The „Tensor Diffusion approach“ for simulating viscoelastic fluids with special emphasis on „no solvent“-case

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Introduction

Simulation of viscoelastic fluids without solvent

Application in mind

- processing of (pure) rubber melts („Kautschuk“), industrial partner: ARLANXEO
- in CFD, consider „realistic“ viscoelastic fluids…
  … consisting of **wide relaxation time spectrum** (over several decades)
  → „multi mode“-approach for adequate modelling
  → High Weissenberg Number Problem (**HWNP**)
  … **without** „solvent contribution“, i.e. *no* polymer *solutions* considered
- **plus:** solver often intended for **direct steady-state** solutions (relevant for applications)
- governing equations for „no solvent“-case

$$\cancel{-2\eta_s \nabla \cdot \boldsymbol{D}(\boldsymbol{u})} - \nabla \cdot \boldsymbol{\sigma} + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

plus constitutive equation for stress tensor $\boldsymbol{\sigma}$

Modelling approaches for constitutive equations

- **differential** material model (for multiple „modes“/relaxation times $\Lambda_k, k = 1, \dots, K$)

$$\boldsymbol{\sigma} = \sum_{k=1}^{K} \boldsymbol{\sigma}_k, \qquad (\boldsymbol{u} \cdot \nabla)\boldsymbol{\sigma}_k - \nabla \boldsymbol{u}^T \cdot \boldsymbol{\sigma}_k - \boldsymbol{\sigma}_k \cdot \nabla \boldsymbol{u} + \boldsymbol{f}(\Lambda_k, \eta_{p,k}, \boldsymbol{\sigma}_k) = 2\frac{\eta_{p,k}}{\Lambda_k}\boldsymbol{D}(\boldsymbol{u})$$

  - model function: $\boldsymbol{f}(\Lambda_k, \eta_{p,k}, \boldsymbol{\sigma}_k) = \frac{1}{\Lambda_k}\left(\boldsymbol{\sigma}_k + \alpha_k \frac{\Lambda_k}{\eta_{p,k}} \boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma}_k\right)$ for $\alpha_k \in [0,1]$

- **integral** material model („Deformation Fields Method“, DFM, c.f. Hulsen et al.)

$$\boldsymbol{\sigma} = \int_0^\infty M(s)\, \boldsymbol{g}(\boldsymbol{B}(s))\, ds, \qquad \frac{\partial}{\partial s}\boldsymbol{B}(s) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B}(s) - \nabla \boldsymbol{u}^T \cdot \boldsymbol{B}(s) - \boldsymbol{B}(s) \cdot \nabla \boldsymbol{u} = 0$$

  - (multi-mode) memory function: $M(s) = \sum_{k=1}^{K} M_k(s) = \sum_{k=1}^{K} \frac{\eta_{p,k}}{\Lambda_k^2} \exp\left(-\frac{s}{\Lambda_k}\right)$

- in both cases for $\eta_s = 0$ („no solvent“):

Operator-Splitting not applicable vs. monolithic approach difficult

Monolithic approach

- „no solvent“ in Stokes equations

$$\cancel{-2\,\eta_s \nabla \cdot \boldsymbol{D}(\boldsymbol{u})} - \nabla \cdot \boldsymbol{\sigma} + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

- **differential** models for **single-mode** (matrix-vector-notation)

$$\begin{pmatrix} 0 & B & -C \\ B^T & 0 & 0 \\ D & 0 & K(\boldsymbol{u}) \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \\ \boldsymbol{\sigma} \end{pmatrix} = \begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_p \\ \boldsymbol{r}_\sigma \end{pmatrix}$$

  - standard Krylov-space methods: no diagonal preconditioning
  - multigrid
    - no diagonal smoother applicable
    - only Vanka-like smoothers work - but not robust
  - stability problems w.r.t. additional LBB for $\boldsymbol{u}, \boldsymbol{\sigma}$
- how to apply for **integral** models?!

