.068	6	83.06

- very good agreement to reference and original approach
- appropriate solver behaviour

Summary and outlook

Summary

- new **„Tensor Diffusion"-approach -** particularly for „no solvent"-case

- direct model for $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{u}, \boldsymbol{D}(\boldsymbol{u}))$ possible? (1D-flows: **YES!**)

→ solve **linear „Stokes"-problem** instead of highly nonlinear system

→ otherwise: numerical calculation from algebraic equation

- validated for simple test cases, evaluated for more complex test cases
- allows modified solvers w.r.t. Operator-Splitting and Newton-multigrid

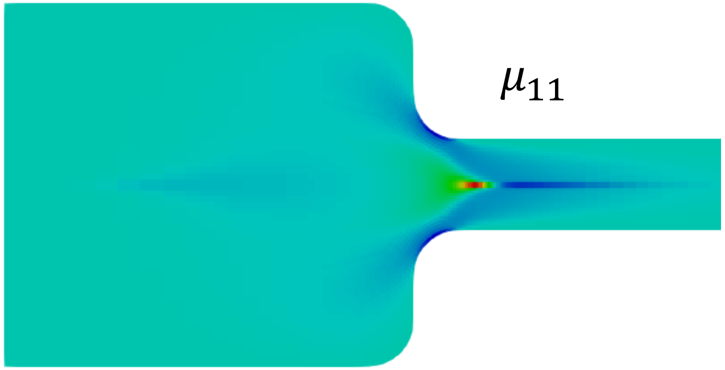
- more numerical tests
- detailed numerical analysis/theoretical background for new „Tensor Stokes"?

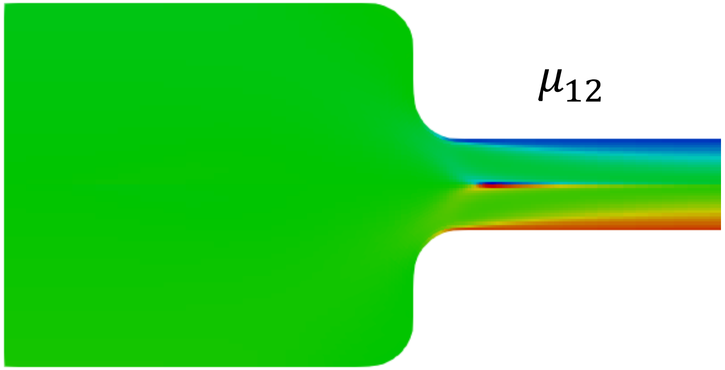
$$-\frac{1}{2}\nabla \cdot \left(\boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{D}(\boldsymbol{u}) \cdot \boldsymbol{\mu}^{T}\right) + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

- application to **integral models** possible?

Outlook

- (smoothed) contraction flow → reasonable $(\boldsymbol{u}, \boldsymbol{\sigma}, p)$-solution
- **but:** discontinuities/singularitites(?) in $\boldsymbol{\mu}$ along symmetry line





→ large amount of EOFEM (w.r.t. $\boldsymbol{\mu}$) to get smooth solution

- **alternative I**: Least-Squares $\frac{1}{2}(\|\boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{\sigma}\|^2 + \delta\|\nabla\boldsymbol{\mu}\|^2) \to \min$
- in four-field formulation: determine $\boldsymbol{\mu}$ from

$$\boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u}) \cdot \boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{\sigma} \cdot \boldsymbol{D}(\boldsymbol{u}) - \delta\Delta\boldsymbol{\mu} = 0$$

Outlook

- **recall:**

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{\sigma} - \nabla\boldsymbol{u}^T\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\nabla\boldsymbol{u} + \boldsymbol{f}\left(\Lambda,\eta_p,\boldsymbol{\sigma}\right) = \boldsymbol{2}\frac{\eta_p}{\Lambda}\boldsymbol{D}(\boldsymbol{u})$$

- inserting $\boldsymbol{\sigma} = \boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})$ into constitutive equation
- **but:** how to treat 2nd-order $\boldsymbol{u}$-derivatives in convective part?

$$(\boldsymbol{u}\cdot\nabla)[\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})] = (\boldsymbol{u}\cdot\nabla)[\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})] + (\nabla\cdot\boldsymbol{u})\cdot\left(\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})\right) = \nabla\cdot\left[\left(\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})\right)\otimes\boldsymbol{u}\right]$$

→ **alternative II:** PDE for $\boldsymbol{\mu}$

- **monolithic 3-field formulation** in $(\boldsymbol{u},\boldsymbol{\mu},p)$

$$-\frac{1}{2}\nabla\cdot\left(\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{D}(\boldsymbol{u})\cdot\boldsymbol{\mu}^T\right) + \nabla p = 0, \qquad \nabla\cdot\boldsymbol{u} = 0$$

$$\nabla\cdot\left[\left(\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})\right)\otimes\boldsymbol{u}\right] - \nabla\boldsymbol{u}^T\cdot[\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})] - [\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})]\cdot\nabla\boldsymbol{u} + \boldsymbol{f}\left(\Lambda,\eta_p,\boldsymbol{\mu},\boldsymbol{u}\right) = \boldsymbol{2}\frac{\eta_p}{\Lambda}\boldsymbol{D}(\boldsymbol{u})$$

→ improved calculation of $\boldsymbol{\mu}$, e.g. regarding „singularities“?