Operator-Splitting

- given $\boldsymbol{u}^n, \boldsymbol{\sigma}^n, p^n$

1. solve **Stokes-problem** for $\boldsymbol{u}^{n+1}, p^{n+1}$ with non-zero RHS

$$-2\,\eta_s \nabla\cdot \boldsymbol{D}\left(\boldsymbol{u}^{n+1}\right) + \nabla p^{n+1} = \nabla\cdot\boldsymbol{\sigma}^n, \qquad \nabla\cdot\boldsymbol{u}^{n+1} = 0$$

2. calculate $\boldsymbol{\sigma}^{n+1}$ from **differential** ($\boldsymbol{\sigma}_k^{n+1}$ for „multi-mode" $\boldsymbol{\sigma} = \sum_{k=1}^{K}\boldsymbol{\sigma}_k$)

$$(\boldsymbol{u}^{n+1}\cdot\nabla)\boldsymbol{\sigma}^{n+1} - \nabla\boldsymbol{u}^{n+1^T}\cdot\boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^{n+1}\cdot\nabla\boldsymbol{u}^{n+1} + \boldsymbol{f}\left(\Lambda, \eta_p, \boldsymbol{\sigma}^{n+1}\right) = 2\frac{\eta_p}{\Lambda}\boldsymbol{D}(\boldsymbol{u}^{n+1})$$

or **integral** constitutive equation / DFM

$$\boldsymbol{\sigma}^{n+1} = \int_0^\infty M(s)\,\boldsymbol{g}\left(\boldsymbol{B}(s)\right)ds$$

$$\frac{\partial}{\partial s}\boldsymbol{B}(s) + (\boldsymbol{u}^{n+1}\cdot\nabla)\boldsymbol{B}(s) - \nabla\boldsymbol{u}^{n+1^T}\cdot\boldsymbol{B}(s) - \boldsymbol{B}(s)\cdot\nabla\boldsymbol{u}^{n+1} = 0$$

- not applicable for „no solvent" - independently of actual model!

→ Stokes problem without diffusive part

The „Tensor Diffusion“ approach

Assumption: „Tensor Diffusion“ $\boldsymbol{\mu}$ via $\boldsymbol{\sigma} = \boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u})$

- idea: in 2D („no solvent“) Stokes equations, replace $\boldsymbol{\sigma}$ by $\boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u})$, $\boldsymbol{\mu} \in \mathbb{R}^{2\times2}$ or $\mathbb{R}^{3\times3}$

$$-\nabla \cdot \boldsymbol{\sigma} + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

→ consider newly defined (symmetrized) „Tensor Stokes“-problem

$$-\frac{1}{2}\nabla \cdot \left(\boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{D}(\boldsymbol{u}) \cdot \boldsymbol{\mu}^T\right) + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

- issues to tackle
  - how to determine „Tensor Diffusion“ $\boldsymbol{\mu}$?
  - (potential) benefits?
  - reasonable assumption / approach?

„Tensor Diffusion“ $\boldsymbol{\mu}$ given/known

- consider (symmetrized) „Tensor Stokes“-problem

$$-\frac{1}{2}\nabla \cdot \left(\boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{D}(\boldsymbol{u}) \cdot \boldsymbol{\mu}^{T}\right) + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

- **approach I:** „Tensor Diffusion“ $\boldsymbol{\mu}$ known from „somewhere“
  - original problem: Stokes equations coupled with nonlinear diff./int. model
  - → fully coupled nonlinear *system* in $(\boldsymbol{u}, \boldsymbol{\sigma}, p)$
  - → but by introducing „Tensor Diffusion“:

direct computation of „nonlinear“ solution via pure $(\boldsymbol{u}, p)$-problem

- $\boldsymbol{\sigma}$ computed in postprocessing, only

„Tensor Diffusion“ $\boldsymbol{\mu}$ from algebraic equation

- **approach II:** determine „Tensor Diffusion“ $\boldsymbol{\mu}$ from algebraic equation
  - consider single-mode differential models for „no-solvent“

$$-\nabla \cdot \boldsymbol{\sigma} + \nabla p = 0 \qquad \textit{or} \qquad -\frac{1}{2}\nabla \cdot \left(\boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{D}(\boldsymbol{u}) \cdot \boldsymbol{\mu}^{T}\right) + \nabla p = 0,$$

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{\sigma} - \nabla \boldsymbol{u}^{T} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \nabla \boldsymbol{u} + \boldsymbol{f}\left(\Lambda, \eta_p, \boldsymbol{\sigma}\right) = 2\frac{\eta_p}{\Lambda}\boldsymbol{D}(\boldsymbol{u}),$$

$$\nabla \cdot \boldsymbol{u} = 0, \qquad \boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{\sigma} = 0$$

  - **1st alternative:** $(\boldsymbol{u}, \boldsymbol{\sigma}, p)$-solution does not depend on $\boldsymbol{\mu}$ („postprocessing fashion“)
  - → **2nd alternative:** *four-field* formulation of symmetrized „Tensor-Stokes“ problem, $(\boldsymbol{u}, \boldsymbol{\sigma}, p)$ coupled with $\boldsymbol{\mu}$
  - discrete operators and nonlinear systems via **FEM** with $Q_2/P_1^{\text{disc}}/\ Q_2/\ Q_0$

„Tensor Diffusion“ $\boldsymbol{\mu}$ from PDE

- **approach III:** determine „Tensor Diffusion“ $\boldsymbol{\mu}$ from PDE
  - insert stress decomposition into constitutive equation for $\boldsymbol{\sigma}$

$$(\boldsymbol{u}\cdot\nabla)[\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})]-\nabla\boldsymbol{u}^T\cdot[\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})]-[\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})]\cdot\nabla\boldsymbol{u}+\frac{1}{\Lambda}[\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})]=\mathbf{2}\frac{\eta_p}{\Lambda}\boldsymbol{D}(\boldsymbol{u})$$

  - suitable treatment of 2nd $\boldsymbol{u}$-derivatives:

$$(\boldsymbol{u}\cdot\nabla)[\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})]=(\boldsymbol{u}\cdot\nabla)[\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})]+(\nabla\cdot\boldsymbol{u})\cdot\big(\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})\big)=\nabla\cdot\big[\big(\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})\big)\otimes\boldsymbol{u}\big]$$

→ monolithic *three-field* formulation in $(\boldsymbol{u},\boldsymbol{\mu},p)$

$$-\frac{1}{2}\nabla\cdot\big(\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})+\boldsymbol{D}(\boldsymbol{u})\cdot\boldsymbol{\mu}^T\big)+\nabla p=0,\qquad \nabla\cdot\boldsymbol{u}=0$$

$$\nabla\cdot\big[\big(\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})\big)\otimes\boldsymbol{u}\big]-\nabla\boldsymbol{u}^T\cdot[\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})]-[\boldsymbol{\mu}\cdot\boldsymbol{D}(\boldsymbol{u})]\cdot\nabla\boldsymbol{u}+\boldsymbol{f}\big(\Lambda,\eta_p,\boldsymbol{\mu},\boldsymbol{u}\big)=\mathbf{2}\frac{\eta_p}{\Lambda}\boldsymbol{D}(\boldsymbol{u})$$

- now, **non-vanishing velocity coupling** in momentum equation – even for „no-solvent“!
- potential benefit regarding numerical difficulties?

Potential benefits

- prototypical **Operator-Splitting** in iterative methods
  1. for given $\boldsymbol{\mu}^n$ (e.g. $\boldsymbol{\mu} = \boldsymbol{I}$ for $n = 0$) determine $(\boldsymbol{u}^{n+1}, p^{n+1})$ from „Tensor Stokes“-problem

$$-\frac{1}{2}\nabla \cdot \left(\boldsymbol{\mu}^n \cdot \boldsymbol{D}(\boldsymbol{u}^{n+1}) + \boldsymbol{D}(\boldsymbol{u}^{n+1}) \cdot \boldsymbol{\mu}^{n^T}\right) + \nabla p^{n+1} = 0, \qquad \nabla \cdot \boldsymbol{u}^{n+1} = 0$$

approach II – four-field formulation in $(\boldsymbol{u}, p, \boldsymbol{\sigma}, \boldsymbol{\mu})$

2. for $\boldsymbol{u}^{n+1}$, determine stress tensor $\boldsymbol{\sigma}^{n+1}$ from **integral** or **differential** model
3. determine (tensor) viscosity $\boldsymbol{\mu}^{n+1}$ via $\boldsymbol{\sigma}^{n+1} = \boldsymbol{\mu}^{n+1} \cdot \boldsymbol{D}(\boldsymbol{u}^{n+1})$

approach III – three-field formulation in $(\boldsymbol{u}, p, \boldsymbol{\mu})$

2. for $\boldsymbol{u}^{n+1}$, calculate $\boldsymbol{\mu}^{n+1}$ from PDE

Potential benefits

- **monolithic** solution approach for „no-solvent“ (differential models only, single-mode)

→ diffusive operator introdcued in „natural way“

approach II – four-field formulation in $(\boldsymbol{u}, p, \boldsymbol{\sigma}, \boldsymbol{\mu})$

$$\begin{pmatrix} -T(\boldsymbol{\mu}) & B & 0 & 0 \\ B^T & 0 & 0 & 0 \\ D & 0 & K(\boldsymbol{u}) & 0 \\ 0 & 0 & M & D(\boldsymbol{u}) \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \\ \boldsymbol{\sigma} \\ \boldsymbol{\mu} \end{pmatrix} = \begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_p \\ \boldsymbol{r}_\sigma \\ \boldsymbol{r}_\mu \end{pmatrix}$$

approach I – pure $(\boldsymbol{u}, p)$-problem

$$\begin{pmatrix} -\tilde{T}(\boldsymbol{u}) & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_p \end{pmatrix}$$

→ Operator-Splitting / standard Stokes-solvers applicable for all approaches

Proof of concept

$\boldsymbol{\mu}$ known analytically

- in the following, consider **fully developed channel flows**

$$\boldsymbol{u} = \binom{u}{v} = \binom{u(y)}{0}, \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial}{\partial x}\sigma_{ij} = \frac{\partial}{\partial x}B_{ij} = 0 \ \text{ for } i,j \in \{1{,}2\}$$

- for **differential version** of UCM, parabolic velocity profile obtained
- corresponding stresses read

$$\boldsymbol{\sigma}_{\mathrm{diff}} = \begin{pmatrix} 2\Lambda\eta_p\left(\frac{\partial u}{\partial y}\right)^2 & \eta_p\frac{\partial u}{\partial y} \\ \eta_p\frac{\partial u}{\partial y} & 0 \end{pmatrix}, \boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{pmatrix} \Rightarrow \boldsymbol{\mu}_{\mathrm{diff}}(\boldsymbol{u}) = 2\eta_p\begin{pmatrix} 1 & 2\Lambda\frac{\partial u}{\partial y} \\ 0 & 1 \end{pmatrix}$$

→ tensor quantity relating $\boldsymbol{\sigma}$ to $\boldsymbol{D}(\boldsymbol{u})$ known analytically

$\boldsymbol{\mu}$ known analytically

- in DFM, analytical solution of Finger tensors for $s \in [0,\infty[$ given

$$B_{22}(s) = 1, \qquad B_{12}(s) = s\frac{\partial u}{\partial y}, \qquad B_{11}(s) = s^2\left(\frac{\partial u}{\partial y}\right)^2 + 1$$

- inserting into **stress integral** for UCM gives

$$\boldsymbol{\sigma} = \int_0^\infty \frac{\eta_p}{\Lambda^2}\exp\left(-\frac{s}{\Lambda}\right)(\boldsymbol{B}(s) - \boldsymbol{I})ds = \left\{2\int_0^\infty \frac{\eta_p}{\Lambda^2}\exp\left(-\frac{s}{\Lambda}\right)\begin{pmatrix} s & s^2\frac{\partial u}{\partial y} \\ 0 & s \end{pmatrix} ds\right\}\left\{\frac{1}{2}\begin{pmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{pmatrix}\right\}$$

→ decomposition of the stress tensor $\boldsymbol{\sigma} = \boldsymbol{\mu}(\boldsymbol{u}) \cdot \boldsymbol{D}(\boldsymbol{u})$

$$\boldsymbol{\mu}_{\text{int}}(\boldsymbol{u}) = 2\eta_p\begin{pmatrix} 1 & 2\Lambda\frac{\partial u}{\partial y} \\ 0 & 1 \end{pmatrix} = \boldsymbol{\mu}_{\text{diff}}(\boldsymbol{u})$$

- similar to differential case: **„Tensor Diffusion“ known analytically**

$\boldsymbol{\mu}$ known „semi-analytically“ - PSM

- for fully developed channel flows, PSM reads

$$\boldsymbol{\sigma} = \int_0^\infty \frac{\eta_p}{\Lambda^2} \exp\left(-\frac{s}{\Lambda}\right) \frac{1}{1 + \gamma\left(s^2\left(\frac{\partial u}{\partial y}\right)^2\right)} \begin{pmatrix} s^2\left(\frac{\partial u}{\partial y}\right)^2 + 1 & s\frac{\partial u}{\partial y} \\ s\frac{\partial u}{\partial y} & 1 \end{pmatrix} ds$$

- similar to UCM, stress integral can be decomposed into

$$\boldsymbol{\sigma} = \begin{pmatrix} g\left(\frac{\partial u}{\partial y}\right) & h\left(\frac{\partial u}{\partial y}\right) \\ 0 & g\left(\frac{\partial u}{\partial y}\right) \end{pmatrix} \boldsymbol{D}(\boldsymbol{u}) + \begin{pmatrix} f\left(\frac{\partial u}{\partial y}\right) & 0 \\ 0 & f\left(\frac{\partial u}{\partial y}\right) \end{pmatrix}$$

→ „Tensor Diffusion“ $\boldsymbol{\mu}$ explicity modelled depending on $\frac{\partial u}{\partial y}$

$\boldsymbol{\mu}$ known „semi-analytically“ - PSM

→ direct modelling of „Tensor Diffusion“ according to $\boldsymbol{\mu} = \boldsymbol{\mu}\left(\frac{\partial u}{\partial y}\right)$ ($\frac{\partial u}{\partial y}$ ≈shear rate)

- for 1D-flows, procedure in principle applicable for „all“ integral models
- isotropic part absorbed into pressure $P = p - f\left(\frac{\partial u}{\partial y}\right)$

→ „transform“ full viscoelastic integral model to **generalized non-Newtonian model**

$$-\nabla \cdot \left(\boldsymbol{\mu}\left(\frac{\partial u}{\partial y}\right) \cdot \boldsymbol{D}(\boldsymbol{u}) \right) + \nabla P = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

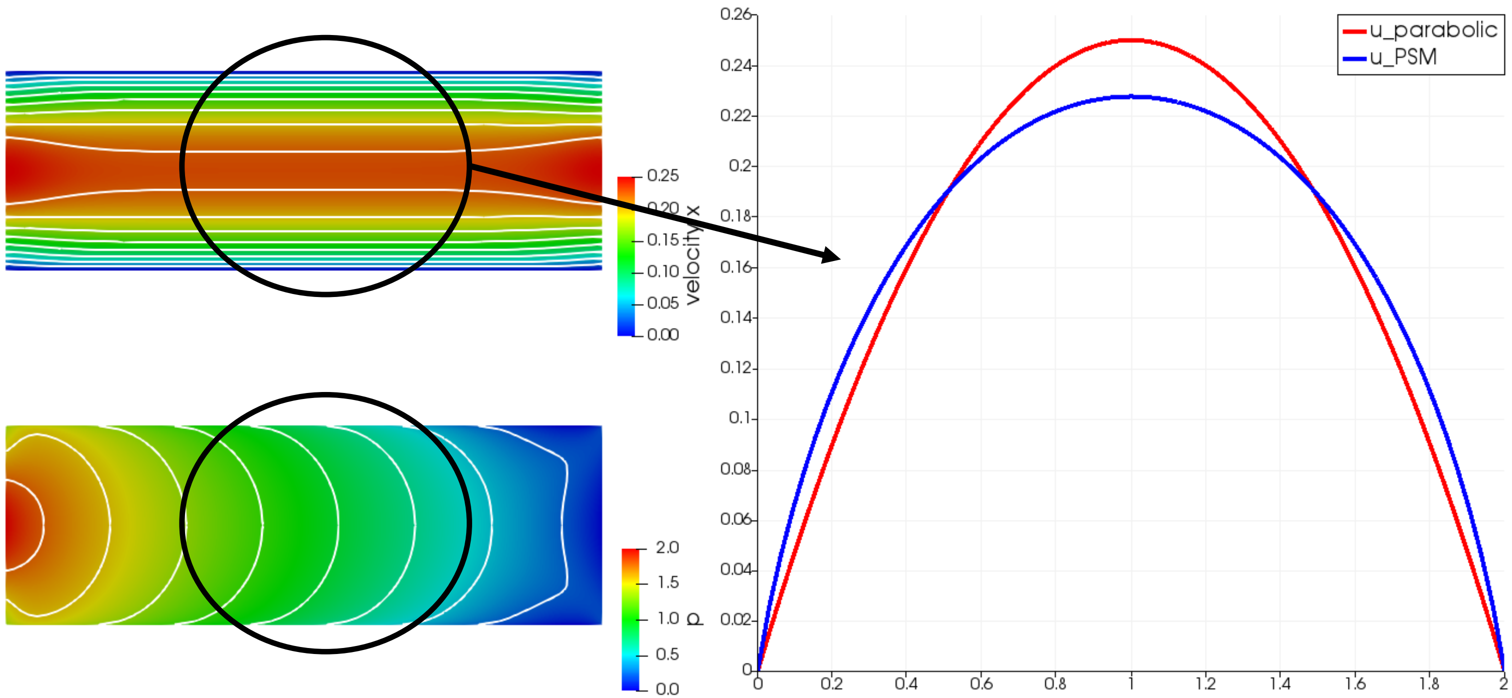
- complex rheology (arising from stress integral) hidden in „Tensor Diffusion“
- now: for PSM-channel flow solve only
  - „generalized tensor-valued“ Stokes problem
  - including non-vanishing **tensor-valued viscosity (NEW!)**

Poiseuille flow - PSM

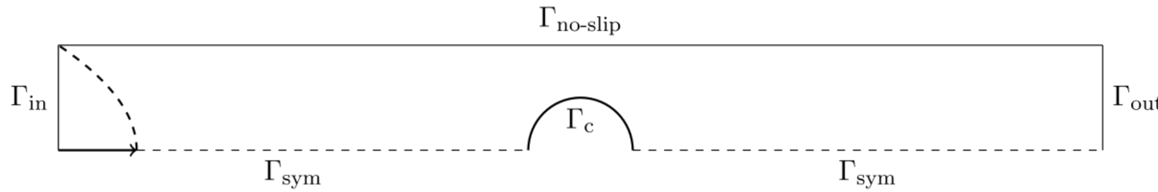
- „wrong“ initial parabolic velocity profile, evaluate „suitable“ $\boldsymbol{\mu}$

$$\boldsymbol{u}_0 \;\rightarrow \boldsymbol{\mu}_0 \;\Rightarrow \boldsymbol{u}_1 \;\rightarrow \boldsymbol{\mu}_1 \;\Rightarrow \boldsymbol{u}_2 \;\ldots$$

→ pure **Stokes-like problem** gives viscoelastic solution from **integral model**



Flow around cylinder



- simulations via *four-field* formulation of „Tensor Stokes"-problem
- key feature: **monolithic Newton-multigrid approach**
- drag coefficient calculated via $C_D(\boldsymbol{T}) = \frac{2}{U_{\text{mean}}^2 R} \int_{E_c} \left(T_{xx} n_1 + T_{xy} n_2\right) \frac{\partial \varphi}{\partial x}\, dx$

→ specific total stress tensor problem-dependent:

$$\boldsymbol{T}_C = -p\boldsymbol{I} + 2\eta_s \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{\sigma}, \qquad \boldsymbol{T}_T = -p\boldsymbol{I} + 2\eta_s \boldsymbol{D}(\boldsymbol{u}) + \frac{1}{2}\left(\boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{D}(\boldsymbol{u}) \cdot \boldsymbol{\mu}^T\right)$$

Oldroyd-B ($\eta_s = 0.59$)

Λ	$C_D(\boldsymbol{T}_C)$	N_C	$C_D(\boldsymbol{T}_T)$	N_T	Ref.
0.1	130.342	2	130.348	3	130.36
0.6	117.695	3	117.970	3	117.78

Giesekus ($\alpha = 0.1, \eta_s = 0.59$)

Λ	$C_D(\boldsymbol{T}_C)$	N_C	$C_D(\boldsymbol{T}_T)$	N_T	Ref.
5.0	85.210	3	85.243	6	85.22
10.0	83.047	4	83.068	6	83.06

- good agreement to original approach and reference
- appropriate solver behaviour

Conclusion

Summary

- original problem: nonlinear system in $(\boldsymbol{u}, \boldsymbol{\sigma}, p)$ for „no-solvent“ in 2D

$$\cancel{-2\eta_s \nabla \cdot \boldsymbol{D}(\boldsymbol{u})} - \nabla \cdot \boldsymbol{\sigma} + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{\sigma} - \nabla \boldsymbol{u}^T \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \nabla \boldsymbol{u} + \boldsymbol{f}\left(\Lambda, \eta_p, \boldsymbol{\sigma}\right) = 2\frac{\eta_p}{\Lambda}\boldsymbol{D}(\boldsymbol{u}) \text{ (or integral model)}$$

- introducing „Tensor Diffusion“ via $\boldsymbol{\sigma} = \boldsymbol{\mu} \cdot \boldsymbol{D}(\boldsymbol{u})$, $\boldsymbol{\mu} \in \mathbb{R}^{2\times 2}$ or $\mathbb{R}^{3\times 3}$

explicit model (1D-flows, semi-analytically)

four-field formulation (algebraic approach)

three-field formulation (PDE approach)

currently most interesting for future work!

- „Tensor Diffusion“ not known
- numerical calculation from algebraic equation/PDE
- validated, evaluated for complex test cases
- more numerical tests, detailed numerical analysis

Outlook

- explicit (semi-analytical) model of „Tensor Diffusion" $\boldsymbol{\mu} = \boldsymbol{\mu}\big(\boldsymbol{u}, \boldsymbol{D}(\boldsymbol{u})\big)$
- for 1D-flows, complex rheology can be hidden in $\boldsymbol{\mu}\left(\frac{\partial u}{\partial y}\right)$
- generalization of direct modelling approach?

→ even for complex 2D-configurations: instead of nonlinear *system*…

…solve „Tensor Stokes"-problem in $(\boldsymbol{u}, p)$ only

$$-\frac{1}{2}\nabla \cdot \left(\boldsymbol{\mu}(\boldsymbol{u}) \cdot \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{D}(\boldsymbol{u}) \cdot \boldsymbol{\mu}(\boldsymbol{u})^T\right) + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0$$

Can steady viscoelastic fluids be modelled as
generalized non-Newtonian Stokes equations including a *tensor-valued viscosity*?